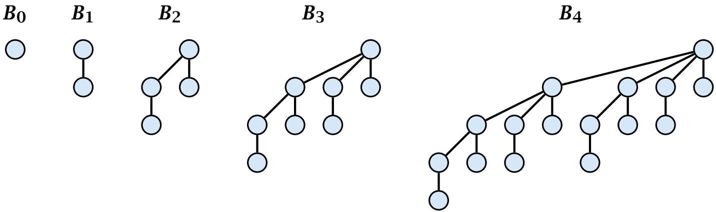
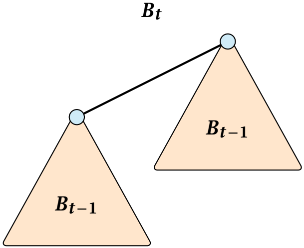
8.2 Binomial Heaps

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$**	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees





Binomial Trees

Properties of Binomial Trees

- B_k has 2^k nodes.

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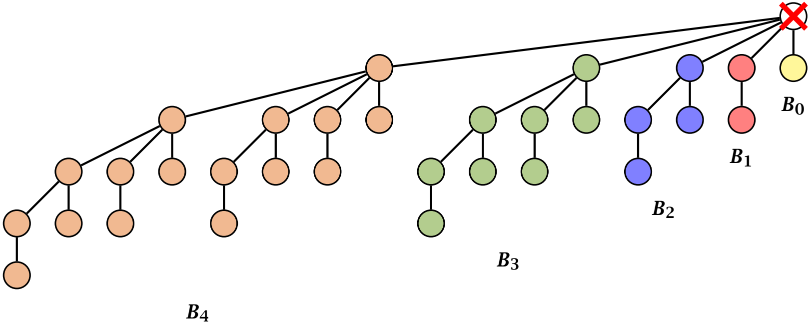
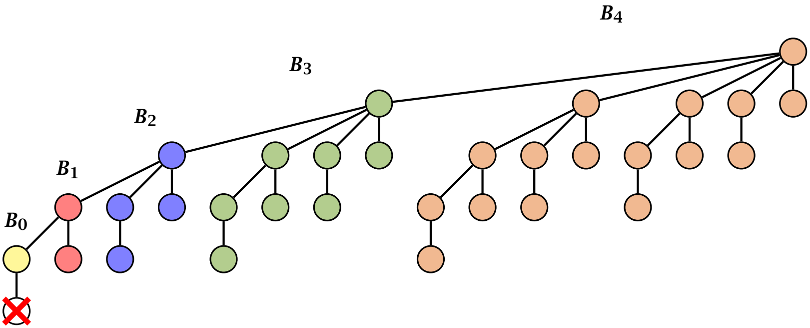
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

Binomial Trees



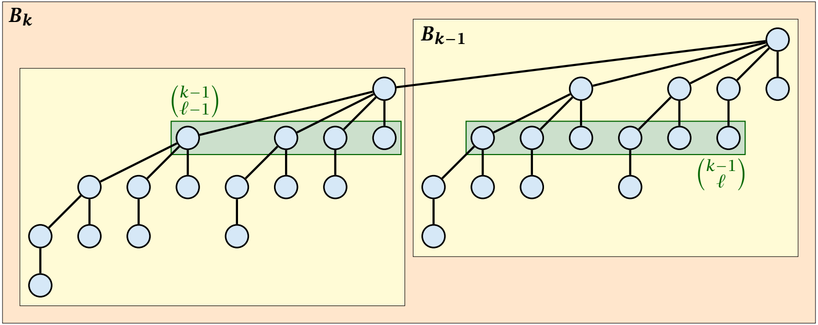
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

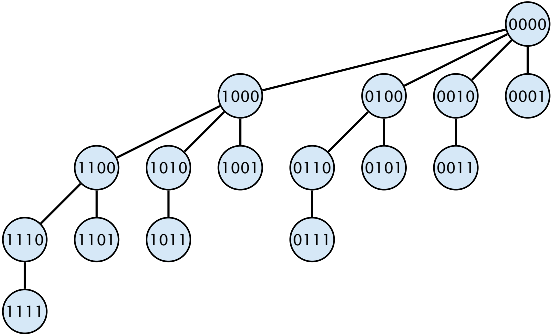
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees



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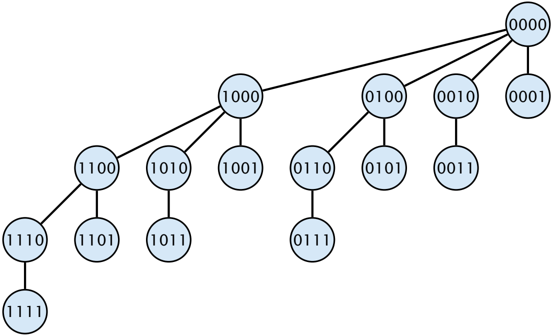
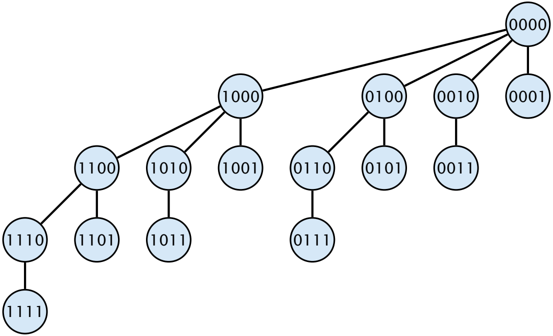


The binomial tree B_k is a sub-graph of the hypercube H_k.

