

## 6 Recurrences

### Algorithm 2 mergesort(list $L$ )

```
1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
8: return  $L$ 
```

## 6 Recurrences

### Algorithm 2 mergesort(list $L$ )

```
1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
8: return  $L$ 
```

This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when  $n > 1$  and 0 comparisons when  $n \leq 1$ .

How do we bring the expression for the number of comparisons ( $\approx$  running time) into a **closed form**?

# Recurrences

How do we bring the expression for the number of comparisons ( $\approx$  running time) into a **closed form**?

For this we need to **solve** the recurrence.

# Methods for Solving Recurrences

## 1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

## 2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

## 3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

## 4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

## 5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

## 6.1 Guessing+Induction

First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

**Informal way:**

## 6.1 Guessing+Induction

First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

**Informal way:**

Assume that instead we have

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$



## 6.1 Guessing+Induction

First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

### Informal way:

Assume that instead we have

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ .

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ . Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ . Then

$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn\end{aligned}$$

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ . Then

$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\&\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\&= dn(\log n - 1) + cn\end{aligned}$$

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ . Then

$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\&\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\&= dn(\log n - 1) + cn \\&= dn \log n + (c - d)n\end{aligned}$$

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ . Then

$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\&\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\&= dn(\log n - 1) + cn \\&= dn \log n + (c - d)n \\&\leq dn \log n\end{aligned}$$

if we choose  $d \geq c$ .

## 6.1 Guessing+Induction

Suppose we guess  $T(n) \leq dn \log n$  for a constant  $d$ . Then

$$\begin{aligned}T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\&\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\&= dn(\log n - 1) + cn \\&= dn \log n + (c - d)n \\&\leq dn \log n\end{aligned}$$

if we choose  $d \geq c$ .

Formally, this is not correct if  $n$  is not a power of 2. Also even in this case one would need to do an induction proof.



## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

## 6.1 Guessing+Induction

Guess:  $T(n) \leq dn \log n$ .

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

## 6.1 Guessing+Induction

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

## 6.1 Guessing+Induction

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ):

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

## 6.1 Guessing+Induction

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

## 6.1 Guessing+Induction

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$



## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \end{aligned}$$

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \end{aligned}$$

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c - d)n \end{aligned}$$

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c - d)n \\ &\leq dn \log n \end{aligned}$$

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

- ▶ **base case** ( $2 \leq n < 16$ ): **true** if we choose  $d \geq b$ .
- ▶ **induction step**  $n/2 \rightarrow n$ :

Let  $n = 2^k \geq 16$ . Suppose statem. is true for  $n' = n/2$ . We prove it for  $n$ :

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c - d)n \\ &\leq dn \log n \end{aligned}$$

Hence, statement is **true** if we choose  $d \geq c$ .

## 6.1 Guessing+Induction

How do we get a result for all values of  $n$ ?

## 6.1 Guessing+Induction

How do we get a result for all values of  $n$ ?

We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

## 6.1 Guessing+Induction

How do we get a result for all values of  $n$ ?

We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant ( $b$  in the above case).



## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n)$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\ &\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn\end{aligned}$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\ &\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn\end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned} T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\ &\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\ \left\lceil \frac{n}{2} \right\rceil &\leq \frac{n}{2} + 1 \quad \leq 2\left(d(n/2 + 1) \log(n/2 + 1)\right) + cn \end{aligned}$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d(n/2 + 1) \log(n/2 + 1)\right) + cn\end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned} T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\ &\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\ &\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\ &\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn \end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$



## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\boxed{\log n \leq \frac{n}{4}} \leq dn \log n + (\log 9 - 3.5)dn + cn$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\&\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn \\&= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\&\leq dn \log n + (\log 9 - 3.5)dn + cn \\&\leq dn \log n - 0.33dn + cn\end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$\log n \leq \frac{n}{4}$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of  $d$ .

## 6.2 Master Theorem

### Lemma 5

Let  $a \geq 1$ ,  $b > 1$  and  $\epsilon > 0$  denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

#### Case 1.

If  $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$  then  $T(n) = \Theta(n^{\log_b a})$ .

#### Case 2.

If  $f(n) = \Theta(n^{\log_b(a)} \log^k n)$  then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ ,  
 $k \geq 0$ .

#### Case 3.

If  $f(n) = \Omega(n^{\log_b(a)+\epsilon})$  and for sufficiently large  $n$   
 $af\left(\frac{n}{b}\right) \leq cf(n)$  for some constant  $c < 1$  then  $T(n) = \Theta(f(n))$ .

## 6.2 Master Theorem

We prove the Master Theorem for the case that  $n$  is of the form  $b^{\ell}$ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



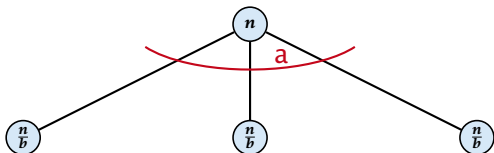
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



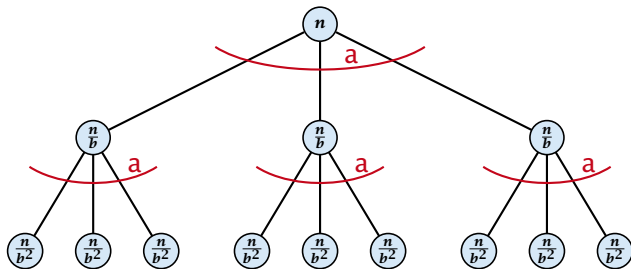
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



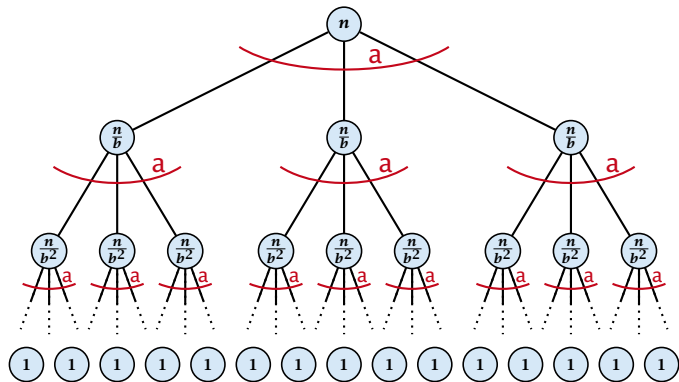
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



