

# 7 Dictionary

## Dictionary:

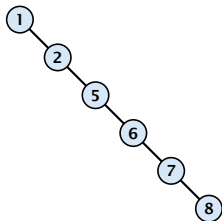
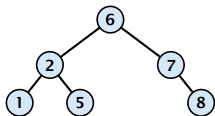
- ▶  **$S$ . insert( $x$ )**: Insert an element  $x$ .
- ▶  **$S$ . delete( $x$ )**: Delete the element pointed to by  $x$ .
- ▶  **$S$ . search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node  $v$  have a smaller key-value than  $\text{key}[v]$  and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

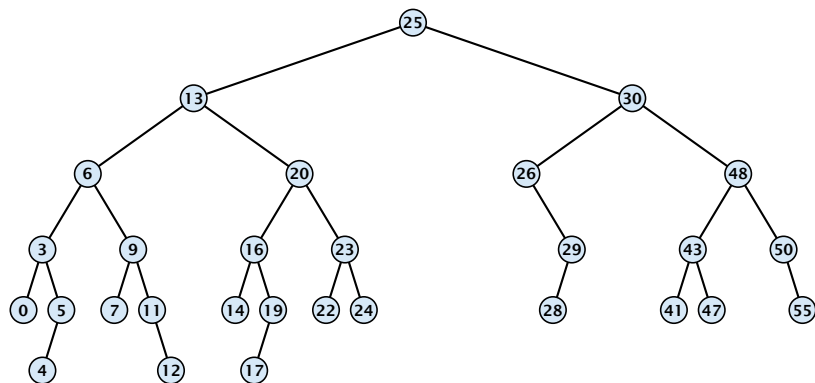


## 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶  $T.\text{insert}(x)$
- ▶  $T.\text{delete}(x)$
- ▶  $T.\text{search}(k)$
- ▶  $T.\text{successor}(x)$
- ▶  $T.\text{predecessor}(x)$
- ▶  $T.\text{minimum}()$
- ▶  $T.\text{maximum}()$

# Binary Search Trees: Searching

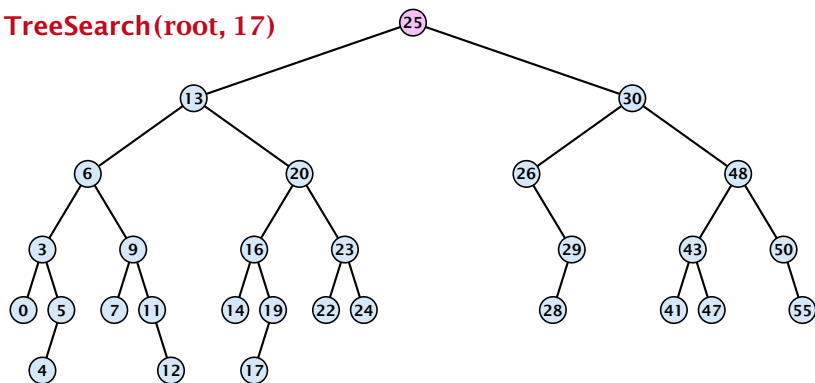


## Algorithm 1 TreeSearch( $x, k$ )

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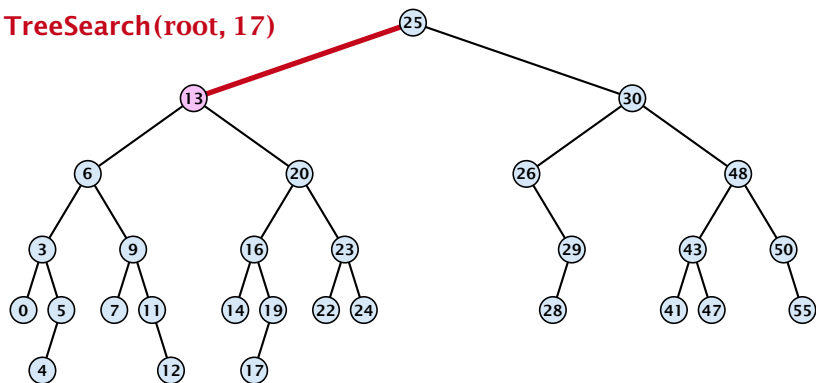


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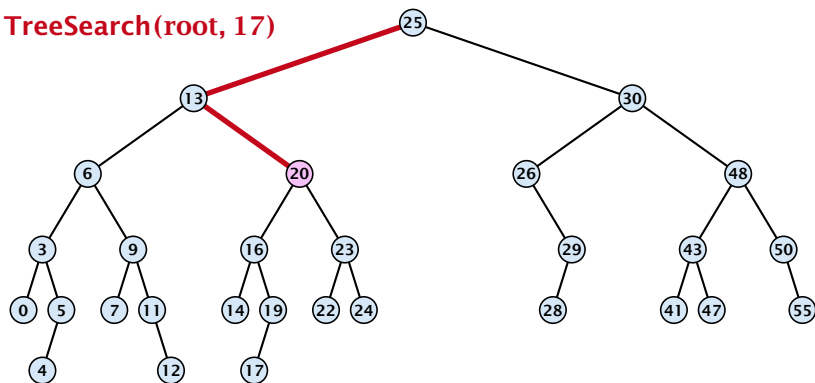


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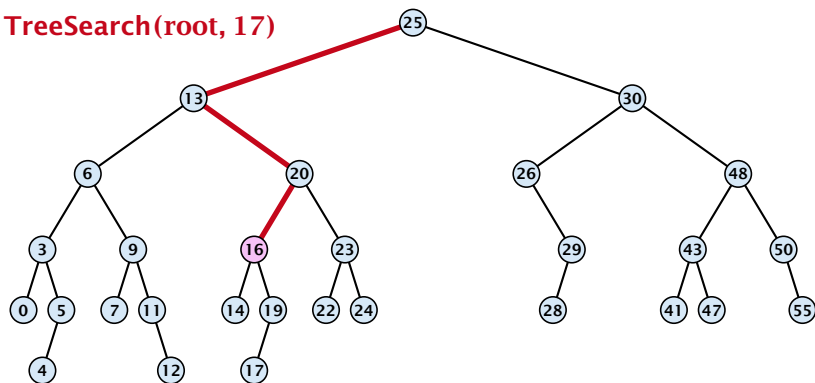


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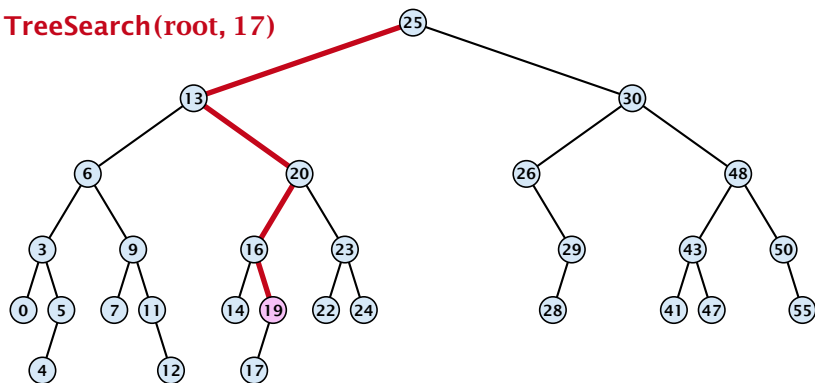
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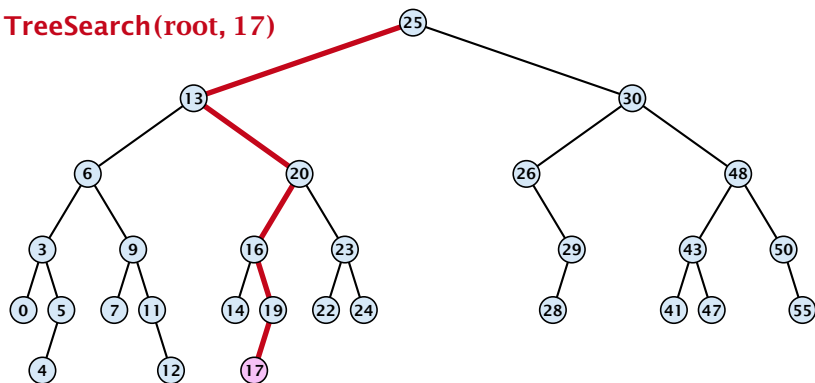