Binomial Trees



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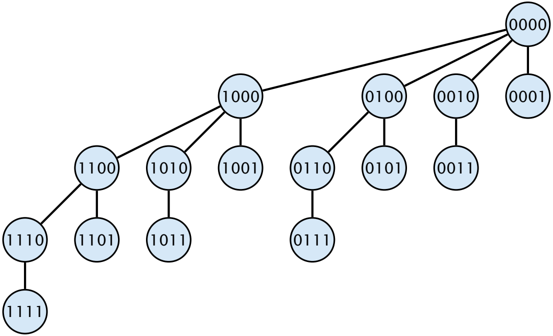
The parent of a node with label $b_k, \ldots, b_1$ is obtained by setting the least significant 1-bit to 0.

Binomial Trees



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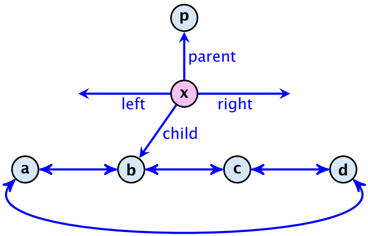
The ℓ-th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.



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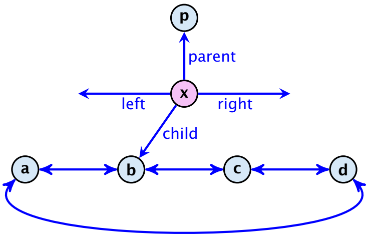

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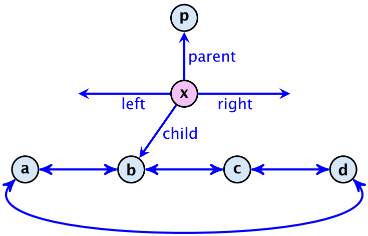

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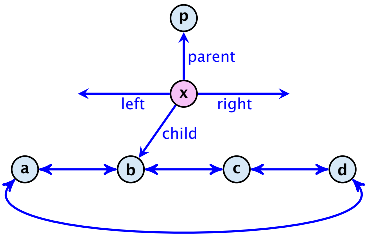
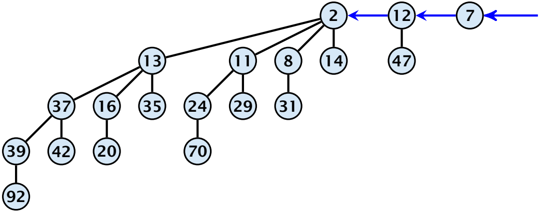
- The children of a node are arranged in a circular linked list.
- A child-pointer points to an arbitrary node within the list.
- A parent-pointer points to the parent node.
- Pointers $x.\mathrm{left}$ and $x.\mathrm{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\mathrm{left} = x.\mathrm{right} = x$).



8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap



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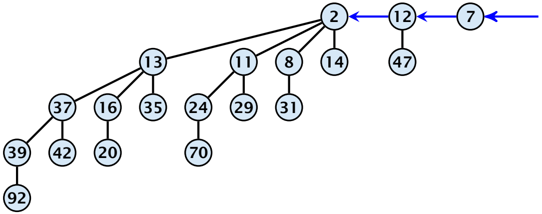
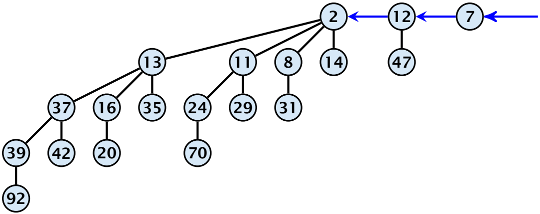


In a binomial heap the keys are arranged in a collection of binomial trees.