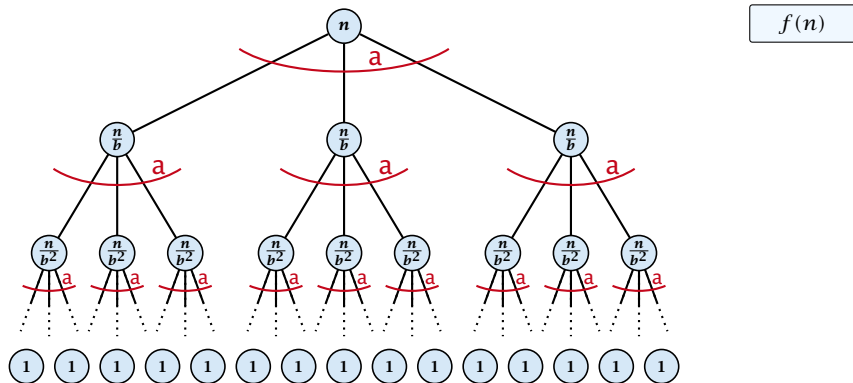
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



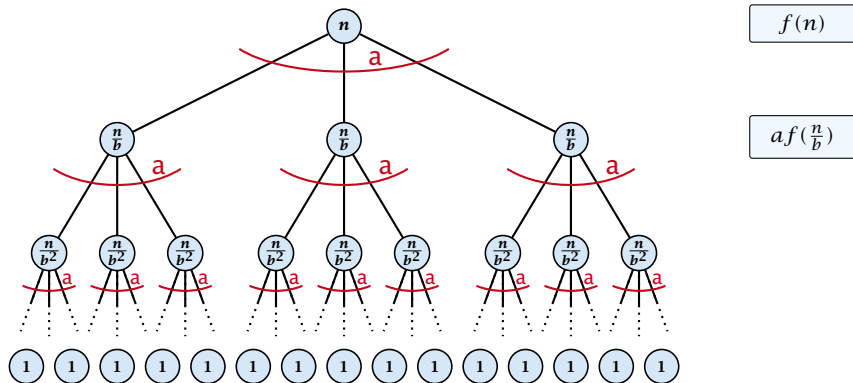
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



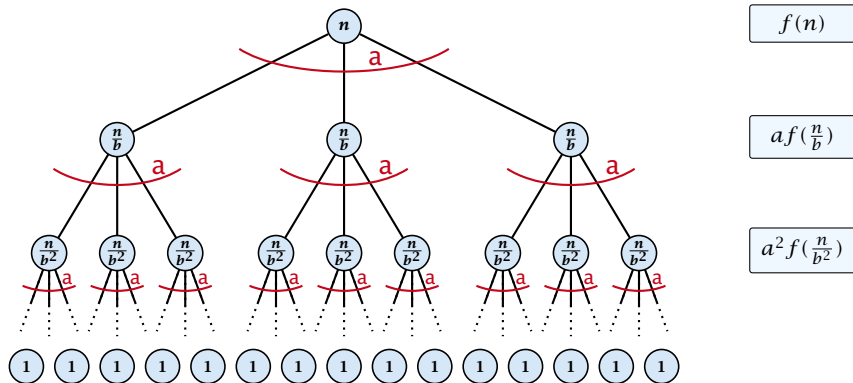
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



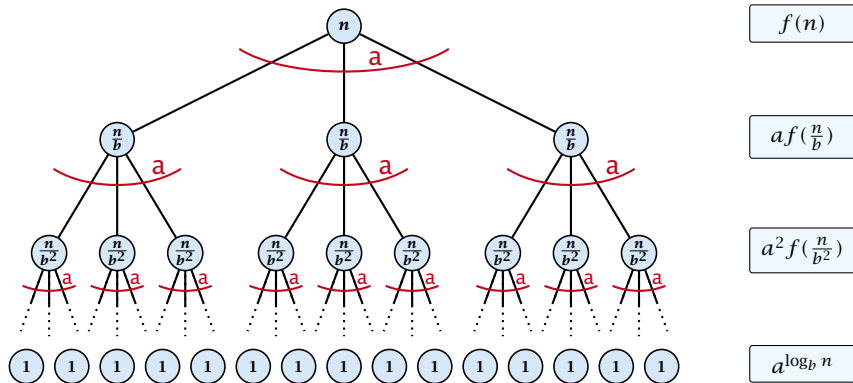
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



# The Recursion Tree

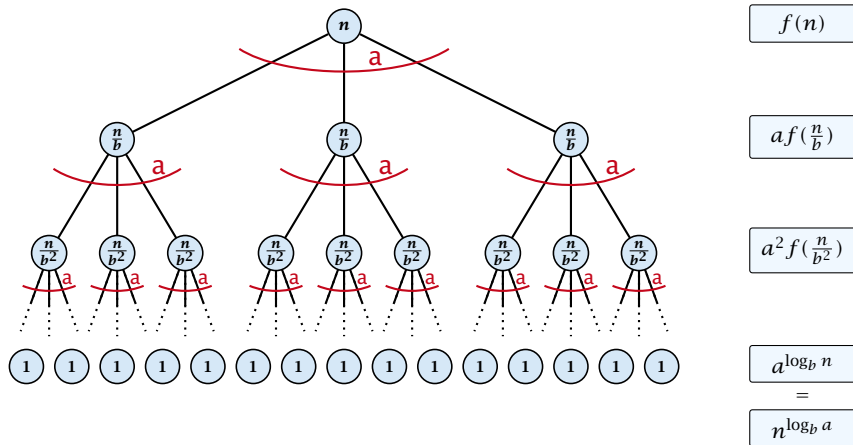
The running time of a recursive algorithm can be visualized by a recursion tree:





# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$T(n) = n^{\log_b a}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$



Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\begin{aligned} \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\ \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \end{aligned}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\begin{aligned} b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i} &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i \\ \sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^\epsilon - 1) \\ &= cn^{\log_b a - \epsilon} (n^\epsilon - 1) / (b^\epsilon - 1) \\ &= \frac{c}{b^\epsilon - 1} n^{\log_b a} (n^\epsilon - 1) / (n^\epsilon) \end{aligned}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\begin{aligned} \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\ \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \\ &= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1) / (n^{\epsilon}) \end{aligned}$$

Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\begin{aligned} \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\ \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \\ &= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1) / (n^{\epsilon}) \end{aligned}$$

Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$T(n) = n^{\log_b a}$$



Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$\begin{aligned}T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\&= cn^{\log_b a} \log_b n\end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).$$

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$T(n) = n^{\log_b a}$$



Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \end{aligned}$$

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \end{aligned}$$

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$\begin{aligned}T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n\end{aligned}$$

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$\begin{aligned}T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n\end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that  $f(n) \geq cn^{\log_b a}$ .

$$\begin{aligned}T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n\end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

$$\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$T(n) = n^{\log_b a}$$



Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$



Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$



**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers, A and B, aligned for addition. Integer A is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer B is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. A vertical light blue box highlights the rightmost bit of A (the least significant bit) and the rightmost bit of B, which are the two bits being added together.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
								0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ } 0 \\ \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ } \boxed{\phantom{0}} \\ \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ } 1 \\ \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ } \phantom{0} \end{array}$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>								0	0

The diagram illustrates the addition of two integers, A and B, using a register of constant size. The integers are represented as binary strings: A = 110110101 and B = 100010011. The addition is performed bit-by-bit from right to left. The result of the addition is shown as 00, indicating that the sum of the two integers is zero. The carry bits are shown as 1 and 1, indicating that the sum of the two integers is zero.



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 1 & 1 & & \\ & & & & & & & & 0 & 0 \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical line separates the 7-bit registers from the 2-bit carry-out. The 7-bit registers contain the values 1101101 (A) and 1000100 (B). The carry-out is 11, and the final result is 00.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two integers, A and B, using a register of constant size. The numbers are represented as binary strings. A vertical box highlights the current bit positions being processed (the 7th, 8th, and 9th bits from the right). The result of the addition is shown below the horizontal line, with the current bit positions highlighted in black.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
					1	1	1		
						0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
					0	1	1	1	
					1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
				1	0	1	1	1	
-----				0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The numbers are aligned to the right. A vertical light blue box highlights the 4th bit position (from the right) in both A and B, and the resulting 0 in the 4th position of the sum. Small subscripts below the bits of B indicate carry-in values: 1 under the 3rd bit, 1 under the 4th bit, 0 under the 5th bit, 1 under the 6th bit, 1 under the 7th bit, and 1 under the 8th bit.



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 1 & 1 & 0 & 1 & 1 & 1 & \\ & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical blue box highlights the third bit position (index 2 from the right), where both A and B have a 0. Below the horizontal line, the carry bits are shown: 1, 1, 0, 1, 1, 1. The resulting sum bits are 0, 0, 1, 0, 0, 0.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

		1	1	0	1	1	0	1	0	1		$A$
		1	0	0	0	1	0	0	0	1	1	$B$
<hr/>												
		1	0	0	1	0	0	0	0			

The diagram illustrates the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1, 1. A vertical box highlights the third bit of A (0) and the second bit of B (0), which are being added together. Below the horizontal line, the result of this addition is shown as 1, 0, 0, 1, 0, 0, 0, 0.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \phantom{1}\phantom{1}\phantom{0}\phantom{1}\phantom{1}\phantom{0}\phantom{1}\phantom{0}\phantom{1}\phantom{1} \\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0		

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

A binary addition diagram showing two 9-bit numbers, A and B, being added. The numbers are written in red and blue respectively, with a horizontal line below them. The result is written in black below the line. A light blue vertical box on the left contains a '1' at the bottom, representing a carry-in. The bits of A are 1 1 0 1 1 0 1 0 1. The bits of B are 1 0 0 0 1 0 0 1 1. The result is 0 1 1 0 0 1 0 0 0. The carry-in '1' is positioned at the bottom of the light blue box, and the result '0' is the first bit of the sum.

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	<hr/>									
	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	1	$A$
	1	0	0	0	1	0	0	1	1	1	$B$
	1	0	1	1	0	0	1	0	0	0	0



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$



## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} \phantom{0\ 0\ 0\ 0\ 0} 0\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$



## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .
- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .



## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .





## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:     return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

$\mathcal{O}(1)$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: **if**  $|A| = |B| = 1$  **then**

$\mathcal{O}(1)$

2:     **return**  $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split**  $A$  into  $A_0$  and  $A_1$

4: **split**  $B$  into  $B_0$  and  $B_1$

5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return**  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: **if**  $|A| = |B| = 1$  **then**

$\mathcal{O}(1)$

2:     **return**  $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split**  $A$  into  $A_0$  and  $A_1$

$\mathcal{O}(n)$

4: **split**  $B$  into  $B_0$  and  $B_1$

5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return**  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: **if**  $|A| = |B| = 1$  **then**

$\mathcal{O}(1)$

2:     **return**  $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split**  $A$  into  $A_0$  and  $A_1$

$\mathcal{O}(n)$

4: **split**  $B$  into  $B_0$  and  $B_1$

$\mathcal{O}(n)$

5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$

$T(\frac{n}{2})$

6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return**  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$



## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: **if**  $|A| = |B| = 1$  **then**

$\mathcal{O}(1)$

2:     **return**  $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split**  $A$  into  $A_0$  and  $A_1$

$\mathcal{O}(n)$

4: **split**  $B$  into  $B_0$  and  $B_1$

$\mathcal{O}(n)$

5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$

$T(\frac{n}{2})$

6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

$2T(\frac{n}{2}) + \mathcal{O}(n)$

7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return**  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

- |  |                                    |
|--|------------------------------------|
| 1: <b>if</b> $ A  =  B  = 1$ <b>then</b>                           | $\mathcal{O}(1)$                   |
| 2: <b>return</b> $a_0 \cdot b_0$                                   | $\mathcal{O}(1)$                   |
| 3: split $A$ into $A_0$ and $A_1$                                  | $\mathcal{O}(n)$                   |
| 4: split $B$ into $B_0$ and $B_1$                                  | $\mathcal{O}(n)$                   |
| 5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$                          | $T(\frac{n}{2})$                   |
| 6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$  | $2T(\frac{n}{2}) + \mathcal{O}(n)$ |
| 7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$                          | $T(\frac{n}{2})$                   |
| 8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ |                                    |

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T\left(\frac{n}{2}\right)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T\left(\frac{n}{2}\right) + \mathcal{O}(n)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T\left(\frac{n}{2}\right)$
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

# Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

## Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

## Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

We get a running time of  $\mathcal{O}(n^2)$  for our algorithm.

## Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

We get a running time of  $\mathcal{O}(n^2)$  for our algorithm.

⇒ Not better than the “school method”.



## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$Z_1 = A_1B_0 + A_0B_1$$

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned}Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0\end{aligned}$$

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
7:  $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1}_{Z_2} - \underbrace{A_0B_0}_{Z_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

```
1: if  $|A| = |B| = 1$  then
2:   return  $a_0 \cdot b_0$ 
3: split  $A$  into  $A_0$  and  $A_1$ 
4: split  $B$  into  $B_0$  and  $B_1$ 
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$ 
6:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$ 
7:  $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$ 
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

$\mathcal{O}(1)$

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: **if**  $|A| = |B| = 1$  **then**

$\mathcal{O}(1)$

2:     **return**  $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: split  $A$  into  $A_0$  and  $A_1$

4: split  $B$  into  $B_0$  and  $B_1$

5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$

7:  $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$

8: **return**  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$



## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: **if**  $|A| = |B| = 1$  **then**

$\mathcal{O}(1)$

2:     **return**  $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: split  $A$  into  $A_0$  and  $A_1$

$\mathcal{O}(n)$

4: split  $B$  into  $B_0$  and  $B_1$

5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$

7:  $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$

8: **return**  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1}_{Z_2} - \underbrace{A_0B_0}_{Z_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

## Example: Multiplying Two Integers

We can use the following identity to compute  $Z_1$ :

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1B_1} && - \underbrace{A_0B_0} \end{aligned}$$

Hence,

### Algorithm 4 mult( $A, B$ )

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

**Master Theorem:** Recurrence:  $T[n] = aT\left(\frac{n}{b}\right) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$



## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

**Master Theorem:** Recurrence:  $T[n] = aT\left(\frac{n}{b}\right) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of  $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$ .

## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

**Master Theorem:** Recurrence:  $T[n] = aT\left(\frac{n}{b}\right) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3:  $f(n) = \Omega(n^{\log_b a + \epsilon})$        $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of  $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$ .

A huge improvement over the “school method”.

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order  $k$**  with constant coefficients ( $c_0, c_k \neq 0$ ).

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order  $k$**  with constant coefficients ( $c_0, c_k \neq 0$ ).

- ▶  $T(n)$  only depends on the  $k$  preceding values. This means the recurrence relation is of **order  $k$** .

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order  $k$**  with constant coefficients ( $c_0, c_k \neq 0$ ).

- ▶  $T(n)$  only depends on the  $k$  preceding values. This means the recurrence relation is of **order  $k$** .
- ▶ The recurrence is linear as there are no products of  $T[n]$ 's.

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order  $k$**  with constant coefficients ( $c_0, c_k \neq 0$ ).

- ▶  $T(n)$  only depends on the  $k$  preceding values. This means the recurrence relation is of **order  $k$** .
- ▶ The recurrence is linear as there are no products of  $T[n]$ 's.
- ▶ If  $f(n) = 0$  then the recurrence relation becomes a linear, **homogenous** recurrence relation of order  $k$ .

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order  $k$**  with constant coefficients ( $c_0, c_k \neq 0$ ).

- ▶  $T(n)$  only depends on the  $k$  preceding values. This means the recurrence relation is of **order  $k$** .
- ▶ The recurrence is linear as there are no products of  $T[n]$ 's.
- ▶ If  $f(n) = 0$  then the recurrence relation becomes a linear, **homogenous** recurrence relation of order  $k$ .

Note that we ignore **boundary conditions** for the moment.



## 6.3 The Characteristic Polynomial

**Observations:**

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.
- ▶  $k$  non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.
- ▶  $k$  non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

### Approach:

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.
- ▶  $k$  non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

### Approach:

- ▶ First determine all solutions that satisfy recurrence relation.

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.
- ▶  $k$  non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

### Approach:

- ▶ First determine all solutions that satisfy recurrence relation.
- ▶ Then pick the right one by analyzing boundary conditions.

## 6.3 The Characteristic Polynomial

### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.
- ▶  $k$  non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

### Approach:

- ▶ First determine all solutions that satisfy recurrence relation.
- ▶ Then pick the right one by analyzing boundary conditions.
- ▶ First consider the homogenous case.



# The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

is a **vector space**.

# The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

is a **vector space**. This means that if  $\mathcal{T}_1, \mathcal{T}_2 \in S$ , then also  $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$ , for arbitrary constants  $\alpha, \beta$ .

# The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

is a **vector space**. This means that if  $\mathcal{T}_1, \mathcal{T}_2 \in S$ , then also  $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$ , for arbitrary constants  $\alpha, \beta$ .

**How do we find a non-trivial solution?**

# The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

is a **vector space**. This means that if  $\mathcal{T}_1, \mathcal{T}_2 \in S$ , then also  $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$ , for arbitrary constants  $\alpha, \beta$ .

**How do we find a non-trivial solution?**

We guess that the solution is of the form  $\lambda^n$ ,  $\lambda \neq 0$ , and see what happens.

# The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

is a **vector space**. This means that if  $\mathcal{T}_1, \mathcal{T}_2 \in S$ , then also  $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$ , for arbitrary constants  $\alpha, \beta$ .

**How do we find a non-trivial solution?**

We guess that the solution is of the form  $\lambda^n$ ,  $\lambda \neq 0$ , and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all  $n \geq k$ .

# The Homogenous Case

Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k = 0$$

# The Homogenous Case

Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

# The Homogenous Case

Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if  $\lambda_i$  is a root (**Nullstelle**) of  $P[\lambda]$  then  $T[n] = \lambda_i^n$  is a solution to the recurrence relation.