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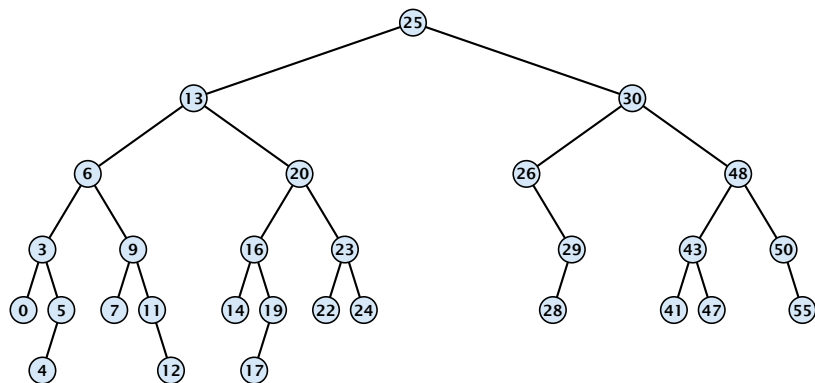
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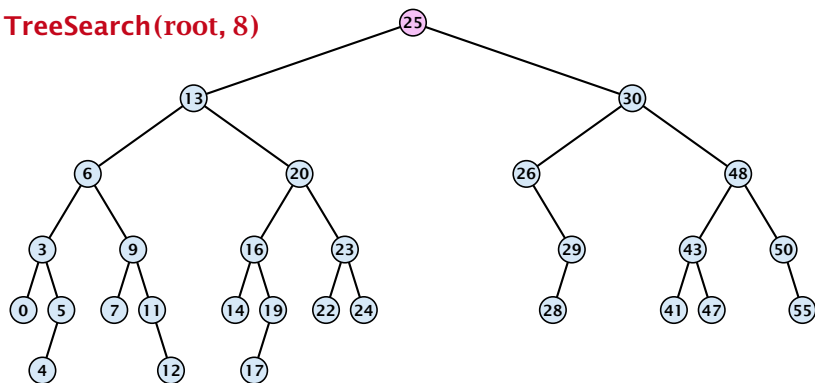


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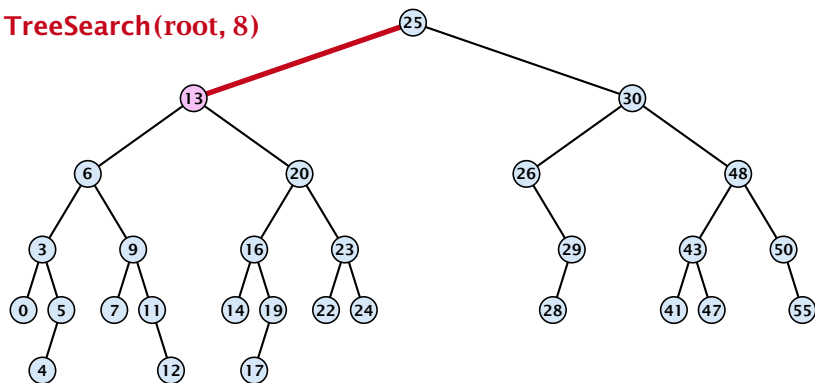


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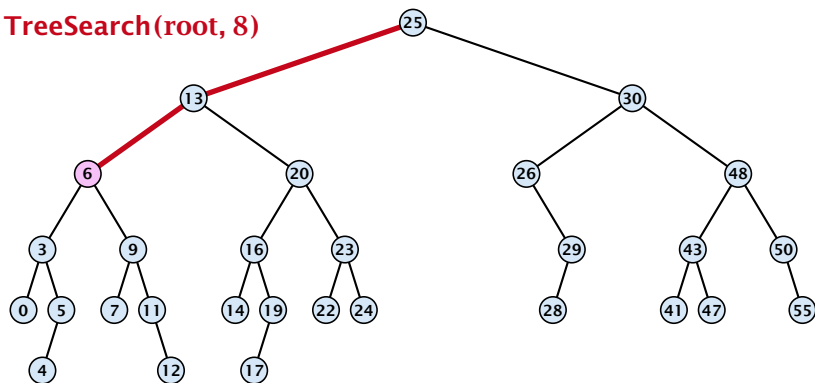


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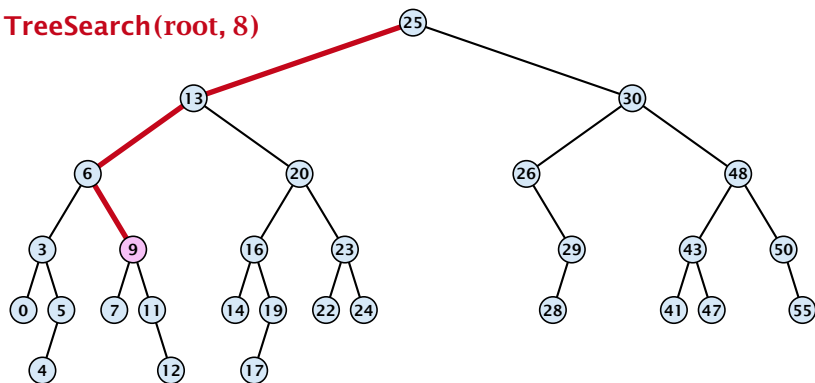


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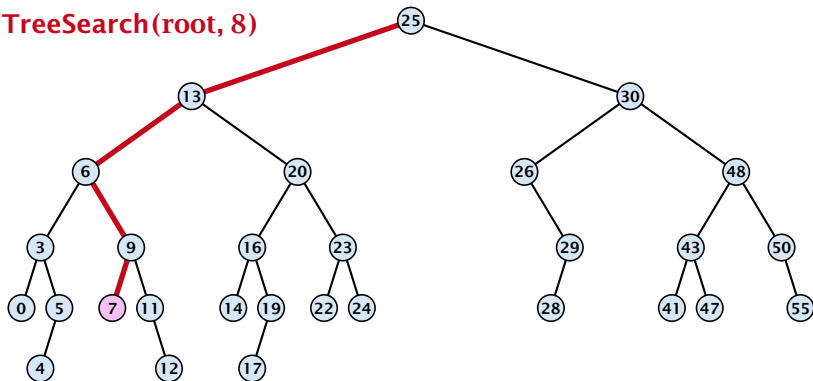