Binomial Heap



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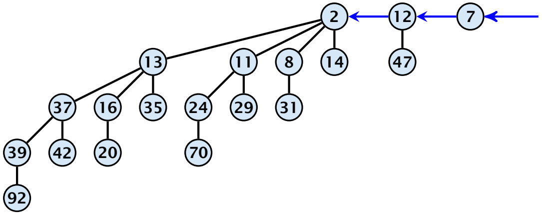

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There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

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Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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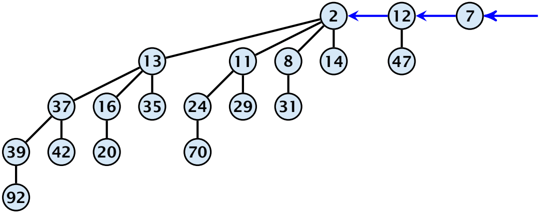
Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

Binomial Heap

Properties of a heap with n keys:



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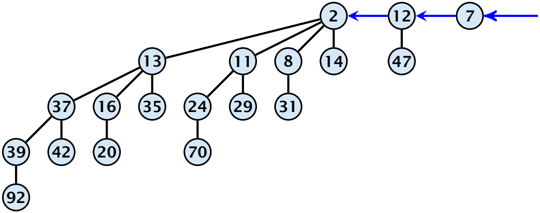

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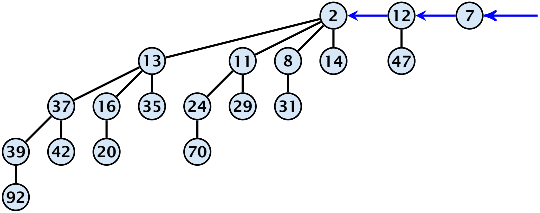

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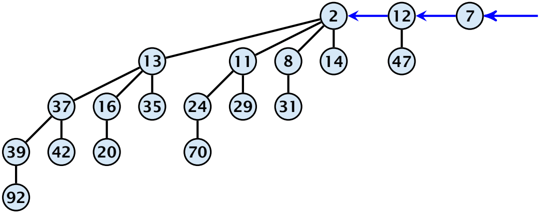

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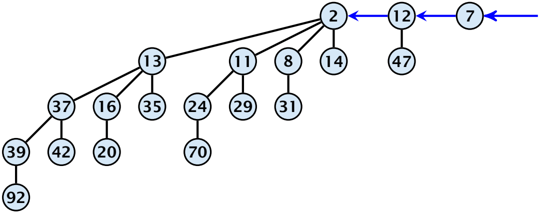

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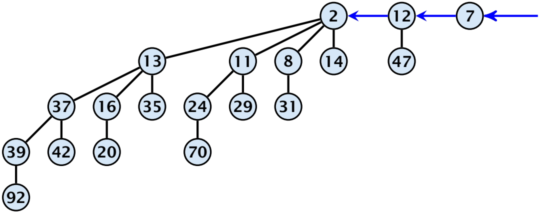

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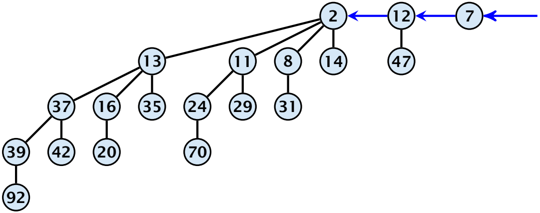
- The trees are stored in a single-linked list; ordered by dimension/size.



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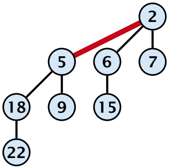

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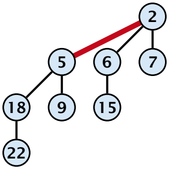
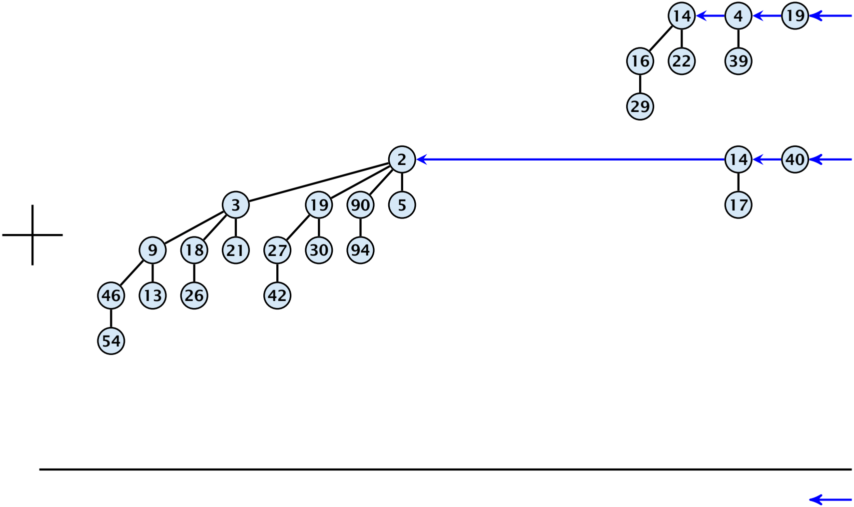
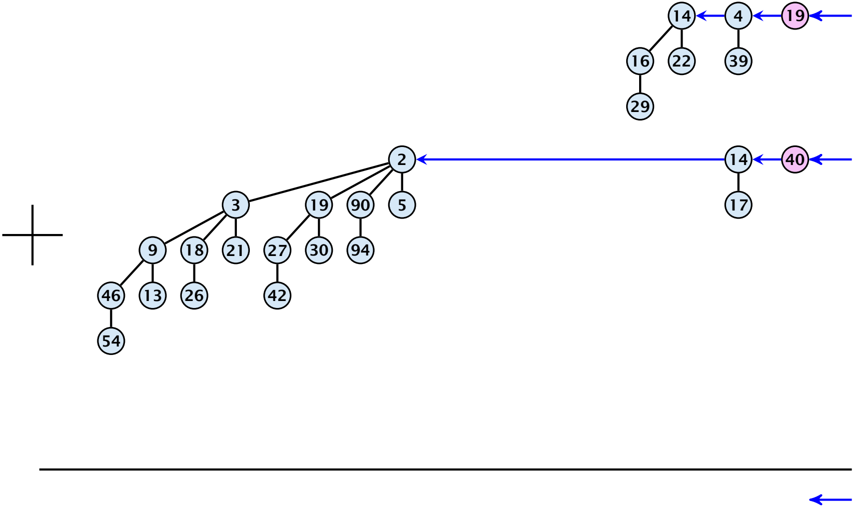


