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- ▶ $\mathcal{P}.\,\text{union}(x, y)$: Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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▶ Kruskals Minimum Spanning Tree Algorithm

9 Union Find
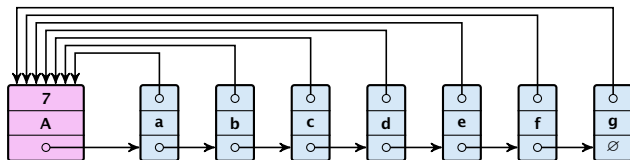

Algorithm 1 Kruskal-MST($G = (V, E), w$)

1: $A \leftarrow \varnothing$;
2: **for all** $v \in V$ **do**
3:      $v.\,\text{set} \leftarrow \mathcal{P}.\,\text{makeset}(v.\,\text{label})$
4: sort edges in non-decreasing order of weight w
5: **for all** $(u, v) \in E$ in non-decreasing order **do**
6:      **if** $\mathcal{P}.\,\text{find}(u.\,\text{set}) \neq \mathcal{P}.\,\text{find}(v.\,\text{set})$ **then**
7:          $A \leftarrow A \cup \{(u, v)\}$
8:          $\mathcal{P}.\,\text{union}(u.\,\text{set}, v.\,\text{set})$

List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

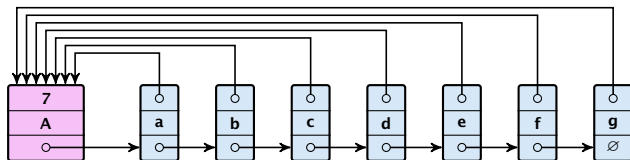# List Implementation

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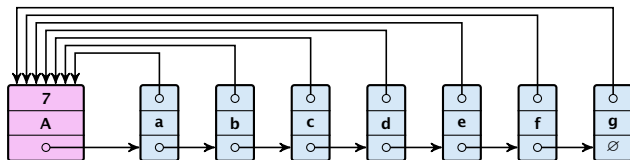
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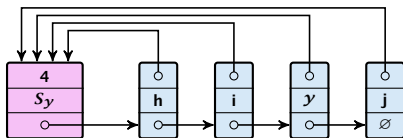
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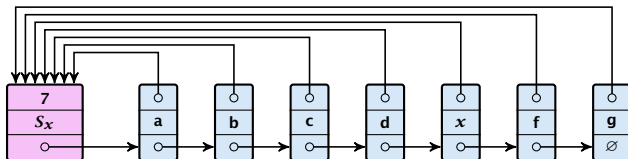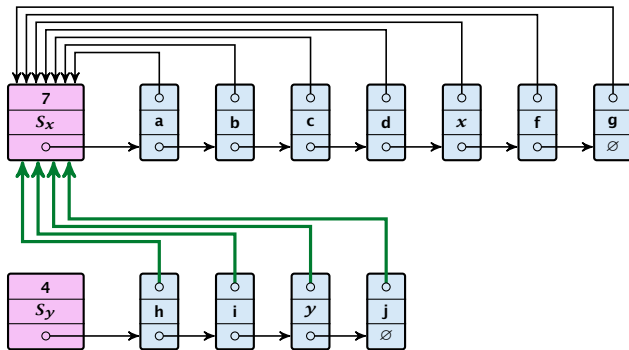
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- ▶ Time: $\min\{|S_x|, |S_y|\}$.

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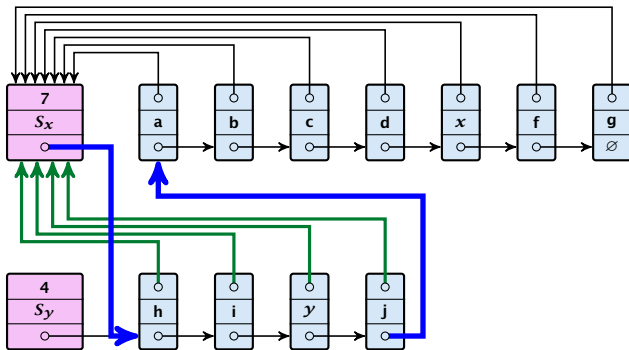
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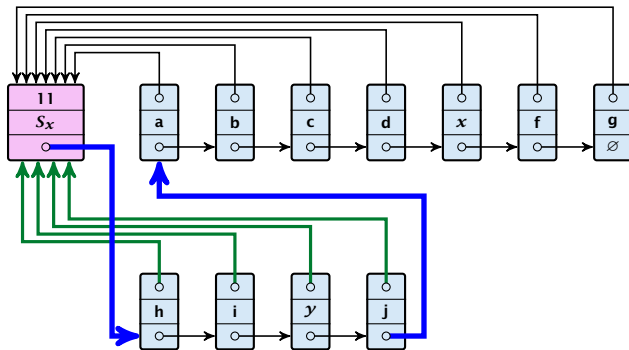
List Implementation

Running times:

- ▶ find(x): constant
- ▶ makeset(x): constant
- ▶ union(x, y): $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- find(x): $\mathcal{O}(1)$.
- makeset(x): $\mathcal{O}(\log n)$.
- union(x, y): $\mathcal{O}(1)$.

- ▶ There is a bank account for every element in the data structure.

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▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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▶ Later operations charge the account but the balance never drops below zero.

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▶ Charge c to every element in set S_x.