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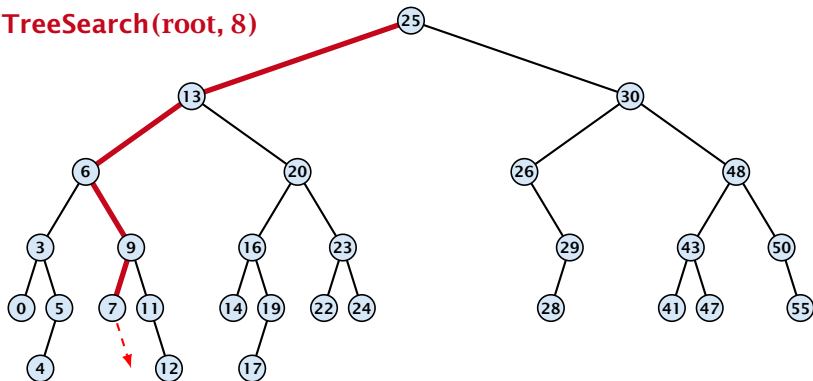
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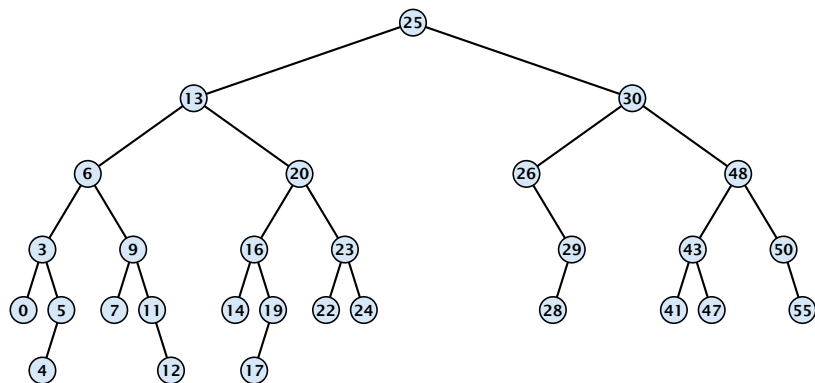
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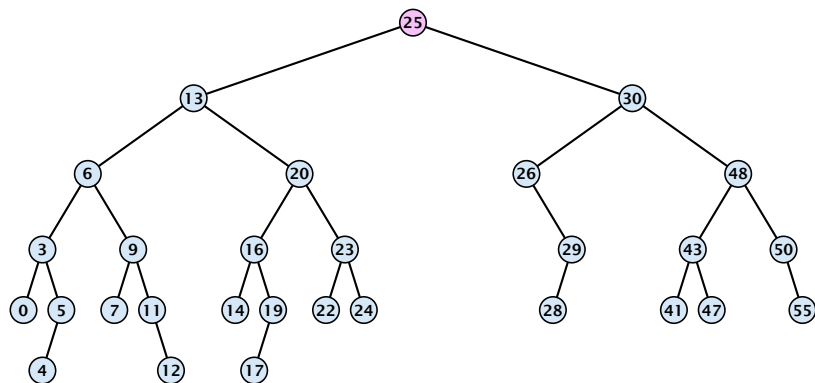
# Binary Search Trees: Minimum



## Algorithm 2 TreeMin( $x$ )

- 1: **if**  $x = \text{null}$  **or**  $\text{left}[x] = \text{null}$  **return**  $x$
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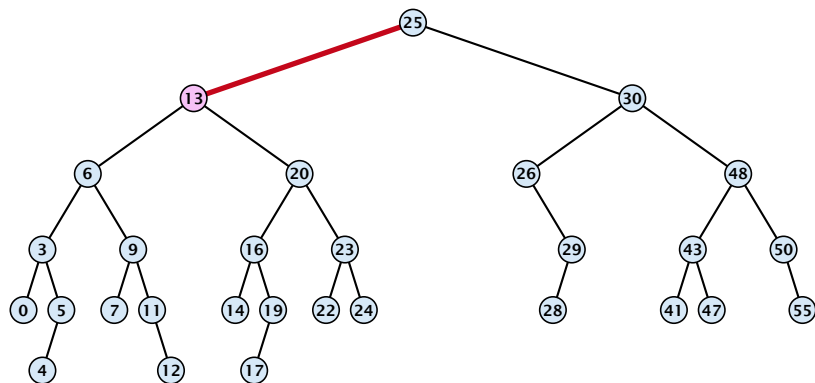
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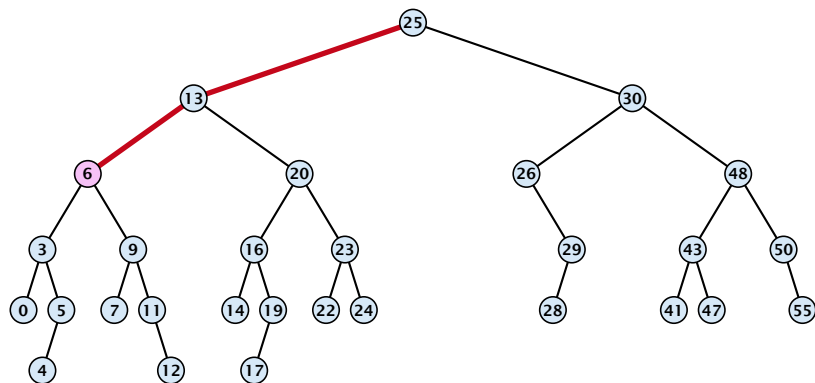
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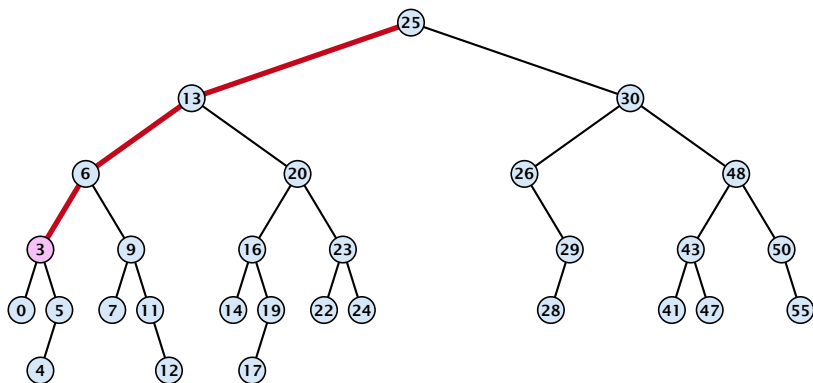
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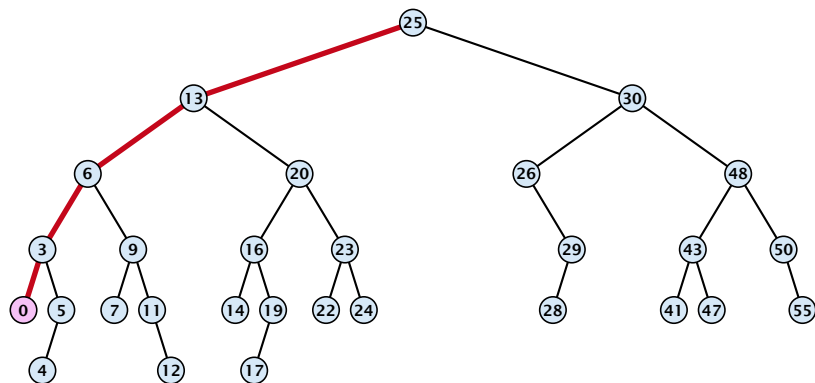
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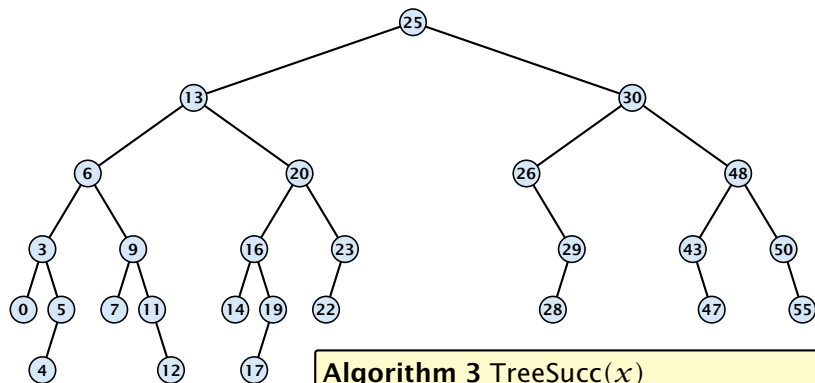
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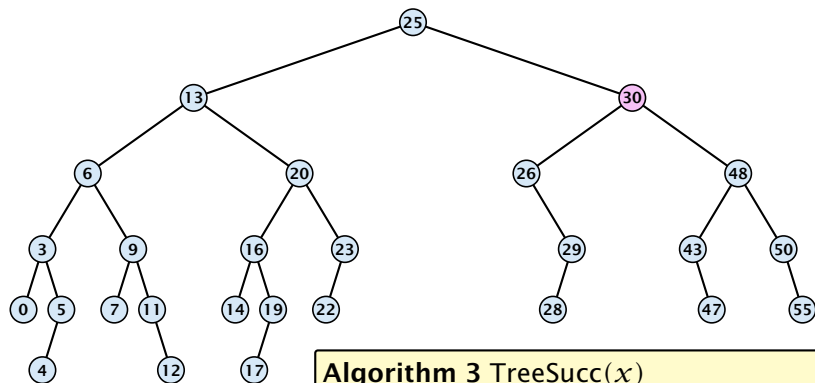