For more trees the technique is analogous to binary addition.

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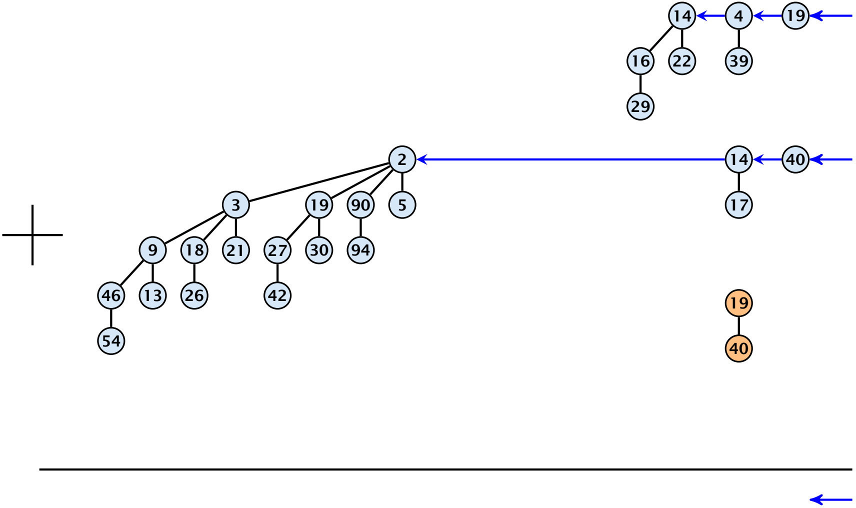

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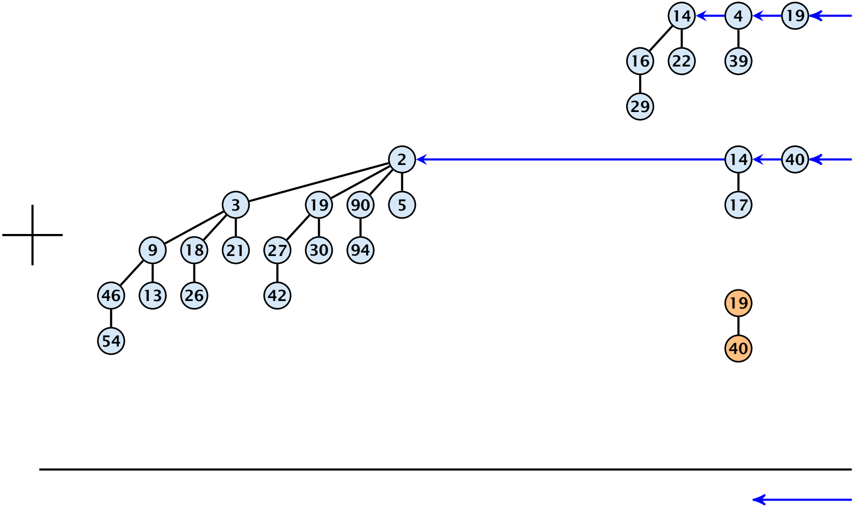
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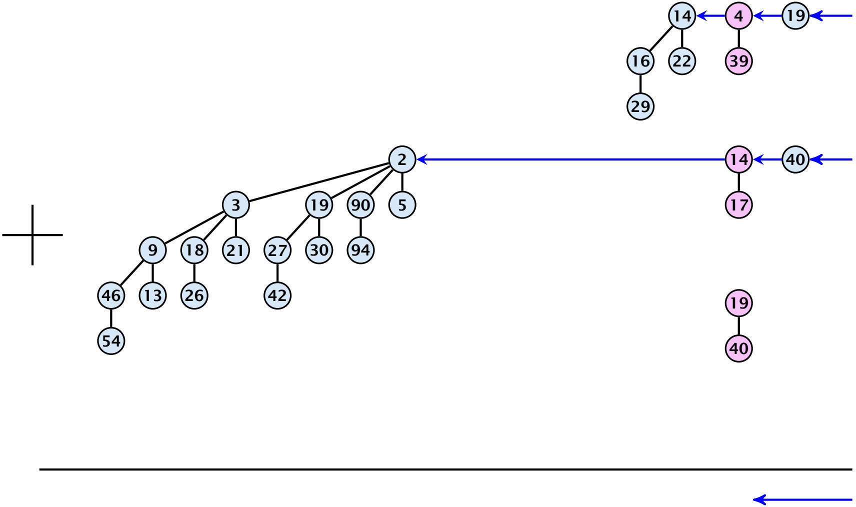