List Implementation

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Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. $\qquad\square$

Implementation via Trees

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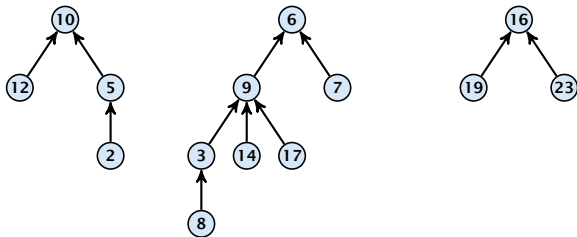
Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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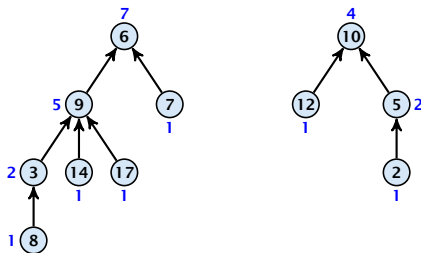
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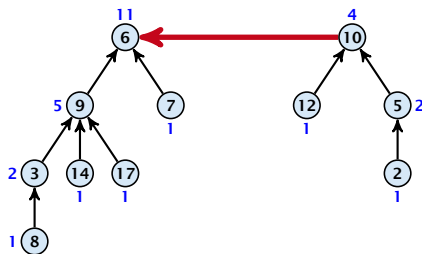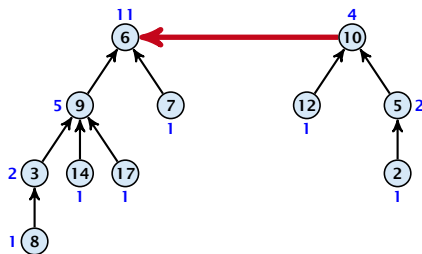- ▶ Time: constant for $\text{link}(a, b)$ plus two find-operations.

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The running time (non-amortized!!!) for $\mathrm{find}(x)$ *is* $\mathcal{O}(\log n)$.

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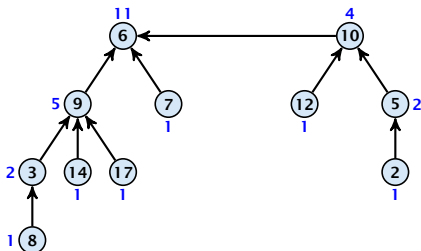▶ Speeds up successive find-operations.

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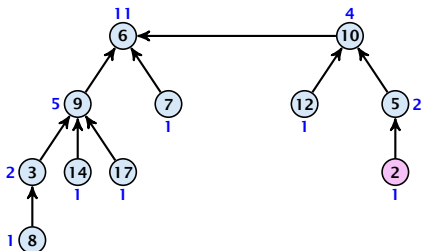
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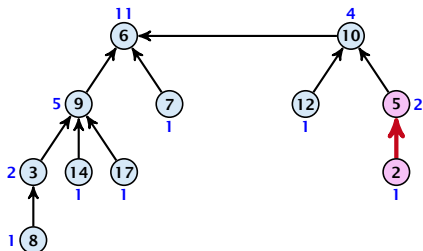
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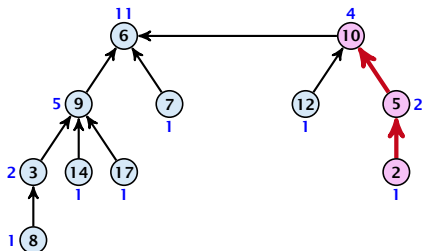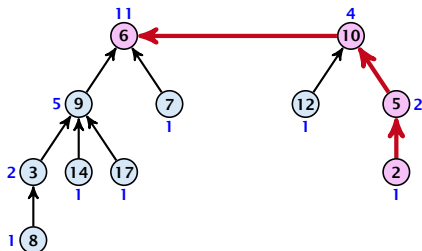
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▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

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Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

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Lemma 37

The rank of a parent must be strictly larger than the rank of a child.

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There are at most $n/2^s$ nodes of rank s.

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▶ Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

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▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

▶ Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Amortized Analysis

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$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases} \qquad \text{tow}(i) = 2^{2^{2^{2^{2^2}}}} \Big\} i \text{ times}$$

and

$$\log^*(n) := \min\{i \mid \text{tow}(i) \geq n\} \ .$$

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Theorem 39
Union find with path compression fulfills the following amortized running times:

▶ $\text{makeset}(x) : \mathcal{O}(\log^*(n))$

▶ $\text{find}(x) : \mathcal{O}(\log^*(n))$

▶ $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

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- ▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- ▶ Hence, the total number of rank-groups is at most $\log^* n$.

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▶ If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.

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- ▶ Otherwise we charge the cost to the find-account.

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- After some charges to v the parent will be in a larger rank-group. $\implies v$ will never be charged again.

- The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g-1) - 1 \leq \text{tow}(g)$.

Amortized Analysis

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▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) \ ,$$

where $n(g)$ is the number of nodes in group g.

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Hence,

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Hence,

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Hence,

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\sum_g n(g) \, \text{tow}(g) \leq n(0) \, \text{tow}(0) + \sum_{g \geq 1} n(g) \, \text{tow}(g) \leq n \log^*(n)
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Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

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makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add
this to the node account of v then the balances of all node
accounts will sum up to a positive value (this is sufficient to
obtain an amortized bound).

Amortized Analysis

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m,n))$.

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$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

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▶ $A(0, y) = y + 1$

▶ $A(1, y) = y + 2$

▶ $A(2, y) = 2y + 3$

▶ $A(3, y) = 2^{y+3} - 3$

▶ $A(4, y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$