## Algorithm 3 TreeSucc( $x$ )

- 1: **if**  $\text{right}[x] \neq \text{null}$  **return**  $\text{TreeMin}(\text{right}[x])$
- 2:  $y \leftarrow \text{parent}[x]$
- 3: **while**  $y \neq \text{null}$  **and**  $x = \text{right}[y]$  **do**
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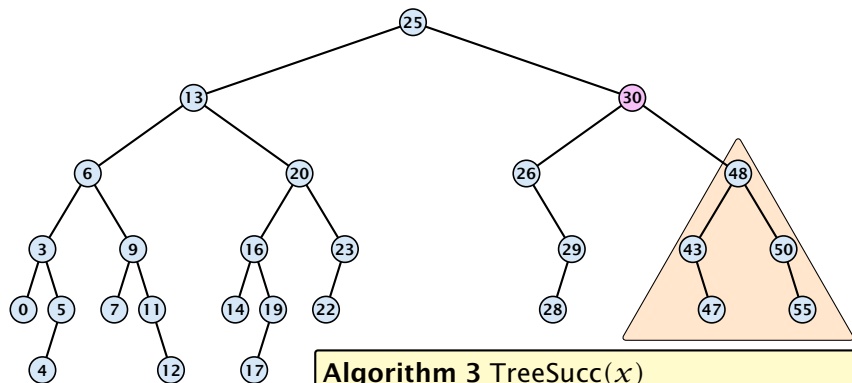
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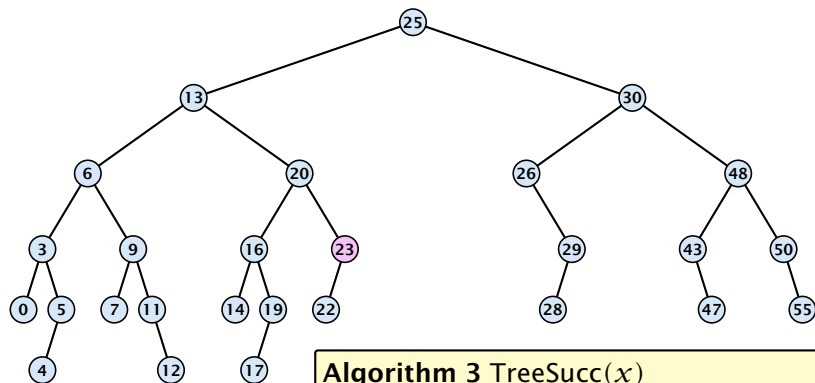
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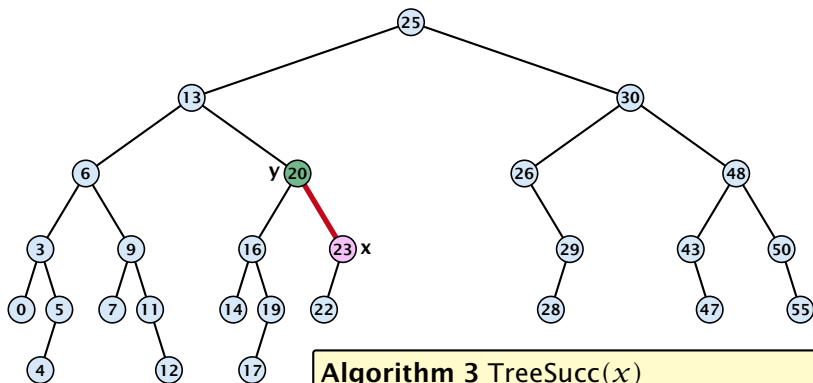
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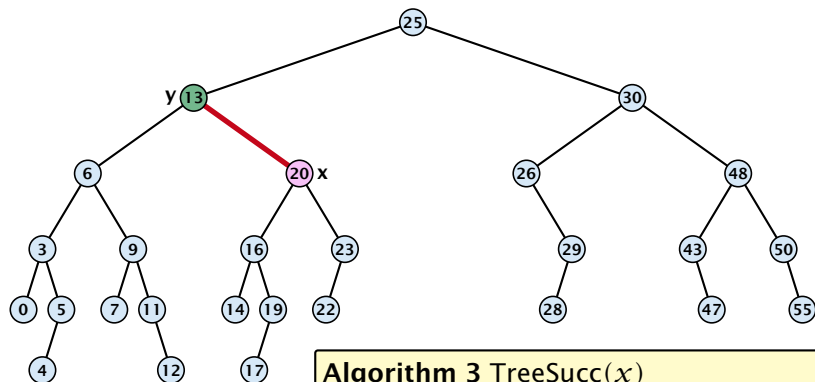
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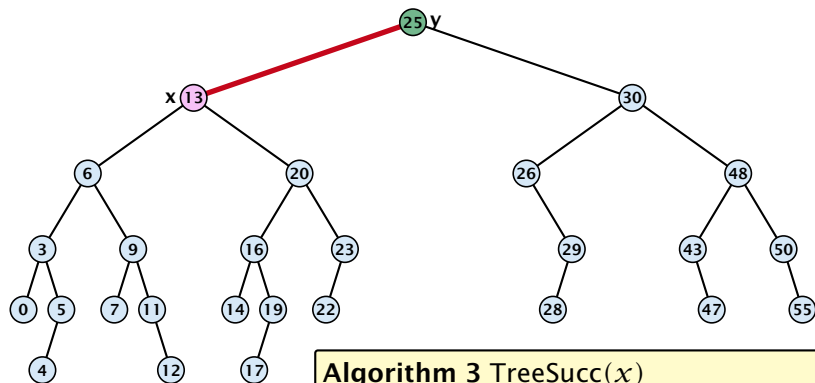
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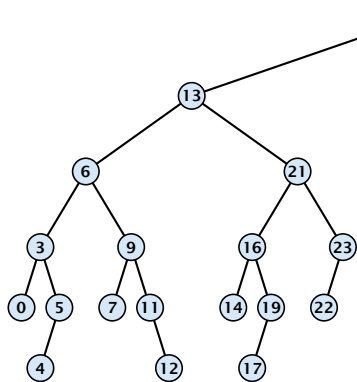
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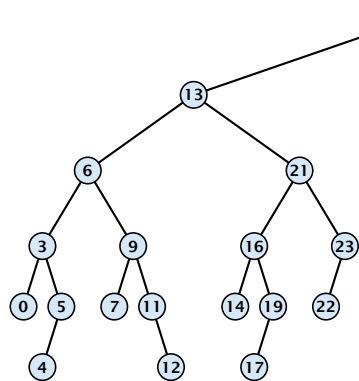