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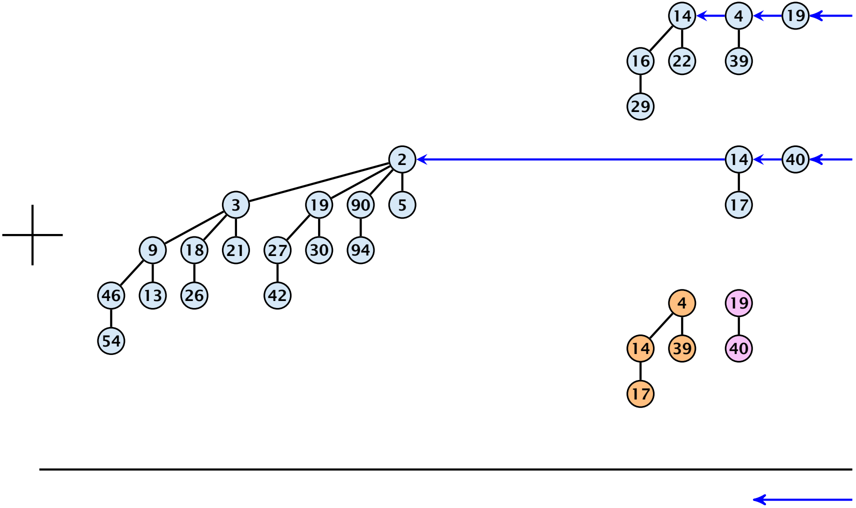

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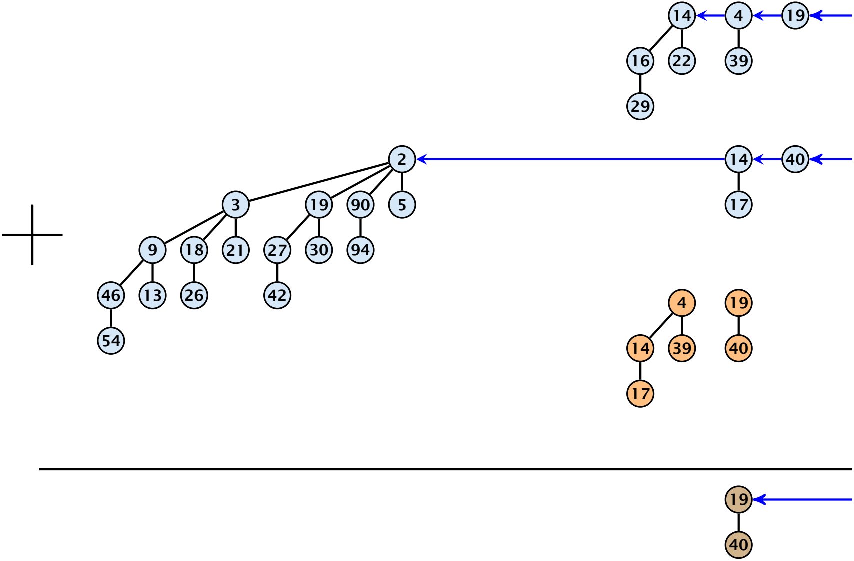
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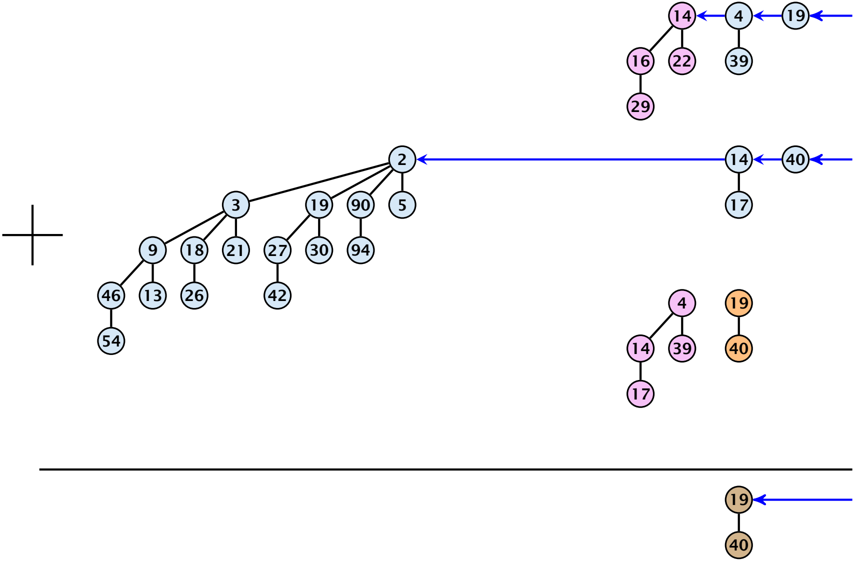
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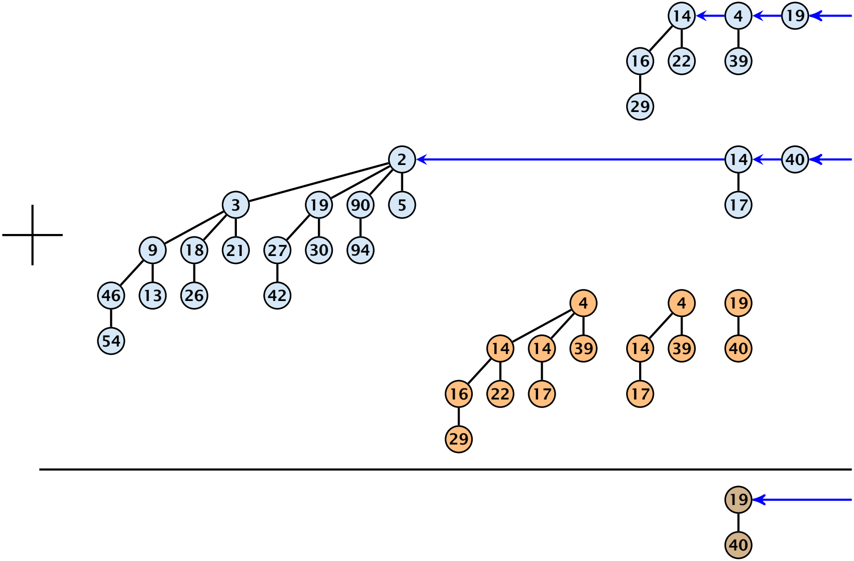

