21 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- ▶ Input: undirected, bipartite graph $G = L \cup R, E$.
- ▶ an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- ▶ find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

**Simplifying Assumptions (wlog [why?]):**

- ▶ assume that $|L| = |R| = n$
- ▶ assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- ▶ can assume goal is to construct maximum weight perfect matching

Weighted Bipartite Matching

Theorem 98 (Halls Theorem)

A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.

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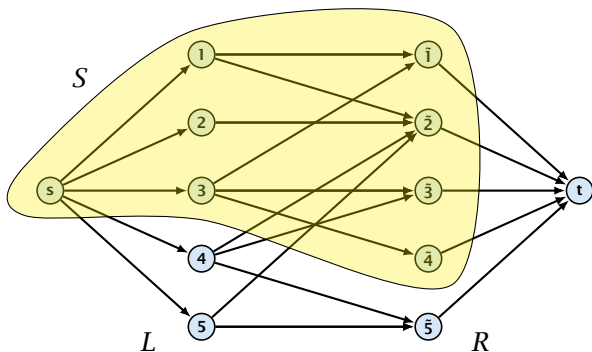
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- Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

Algorithm Outline

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▶ Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.

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- ▶ Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

Algorithm Outline

Reason:

▶ The weight of your matching M^* is

$$\sum_{(u,v)\in M^*} w_{(u,v)} = \sum_{(u,v)\in M^*} (x_u + x_v) = \sum_v x_v \ .$$

▶ Any other perfect matching M (in G, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v)\in M} w_{(u,v)} \leq \sum_{(u,v)\in M} (x_u + x_v) = \sum_v x_v \ .$$

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If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

Changing Node Weights
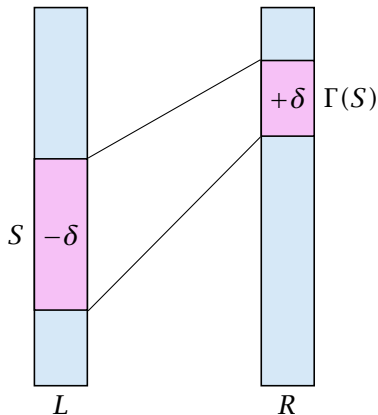
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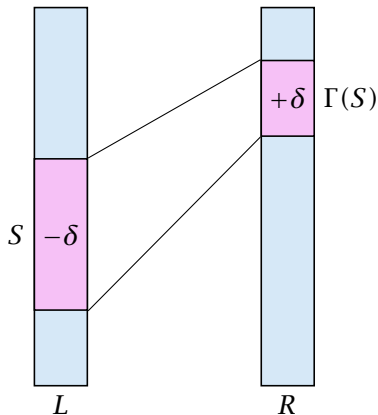
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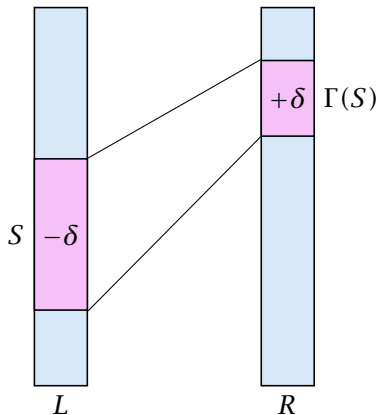
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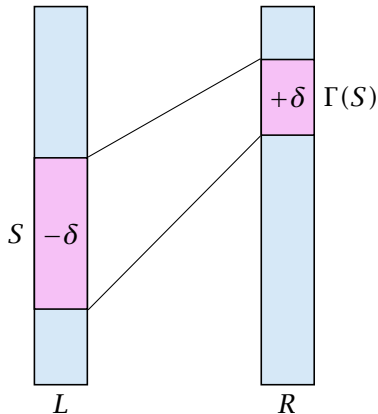
Changing Node Weights

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- ▶ Total node-weight decreases.
- ▶ Only edges from S to $R - \Gamma(S)$
  decrease in their weight.
- ▶ Since, none of these edges is
  tight (otw. the edge would be
  contained in $H(\vec{x})$, and hence
  would go between S and $\Gamma(S)$)
  we can do this decrement for
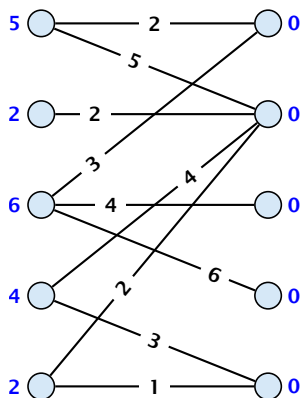 small enough $\delta > 0$ until a new
  edge gets tight.