### Algorithm 4 TreeInsert( $x, z$ )

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1: if  $x = \text{null}$  then
2:      $\text{root}[T] \leftarrow z$ ;  $\text{parent}[z] \leftarrow \text{null}$ ;
3:     return;
4: if  $\text{key}[x] > \text{key}[z]$  then
5:     if  $\text{left}[x] = \text{null}$  then
6:          $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
7:     else TreeInsert( $\text{left}[x], z$ );
8: else
9:     if  $\text{right}[x] = \text{null}$  then
10:         $\text{right}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
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Insert element **not** in the tree.



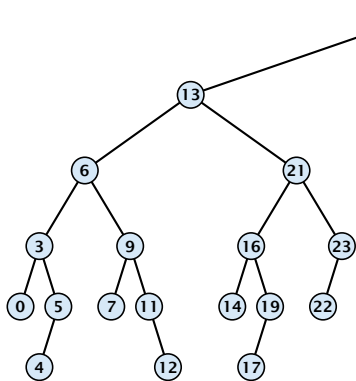
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Search for  $z$ . At some point the search stops at a null-pointer. This is the place to insert  $z$ .

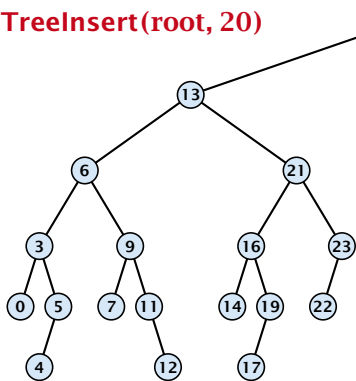
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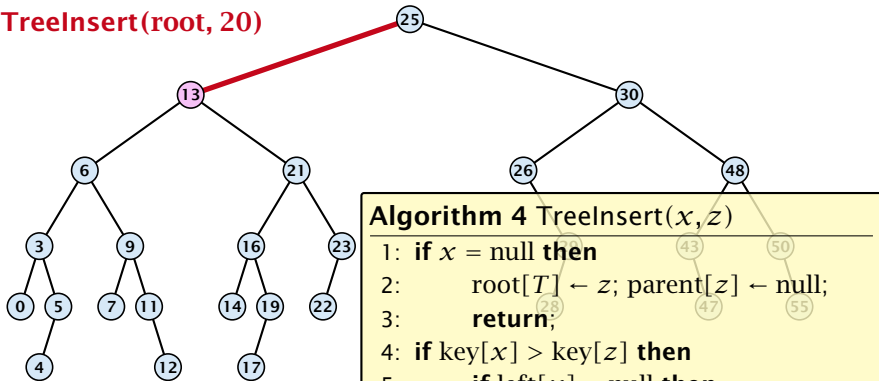
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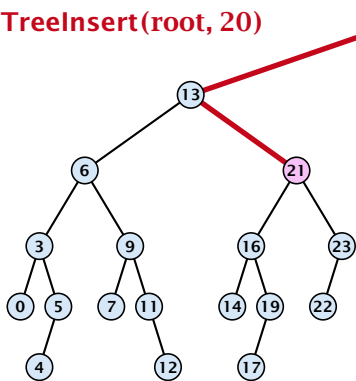
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**TreeInsert**(root, 20)



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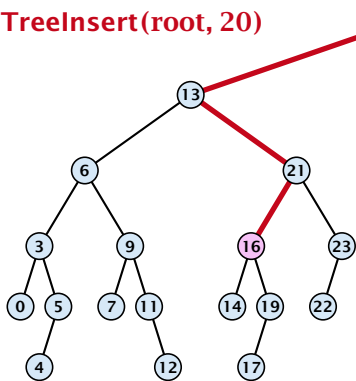
## Algorithm 4 TreeInsert( $x, z$ )

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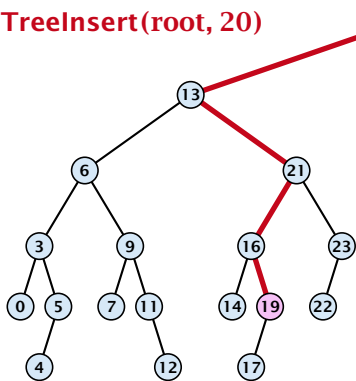
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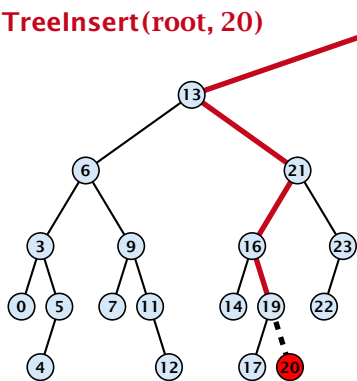
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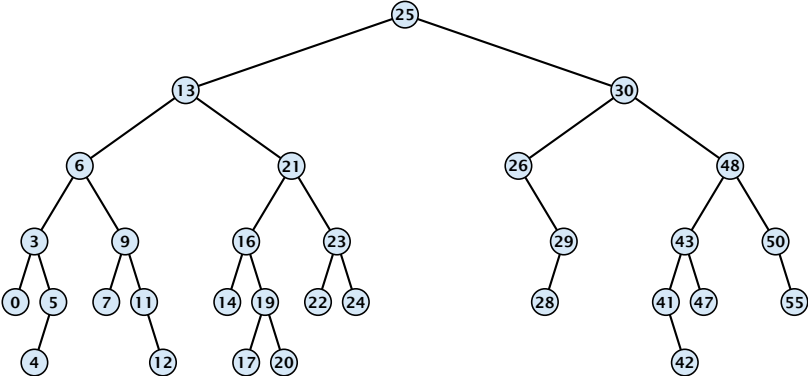