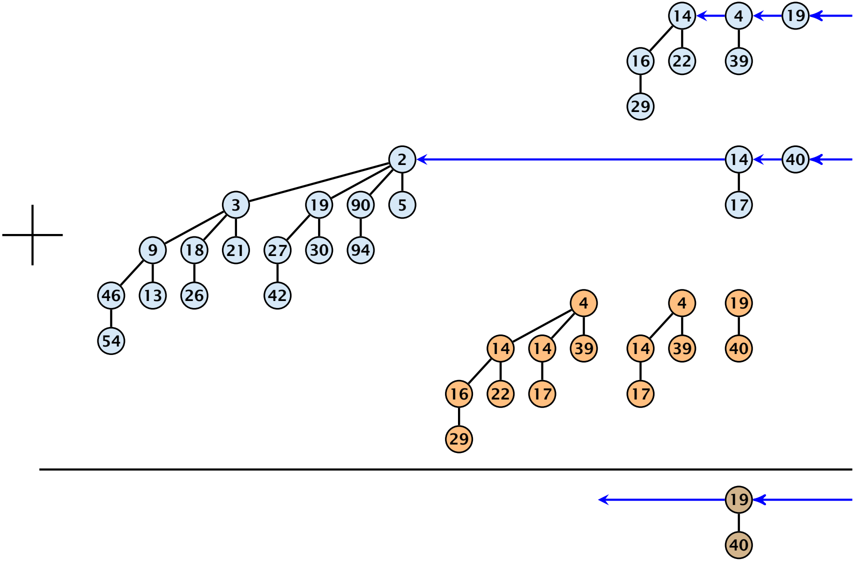

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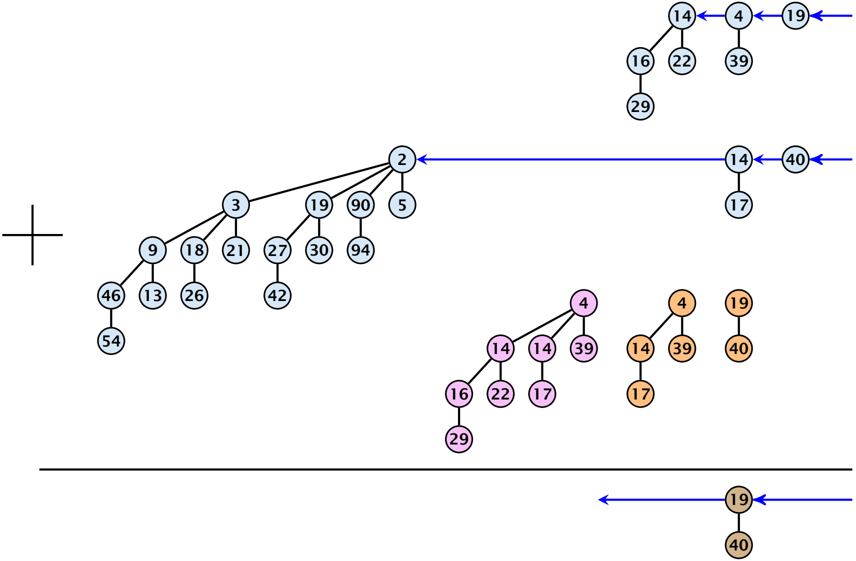