# The Homogenous Case

Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if  $\lambda_i$  is a root (Nullstelle) of  $P[\lambda]$  then  $T[n] = \lambda_i^n$  is a solution to the recurrence relation.

Let  $\lambda_1, \dots, \lambda_k$  be the  $k$  (complex) roots of  $P[\lambda]$ . Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values  $\alpha_i$ .

# The Homogenous Case

## Lemma 6

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \dots + \alpha_k \lambda_k^n .$$

# The Homogenous Case

## Lemma 6

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \dots + \alpha_k \lambda_k^n .$$

## Proof.

There is one solution for every possible choice of boundary conditions for  $T[1], \dots, T[k]$ .

# The Homogenous Case

## Lemma 6

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \dots + \alpha_k \lambda_k^n .$$

## Proof.

There is one solution for every possible choice of boundary conditions for  $T[1], \dots, T[k]$ .

We show that the above set of solutions contains one solution for every choice of boundary conditions.

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ s such that these conditions are met:

$$\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \dots + \alpha_k \cdot \lambda_k = T[1]$$

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ s such that these conditions are met:

$$\begin{aligned}\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \dots + \alpha_k \cdot \lambda_k &= T[1] \\ \alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \dots + \alpha_k \cdot \lambda_k^2 &= T[2]\end{aligned}$$

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ s such that these conditions are met:

$$\begin{aligned}\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \dots + \alpha_k \cdot \lambda_k &= T[1] \\ \alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \dots + \alpha_k \cdot \lambda_k^2 &= T[2] \\ &\vdots\end{aligned}$$



# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

$$\begin{aligned}\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \cdots + \alpha_k \cdot \lambda_k &= T[1] \\ \alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \cdots + \alpha_k \cdot \lambda_k^2 &= T[2] \\ &\vdots \\ \alpha_1 \cdot \lambda_1^k + \alpha_2 \cdot \lambda_2^k + \cdots + \alpha_k \cdot \lambda_k^k &= T[k]\end{aligned}$$

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

# The Homogenous Case

## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$
$$= \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

# Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix} =$$

# Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix} =$$

$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$



# Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix} =$$

# Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix} =$$

# Computing the Determinant

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix} =$$

$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all  $\lambda_i$ 's are different, then the determinant is non-zero.

# The Homogeneous Case

**What happens if the roots are not all distinct?**

# The Homogeneous Case

**What happens if the roots are not all distinct?**

Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

# The Homogeneous Case

## What happens if the roots are not all distinct?

Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$



# The Homogeneous Case

## What happens if the roots are not all distinct?

Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

Since  $\lambda_i$  is a root we can write this as  $Q[\lambda] \cdot (\lambda - \lambda_i)^2$ . Calculating the derivative gives a polynomial that still has root  $\lambda_i$ .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

Hence,

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

# The Homogeneous Case

Suppose  $\lambda_i$  has multiplicity  $j$ .

# The Homogeneous Case

Suppose  $\lambda_i$  has multiplicity  $j$ . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with  $\lambda$ ; plugging in  $\lambda_i$ )

# The Homogeneous Case

Suppose  $\lambda_i$  has multiplicity  $j$ . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with  $\lambda$ ; plugging in  $\lambda_i$ )

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

# The Homogeneous Case

Suppose  $\lambda_i$  has multiplicity  $j$ . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with  $\lambda$ ; plugging in  $\lambda_i$ )

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue  $j-1$  times.



# The Homogeneous Case

Suppose  $\lambda_i$  has multiplicity  $j$ . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with  $\lambda$ ; plugging in  $\lambda_i$ )

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue  $j-1$  times.

Hence,  $n^\ell \lambda_i^n$  is a solution for  $\ell \in 0, \dots, j-1$ .

# The Homogeneous Case

## Lemma 7

Let  $P[\lambda]$  denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let  $\lambda_i$ ,  $i = 1, \dots, m$  be the (complex) roots of  $P[\lambda]$  with multiplicities  $\ell_i$ . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of  $\alpha_{ij}$ 's is a solution to the recurrence.

## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

$T[0] = 0$  gives  $\alpha + \beta = 0$ .

## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

$T[0] = 0$  gives  $\alpha + \beta = 0$ .

$T[1] = 1$  gives

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1$$



## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

$T[0] = 0$  gives  $\alpha + \beta = 0$ .

$T[1] = 1$  gives

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

## Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with  $f(n) \neq 0$ .

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where  $T_h$  is **any** solution to the homogeneous equation, and  $T_p$  is **one** particular solution to the inhomogeneous equation.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where  $T_h$  is **any** solution to the homogeneous equation, and  $T_p$  is **one** particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

# The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

# The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

# The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$



# The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

# The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

I get a completely determined recurrence if I add  $T[0] = 1$  and  $T[1] = 2$ .

# The Inhomogeneous Case

Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

# The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

# The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

# The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$  gives  $\alpha = 1$ .

# The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$  gives  $\alpha = 1$ .

$T[1] = 2$  gives  $1 + \beta = 2 \Rightarrow \beta = 1$ .

## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:



## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2$$

## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] + 2n - 1$$

## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] + 2n - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$



$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned}T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3\end{aligned}$$

Difference:

$$\begin{aligned}T[n] - T[n - 1] &= 2T[n - 1] - T[n - 2] + 2n - 1 \\ &\quad - 2T[n - 2] + T[n - 3] - 2n + 3\end{aligned}$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned}T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3\end{aligned}$$

Difference:

$$\begin{aligned}T[n] - T[n - 1] &= 2T[n - 1] - T[n - 2] + 2n - 1 \\ &\quad - 2T[n - 2] + T[n - 3] - 2n + 3\end{aligned}$$

$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned}T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3\end{aligned}$$

Difference:

$$\begin{aligned}T[n] - T[n - 1] &= 2T[n - 1] - T[n - 2] + 2n - 1 \\ &\quad - 2T[n - 2] + T[n - 3] - 2n + 3\end{aligned}$$

$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

and so on...

## 6.4 Generating Functions

### Definition 8 (Generating Function)

Let  $(a_n)_{n \geq 0}$  be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} a_n z^n ;$$

## 6.4 Generating Functions

### Definition 8 (Generating Function)

Let  $(a_n)_{n \geq 0}$  be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} a_n z^n ;$$

- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n .$$

## 6.4 Generating Functions

### Example 9

1. The generating function of the sequence  $(1, 0, 0, \dots)$  is

$$F(z) = 1.$$

## 6.4 Generating Functions

### Example 9

1. The generating function of the sequence  $(1, 0, 0, \dots)$  is

$$F(z) = 1.$$

2. The generating function of the sequence  $(1, 1, 1, \dots)$  is

$$F(z) = \frac{1}{1-z}.$$

## 6.4 Generating Functions

There are two different views:



## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

Let  $f = \sum_{n \geq 0} a_n z^n$  and  $g = \sum_{n \geq 0} b_n z^n$ .

## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

Let  $f = \sum_{n \geq 0} a_n z^n$  and  $g = \sum_{n \geq 0} b_n z^n$ .

- ▶ **Equality:**  $f$  and  $g$  are equal if  $a_n = b_n$  for all  $n$ .

## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

Let  $f = \sum_{n \geq 0} a_n z^n$  and  $g = \sum_{n \geq 0} b_n z^n$ .

- ▶ **Equality:**  $f$  and  $g$  are equal if  $a_n = b_n$  for all  $n$ .
- ▶ **Addition:**  $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$ .

## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

Let  $f = \sum_{n \geq 0} a_n z^n$  and  $g = \sum_{n \geq 0} b_n z^n$ .

- ▶ **Equality:**  $f$  and  $g$  are equal if  $a_n = b_n$  for all  $n$ .
- ▶ **Addition:**  $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$ .
- ▶ **Multiplication:**  $f \cdot g := \sum_{n \geq 0} c_n z^n$  with  $c_n = \sum_{p=0}^n a_p b_{n-p}$ .

## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

Let  $f = \sum_{n \geq 0} a_n z^n$  and  $g = \sum_{n \geq 0} b_n z^n$ .

- ▶ **Equality:**  $f$  and  $g$  are equal if  $a_n = b_n$  for all  $n$ .
- ▶ **Addition:**  $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$ .
- ▶ **Multiplication:**  $f \cdot g := \sum_{n \geq 0} c_n z^n$  with  $c_n = \sum_{p=0}^n a_p b_{n-p}$ .

There are no convergence issues here.

## 6.4 Generating Functions

The arithmetic view:



## 6.4 Generating Functions

The arithmetic view:

We view a power series as a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ .

## 6.4 Generating Functions

The arithmetic view:

We view a power series as a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ .

Then, it is important to think about convergence/convergence radius etc.

## 6.4 Generating Functions

What does  $\sum_{n \geq 0} z^n = \frac{1}{1-z}$  mean in the algebraic view?

## 6.4 Generating Functions

What does  $\sum_{n \geq 0} z^n = \frac{1}{1-z}$  mean in the **algebraic view**?

It means that the power series  $1 - z$  and the power series  $\sum_{n \geq 0} z^n$  are invers, i.e.,

$$(1 - z) \cdot \left( \sum_{n \geq 0} z^n \right) = 1 .$$

## 6.4 Generating Functions

What does  $\sum_{n \geq 0} z^n = \frac{1}{1-z}$  mean in the algebraic view?

It means that the power series  $1 - z$  and the power series  $\sum_{n \geq 0} z^n$  are invers, i.e.,

$$(1 - z) \cdot \left( \sum_{n \geq 0} z^n \right) = 1 .$$

This is well-defined.

## 6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} .$$

## 6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} .$$

We can compute the derivative:

$$\sum_{n \geq 1} n z^{n-1} = \frac{1}{(1-z)^2}$$

## 6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} .$$

We can compute the derivative:

$$\underbrace{\sum_{n \geq 1} n z^{n-1}}_{\sum_{n \geq 0} (n+1) z^n} = \frac{1}{(1-z)^2}$$



## 6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} .$$

We can compute the derivative:

$$\underbrace{\sum_{n \geq 1} n z^{n-1}}_{\sum_{n \geq 0} (n+1) z^n} = \frac{1}{(1-z)^2}$$

Hence, the generating function of the sequence  $a_n = n + 1$  is  $1/(1-z)^2$ .

## 6.4 Generating Functions

We can repeat this

## 6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n + 1)z^n = \frac{1}{(1 - z)^2} .$$

## 6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2} .$$

Derivative:

$$\sum_{n \geq 1} n(n+1)z^{n-1} = \frac{2}{(1-z)^3}$$

## 6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2} .$$

Derivative:

$$\underbrace{\sum_{n \geq 1} n(n+1)z^{n-1}}_{\sum_{n \geq 0} (n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

## 6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2} .$$

Derivative:

$$\underbrace{\sum_{n \geq 1} n(n+1)z^{n-1}}_{\sum_{n \geq 0} (n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

Hence, the generating function of the sequence

$$a_n = (n+1)(n+2) \text{ is } \frac{2}{(1-z)^3} .$$

## 6.4 Generating Functions

Computing the  $k$ -th derivative of  $\sum z^n$ .

## 6.4 Generating Functions

Computing the  $k$ -th derivative of  $\sum z^n$ .

$$\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1) z^{n-k}$$



## 6.4 Generating Functions

Computing the  $k$ -th derivative of  $\sum z^n$ .

$$\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1) z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1) z^n$$

## 6.4 Generating Functions

Computing the  $k$ -th derivative of  $\sum z^n$ .

$$\begin{aligned}\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1)z^{n-k} &= \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}} \cdot\end{aligned}$$

## 6.4 Generating Functions

Computing the  $k$ -th derivative of  $\sum z^n$ .

$$\begin{aligned}\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1)z^{n-k} &= \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}} \cdot\end{aligned}$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \cdot$$

## 6.4 Generating Functions

Computing the  $k$ -th derivative of  $\sum z^n$ .

$$\begin{aligned}\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1)z^{n-k} &= \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}} \cdot\end{aligned}$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \cdot$$

The generating function of the sequence  $a_n = \binom{n+k}{k}$  is  $\frac{1}{(1-z)^{k+1}}$ .

## 6.4 Generating Functions

$$\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n$$

## 6.4 Generating Functions

$$\begin{aligned}\sum_{n \geq 0} n z^n &= \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z}\end{aligned}$$

## 6.4 Generating Functions

$$\begin{aligned}\sum_{n \geq 0} n z^n &= \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\end{aligned}$$

## 6.4 Generating Functions

$$\begin{aligned}\sum_{n \geq 0} n z^n &= \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\end{aligned}$$

The generating function of the sequence  $a_n = n$  is  $\frac{z}{(1-z)^2}$ .