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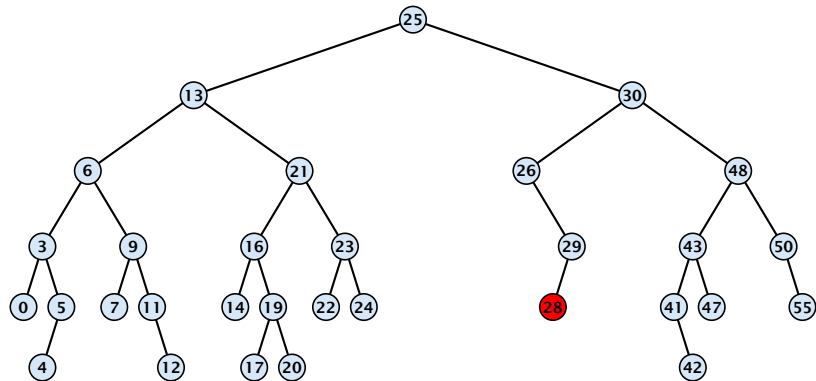
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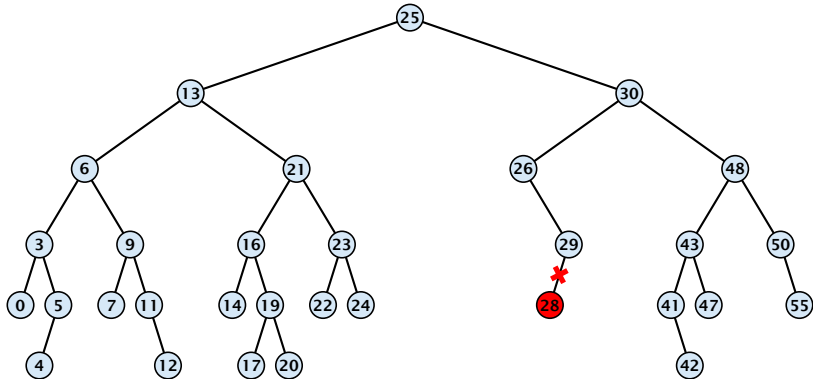


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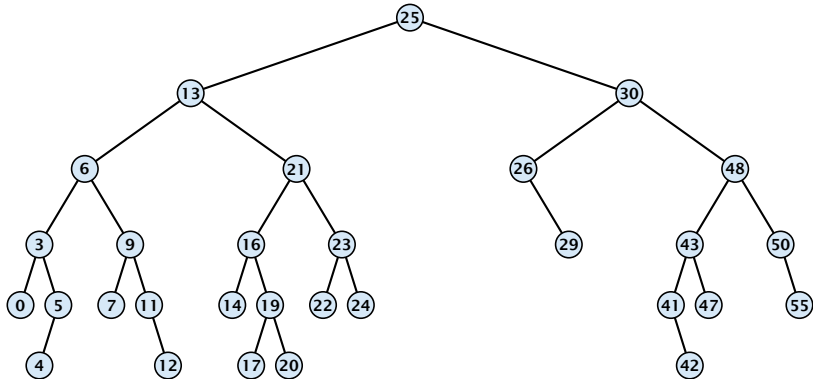


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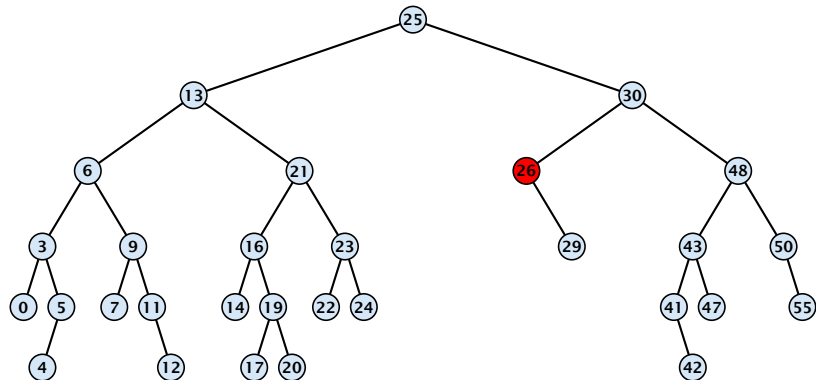


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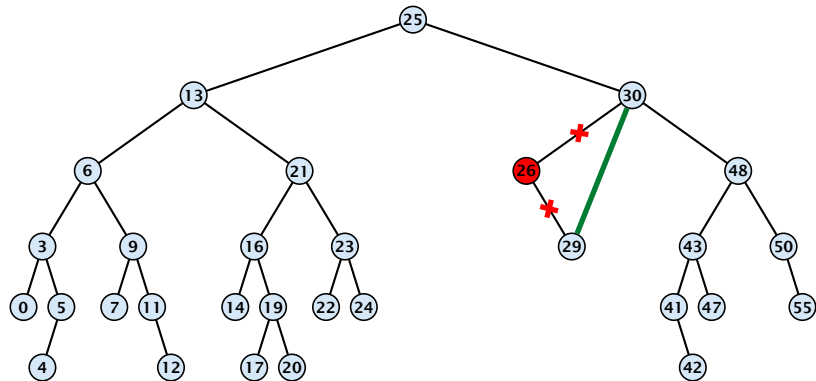


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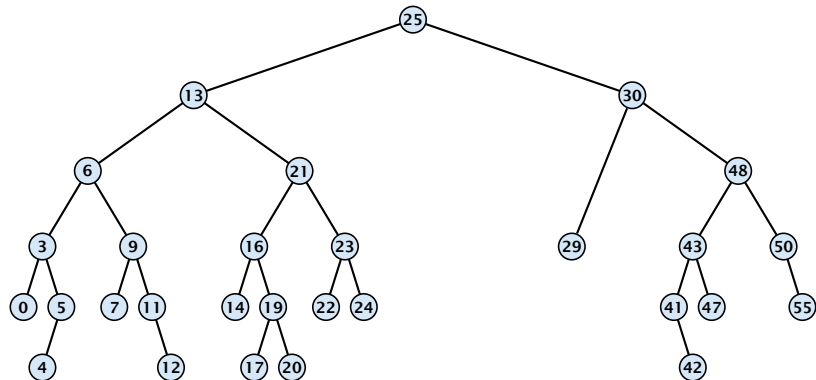


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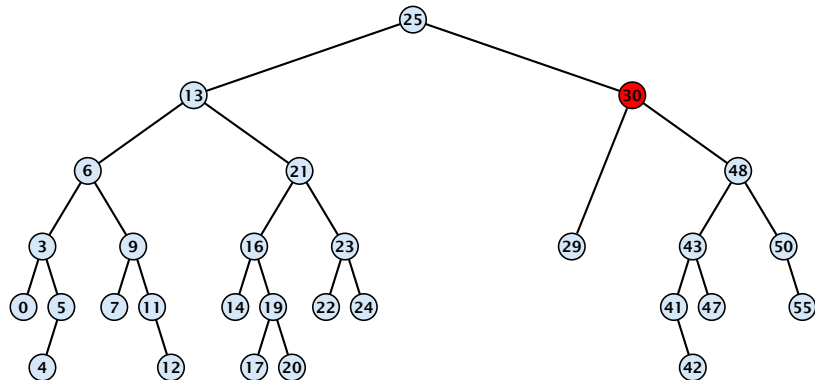