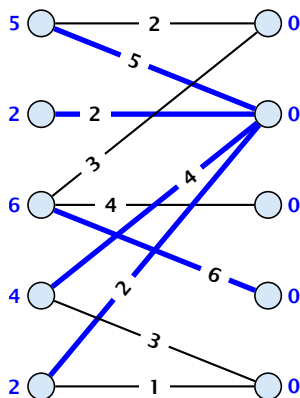# Weighted Bipartite Matching
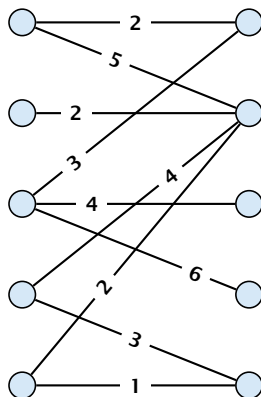
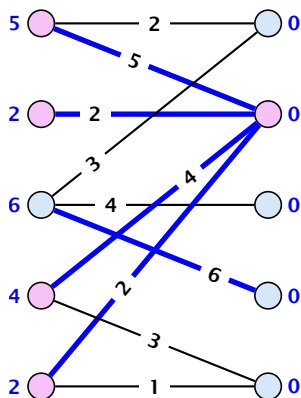Edges not drawn have weight 0.

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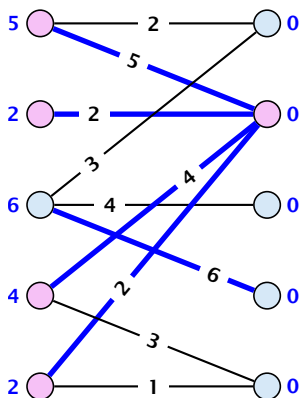
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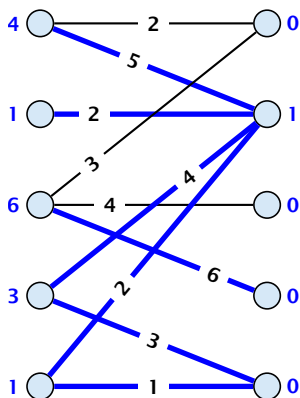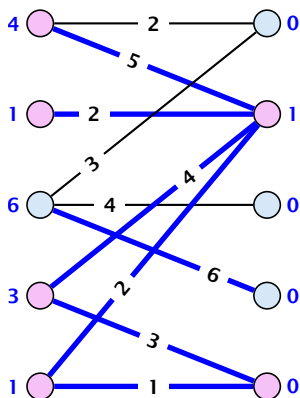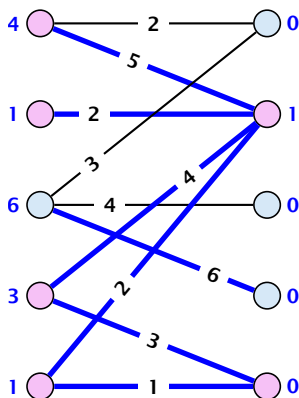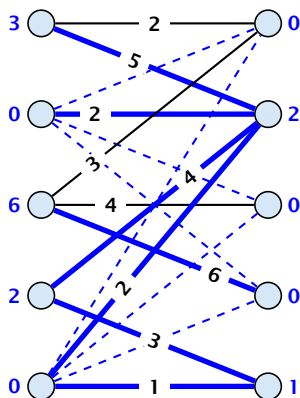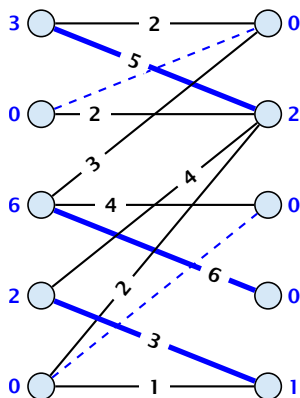
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▶ This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.

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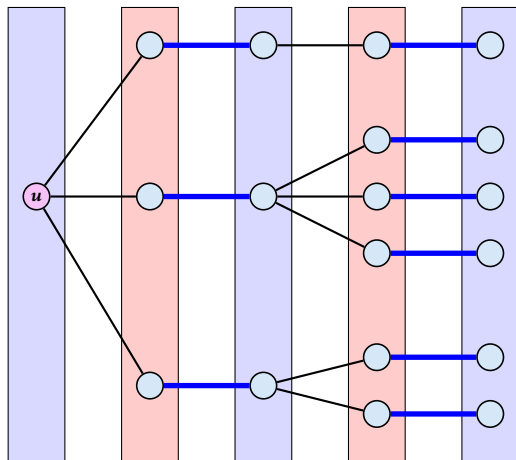
▶ Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

Analysis

▶ We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.

▶ This gives a polynomial running time.

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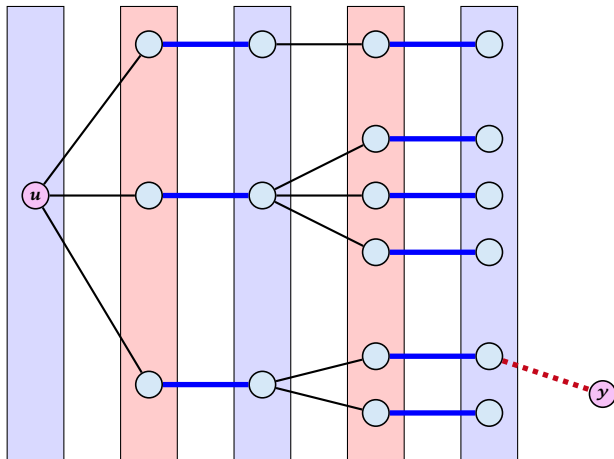
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▶ All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

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- ▶ A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.