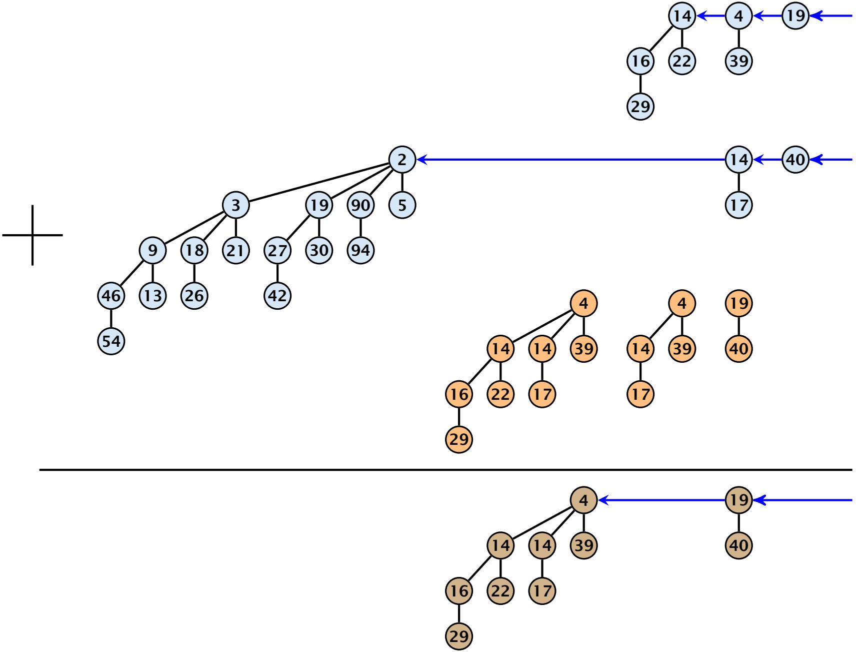

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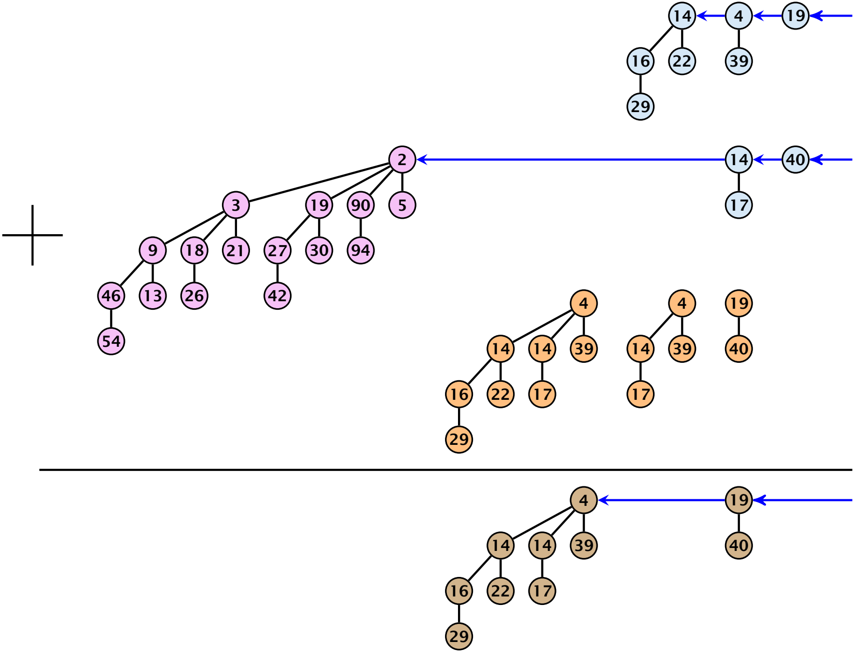