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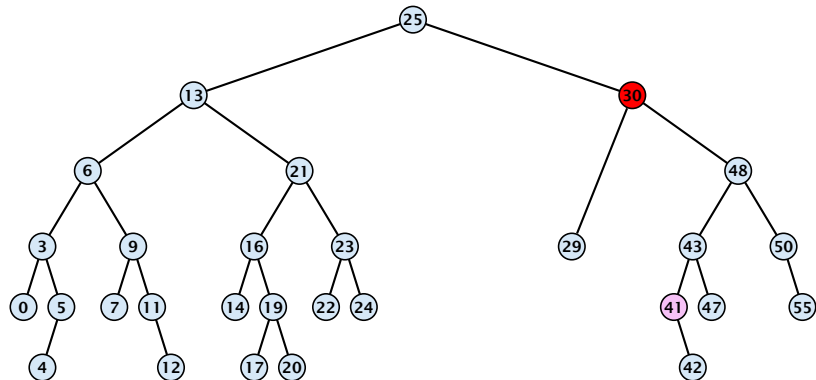


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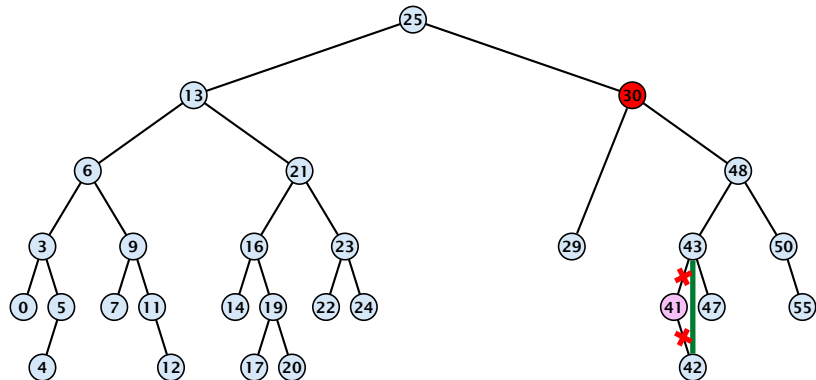
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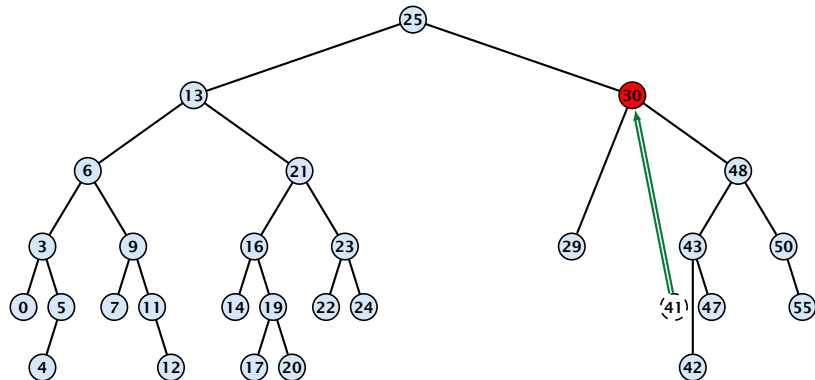


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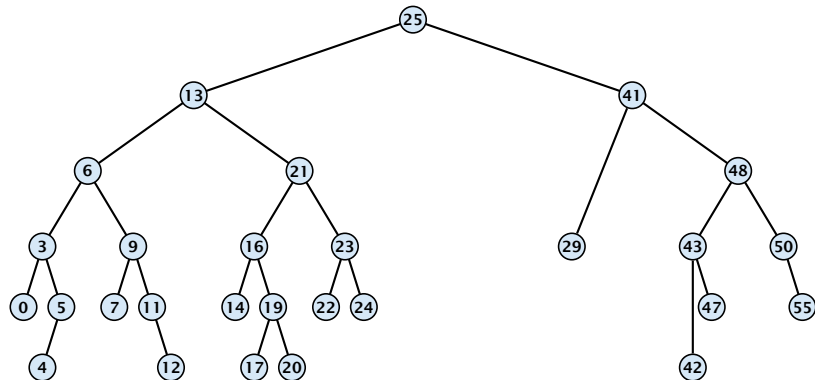


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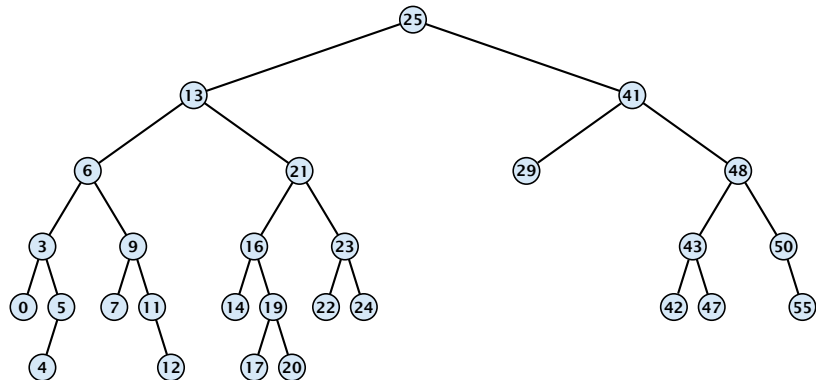


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# Binary Search Trees: Delete

## Algorithm 9 TreeDelete( $z$ )

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3: if left[ $y$ ]  $\neq$  null
4:   then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5: if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6: if parent[ $y$ ] = null then
7:   root[ $T$ ]  $\leftarrow x$ 
8: else
9:   if  $y = \text{left}[\text{parent}[y]]$  then
10:    left[parent[ $y$ ]]  $\leftarrow x$ 
11:   else
12:    right[parent[ $y$ ]]  $\leftarrow x$ 
13: if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to  $x$

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

## 7.2 Red Black Trees

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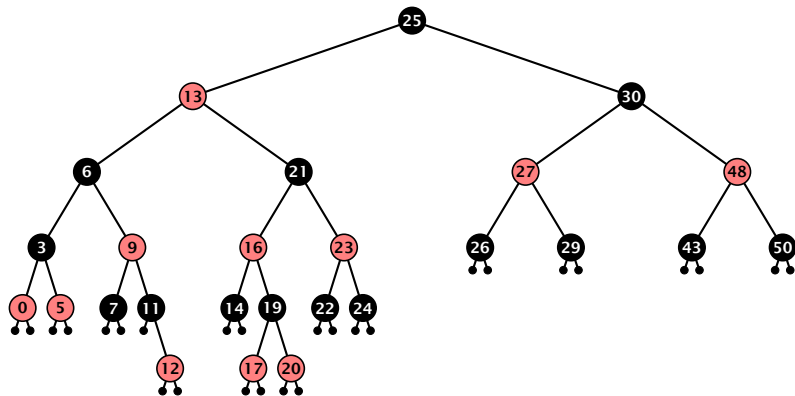
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# Red Black Trees: Example



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The **black height**  $\text{bh}(v)$  of a node  $v$  in a red black tree is the number of black nodes on a path from  $v$  to a leaf vertex (not counting  $v$ ).

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We first show:

### Lemma 15

A sub-tree of black height  $\text{bh}(v)$  in a red black tree contains at least  $2^{\text{bh}(v)} - 1$  internal vertices.

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- ▶ Then  $T_v$  contains at least  $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$  vertices.