## 6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

## 6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1-az}$$

## 6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence  $f_n = a^n$  is  $\frac{1}{1-az}$ .

**Example:**  $a_n = a_{n-1} + 1$ ,  $a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$A(z)$

## Example: $a_n = a_{n-1} + 1$ , $a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$$A(z) = \sum_{n \geq 0} a_n z^n$$

## Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$$\begin{aligned} A(z) &= \sum_{n \geq 0} a_n z^n \\ &= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \end{aligned}$$

## Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$$\begin{aligned} A(z) &= \sum_{n \geq 0} a_n z^n \\ &= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\ &= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \end{aligned}$$

## Example: $a_n = a_{n-1} + 1$ , $a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \\&= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n\end{aligned}$$



## Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \\&= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n \\&= zA(z) + \sum_{n \geq 0} z^n\end{aligned}$$

## Example: $a_n = a_{n-1} + 1$ , $a_0 = 1$

Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \\&= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n \\&= zA(z) + \sum_{n \geq 0} z^n \\&= zA(z) + \frac{1}{1-z}\end{aligned}$$

**Example:**  $a_n = a_{n-1} + 1, a_0 = 1$

Solving for  $A(z)$  gives

**Example:**  $a_n = a_{n-1} + 1, a_0 = 1$

Solving for  $A(z)$  gives

$$A(z) = \frac{1}{(1-z)^2}$$

**Example:**  $a_n = a_{n-1} + 1, a_0 = 1$

Solving for  $A(z)$  gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2}$$

**Example:**  $a_n = a_{n-1} + 1, a_0 = 1$

Solving for  $A(z)$  gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n \geq 0} (n+1)z^n$$

**Example:**  $a_n = a_{n-1} + 1, a_0 = 1$

Solving for  $A(z)$  gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n \geq 0} (n+1)z^n$$

Hence,  $a_n = n + 1$ .

# Some Generating Functions

<i>n</i> -th sequence element	generating function



# Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$

## Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$

## Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$

## Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
$n$	$\frac{z}{(1-z)^2}$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
$n$	$\frac{z}{(1-z)^2}$
$a^n$	$\frac{1}{1-az}$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
$n$	$\frac{z}{(1-z)^2}$
$a^n$	$\frac{1}{1-az}$
$n^2$	$\frac{z(1+z)}{(1-z)^3}$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
$n$	$\frac{z}{(1-z)^2}$
$a^n$	$\frac{1}{1-az}$
$n^2$	$\frac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	$e^z$

# Some Generating Functions

<i>n</i> -th sequence element	generating function



# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$
$f_n + g_n$	$F + G$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$
$f_n + g_n$	$F + G$
$\sum_{i=0}^n f_i g_{n-i}$	$F \cdot G$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$
$f_n + g_n$	$F + G$
$\sum_{i=0}^n f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \ (n \geq k); \ 0 \text{ otw.}$	$z^k F$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$
$f_n + g_n$	$F + G$
$\sum_{i=0}^n f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \ (n \geq k); \ 0 \text{ otw.}$	$z^k F$
$\sum_{i=0}^n f_i$	$\frac{F(z)}{1-z}$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$
$f_n + g_n$	$F + G$
$\sum_{i=0}^n f_i g_{n-i}$	$F \cdot G$
$f_{n-k}$ ( $n \geq k$ ); 0 otw.	$z^k F$
$\sum_{i=0}^n f_i$	$\frac{F(z)}{1-z}$
$nf_n$	$z \frac{dF(z)}{dz}$

# Some Generating Functions

<i>n</i> -th sequence element	generating function
$cf_n$	$cF$
$f_n + g_n$	$F + G$
$\sum_{i=0}^n f_i g_{n-i}$	$F \cdot G$
$f_{n-k}$ ( $n \geq k$ ); 0 otw.	$z^k F$
$\sum_{i=0}^n f_i$	$\frac{F(z)}{1-z}$
$nf_n$	$z \frac{dF(z)}{dz}$
$c^n f_n$	$F(cz)$

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .



# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by  $A(z)$ .

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by  $A(z)$ .
4. Solving for  $A(z)$  gives an equation of the form  $A(z) = f(z)$ , where hopefully  $f(z)$  is a simple function.

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by  $A(z)$ .
4. Solving for  $A(z)$  gives an equation of the form  $A(z) = f(z)$ , where hopefully  $f(z)$  is a simple function.
5. Write  $f(z)$  as a formal power series.  
Techniques:

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by  $A(z)$ .
4. Solving for  $A(z)$  gives an equation of the form  $A(z) = f(z)$ , where hopefully  $f(z)$  is a simple function.
5. Write  $f(z)$  as a formal power series.  
Techniques:
  - ▶ partial fraction decomposition (**Partialbruchzerlegung**)

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by  $A(z)$ .
4. Solving for  $A(z)$  gives an equation of the form  $A(z) = f(z)$ , where hopefully  $f(z)$  is a simple function.
5. Write  $f(z)$  as a formal power series.  
Techniques:
  - ▶ partial fraction decomposition (**Partialbruchzerlegung**)
  - ▶ lookup in tables

# Solving Recursions with Generating Functions

1. Set  $A(z) = \sum_{n \geq 0} a_n z^n$ .
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by  $A(z)$ .
4. Solving for  $A(z)$  gives an equation of the form  $A(z) = f(z)$ , where hopefully  $f(z)$  is a simple function.
5. Write  $f(z)$  as a formal power series.  
Techniques:
  - ▶ partial fraction decomposition (**Partialbruchzerlegung**)
  - ▶ lookup in tables
6. The coefficients of the resulting power series are the  $a_n$ .

**Example:**  $a_n = 2a_{n-1}$ ,  $a_0 = 1$

1. Set up generating function:



**Example:**  $a_n = 2a_{n-1}$ ,  $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

## Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

## Example: $a_n = 2a_{n-1}$ , $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n \geq 1} a_n z^n$$

## Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n \geq 1} a_n z^n$$

2. Plug in:

## Example: $a_n = 2a_{n-1}$ , $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n \geq 1} a_n z^n$$

2. Plug in:

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

**Example:**  $a_n = 2a_{n-1}$ ,  $a_0 = 1$

**Example:**  $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

**Example:**  $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$



## Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n \geq 1} (2a_{n-1})z^n \\ &= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1} \end{aligned}$$

## Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

$$\begin{aligned}A(z) &= 1 + \sum_{n \geq 1} (2a_{n-1})z^n \\&= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1} \\&= 1 + 2z \sum_{n \geq 0} a_n z^n\end{aligned}$$

## Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

$$\begin{aligned}A(z) &= 1 + \sum_{n \geq 1} (2a_{n-1})z^n \\&= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1} \\&= 1 + 2z \sum_{n \geq 0} a_n z^n \\&= 1 + 2z \cdot A(z)\end{aligned}$$

## Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

$$\begin{aligned}A(z) &= 1 + \sum_{n \geq 1} (2a_{n-1})z^n \\&= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1} \\&= 1 + 2z \sum_{n \geq 0} a_n z^n \\&= 1 + 2z \cdot A(z)\end{aligned}$$

4. Solve for  $A(z)$ .

## Example: $a_n = 2a_{n-1}$ , $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by  $A(z)$  or by simple function.

$$\begin{aligned}A(z) &= 1 + \sum_{n \geq 1} (2a_{n-1})z^n \\&= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1} \\&= 1 + 2z \sum_{n \geq 0} a_n z^n \\&= 1 + 2z \cdot A(z)\end{aligned}$$