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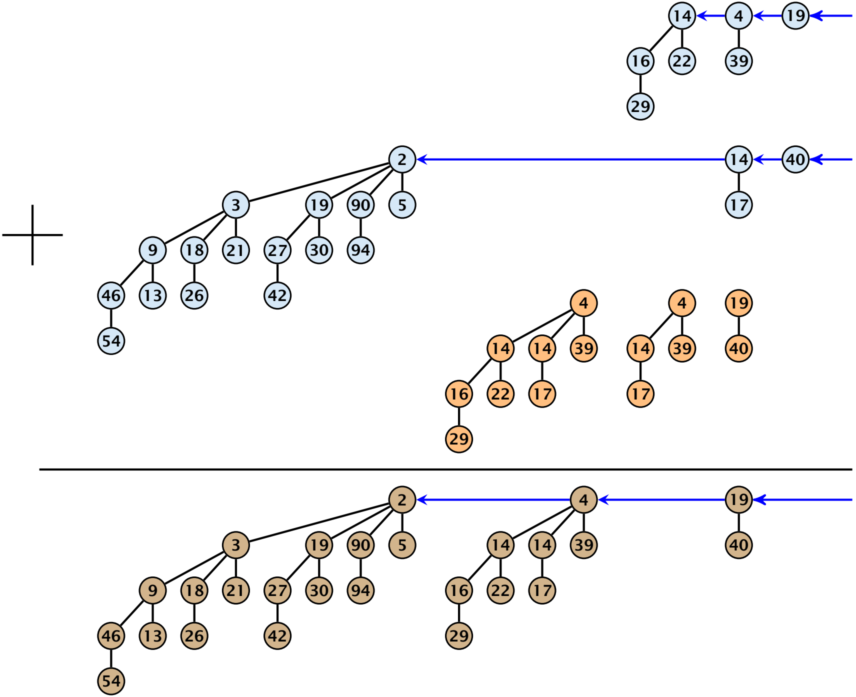
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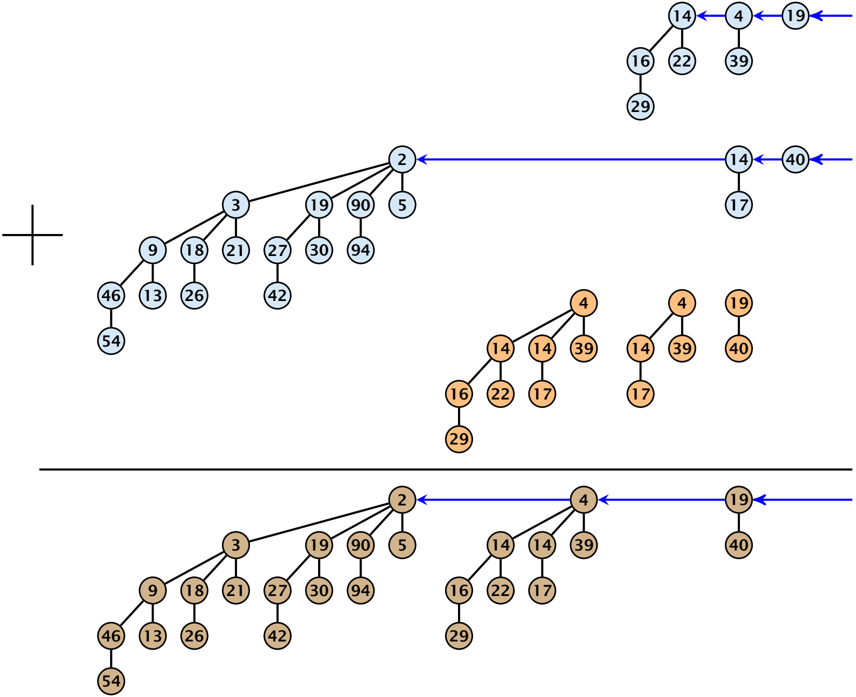

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- Time: $\mathcal{O}(\log n)$.

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All other operations can be reduced to $\mathrm{merge}()$.

$S.\mathrm{insert}(x)$:

- Create a new heap S' that contains just the element x.

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$S.\mathrm{insert}(x)$:

- Create a new heap S' that contains just the element x.
- Execute $S.\mathrm{merge}(S')$.

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- Create a new heap S' that contains just the element x.
- Execute $S.\mathrm{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

- Find the minimum key-value among all roots.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete-min():

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- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.

8.2 Binomial Heaps

S. delete-min():

- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.
- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).

8.2 Binomial Heaps

$S.$ delete-min():

- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.
- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- Compute $S.\mathrm{merge}(S')$.

8.2 Binomial Heaps

S. delete-min():

- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.
- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- Compute $S.\mathrm{merge}(S')$.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. decrease-key(handle h):

8.2 Binomial Heaps

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- Decrease the key of the element pointed to by h.

8.2 Binomial Heaps

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- Bubble the element up in the tree until the heap property is fulfilled.

8.2 Binomial Heaps

$S.$ **decrease-key(handle h):**

- Decrease the key of the element pointed to by h.
- Bubble the element up in the tree until the heap property is fulfilled.
- Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete(handle h):

8.2 Binomial Heaps

S. delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.

8.2 Binomial Heaps

S. delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.

8.2 Binomial Heaps

$S.\text{delete}(\text{handle } h)$:

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.
- Time: $\mathcal{O}(\log n)$.