4. Solve for  $A(z)$ .

$$A(z) = \frac{1}{1 - 2z}$$

**Example:**  $a_n = 2a_{n-1}$ ,  $a_0 = 1$

5. Rewrite  $f(z)$  as a power series:

$$A(z) = \frac{1}{1 - 2z}$$

**Example:**  $a_n = 2a_{n-1}$ ,  $a_0 = 1$

5. Rewrite  $f(z)$  as a power series:

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{1 - 2z}$$

**Example:**  $a_n = 2a_{n-1}$ ,  $a_0 = 1$

5. Rewrite  $f(z)$  as a power series:

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{1 - 2z} = \sum_{n \geq 0} 2^n z^n$$



**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

1. Set up generating function:

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned} A(z) &= \sum_{n \geq 0} a_n z^n \\ &= a_0 + \sum_{n \geq 1} a_n z^n \end{aligned}$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} a_n z^n \\&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n\end{aligned}$$

## Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} a_n z^n \\&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n\end{aligned}$$

## Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} a_n z^n \\&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n \\&= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n\end{aligned}$$



## Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} a_n z^n \\&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n \\&= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n \\&= 1 + 3zA(z) + \frac{z}{(1-z)^2}\end{aligned}$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

4. Solve for  $A(z)$ :

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

4. Solve for  $A(z)$ :

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

4. Solve for  $A(z)$ :

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2}$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

4. Solve for  $A(z)$ :

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

We use partial fraction decomposition:

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write  $f(z)$  as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2}$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write  $f(z)$  as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$



**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

This gives

$$z^2 - z + 1 = A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z)$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

This gives

$$\begin{aligned} z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\ &= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \end{aligned}$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

This gives

$$\begin{aligned} z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\ &= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \\ &= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C) \end{aligned}$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$

$$2A + 4B + 3C = 1$$

$$A + 3B = 1$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$

$$2A + 4B + 3C = 1$$

$$A + 3B = 1$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write  $f(z)$  as a formal power series:

**Example:**  $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write  $f(z)$  as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

$$\begin{aligned} A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\ &= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1)z^n \end{aligned}$$



**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

$$\begin{aligned}A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\&= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1)z^n \\&= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) z^n\end{aligned}$$

## Example: $a_n = 3a_{n-1} + n$ , $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

$$\begin{aligned}A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\&= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1)z^n \\&= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) z^n \\&= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{2}n - \frac{3}{4} \right) z^n\end{aligned}$$

**Example:**  $a_n = 3a_{n-1} + n$ ,  $a_0 = 1$

5. Write  $f(z)$  as a formal power series:

$$\begin{aligned}A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\&= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1)z^n \\&= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) z^n \\&= \sum_{n \geq 0} \left( \frac{7}{4} \cdot 3^n - \frac{1}{2}n - \frac{3}{4} \right) z^n\end{aligned}$$

6. This means  $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$ .

## 6.5 Transformation of the Recurrence

### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

## 6.5 Transformation of the Recurrence

### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

## 6.5 Transformation of the Recurrence

### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

## 6.5 Transformation of the Recurrence

### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), g_0 = 0$$

## 6.5 Transformation of the Recurrence

### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), \quad g_0 = 0$$

$$g_n = F_n \text{ (} n\text{-th Fibonacci number)}$$



## 6.5 Transformation of the Recurrence

### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), \quad g_0 = 0$$

$$g_n = F_n \text{ (} n\text{-th Fibonacci number)}$$

$$f_n = 2^{F_n}$$

## 6.5 Transformation of the Recurrence

### Example 11

$$f_1 = 1$$

$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 ;$$

## 6.5 Transformation of the Recurrence

### Example 11

$$f_1 = 1$$

$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 ;$$

Define

$$g_k := f_{2^k} .$$

## 6.5 Transformation of the Recurrence

### Example 11

$$f_1 = 1$$

$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 ;$$

Define

$$g_k := f_{2^k} .$$

Then:

$$g_0 = 1$$

## 6.5 Transformation of the Recurrence

### Example 11

$$f_1 = 1$$

$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 ;$$

Define

$$g_k := f_{2^k} .$$

Then:

$$g_0 = 1$$

$$g_k = 3g_{k-1} + 2^k, k \geq 1$$

## 6 Recurrences

We get

$$g_k = 3 [g_{k-1}] + 2^k$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3[g_{k-1}] + 2^k \\ &= 3[3g_{k-2} + 2^{k-1}] + 2^k\end{aligned}$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k\end{aligned}$$



## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k\end{aligned}$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k\end{aligned}$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i\end{aligned}$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i \\&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2}\end{aligned}$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i \\&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}\end{aligned}$$

# 6 Recurrences

Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

# 6 Recurrences

Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$\begin{aligned} f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log_2 3})^k - 2 \cdot 2^k \end{aligned}$$

# 6 Recurrences

Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$\begin{aligned} f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log 3} - 2 \cdot 2^k \end{aligned}$$



# 6 Recurrences

Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$\begin{aligned} f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log_3 2})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log_3 2} - 2 \cdot 2^k \\ &= 3n^{\log_3 2} - 2n . \end{aligned}$$