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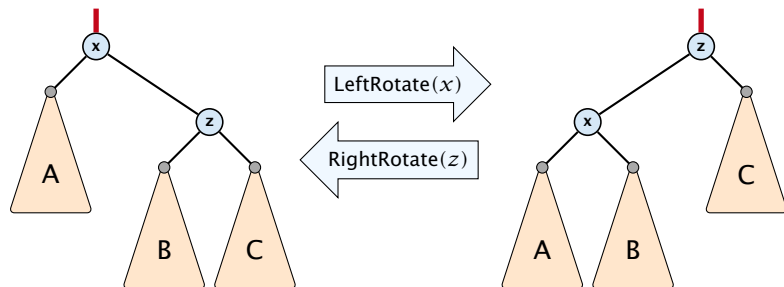
The **null**-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

## 7.2 Red Black Trees

We need to adapt the insert and delete operations so that the red black properties are maintained.

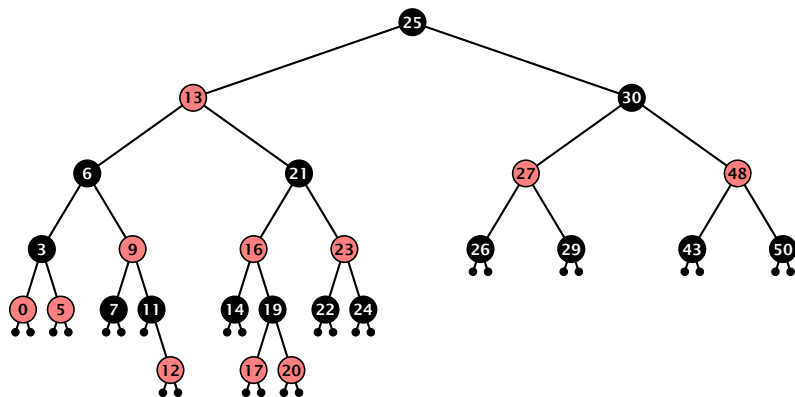
# Rotations

The properties will be maintained through rotations:





# Red Black Trees: Insert

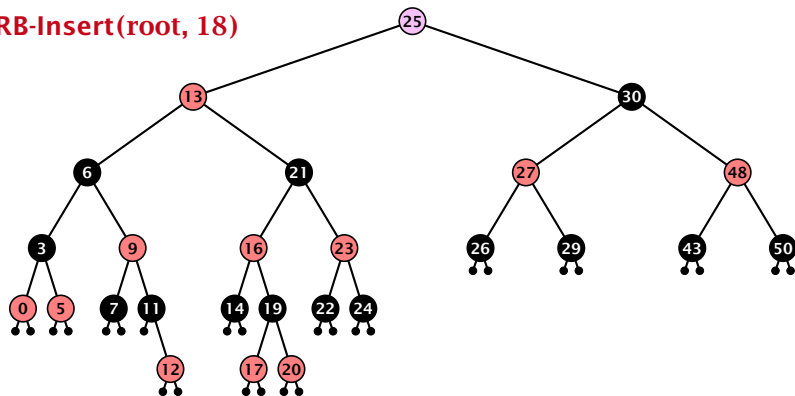


## Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

# Red Black Trees: Insert

RB-Insert(root, 18)

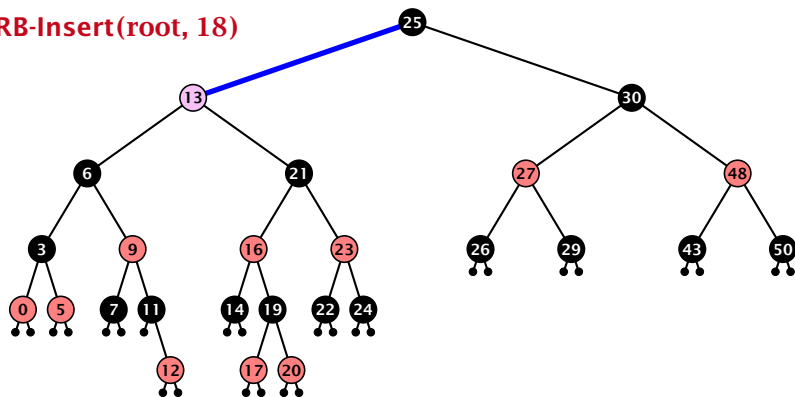


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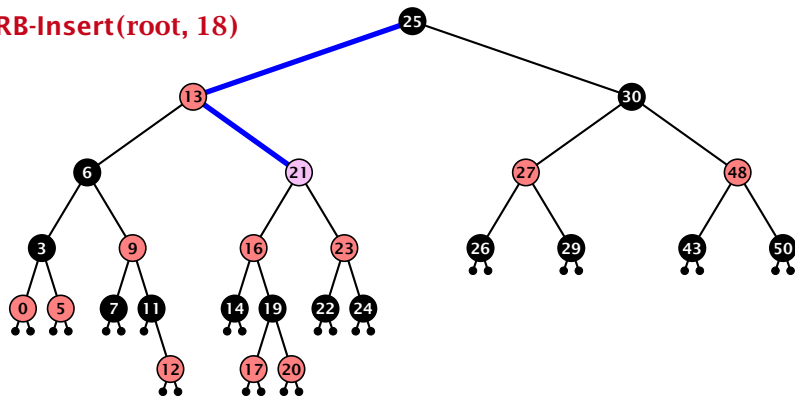


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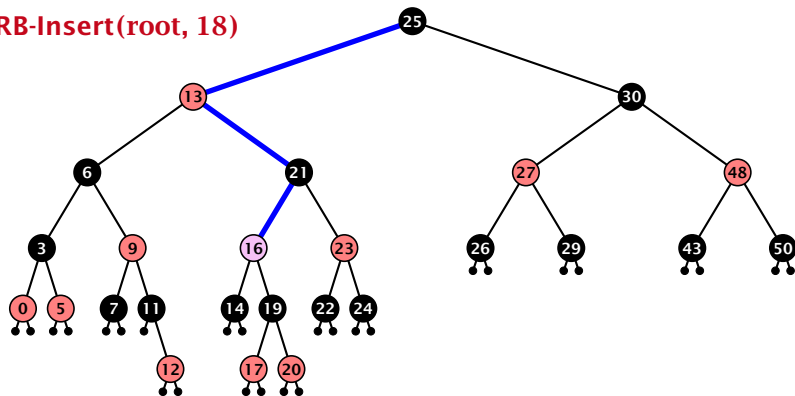


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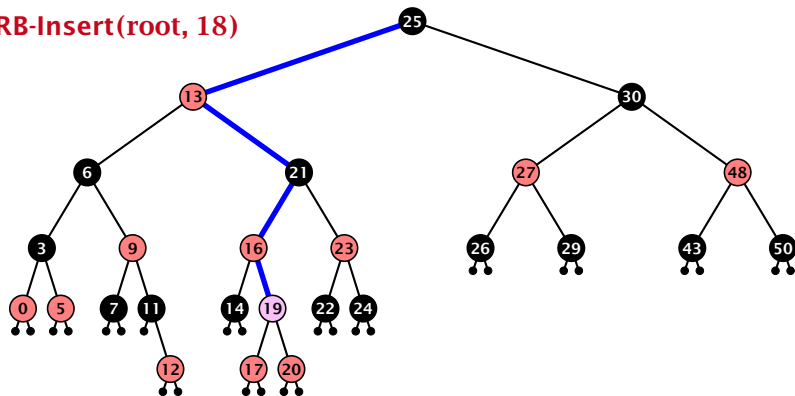


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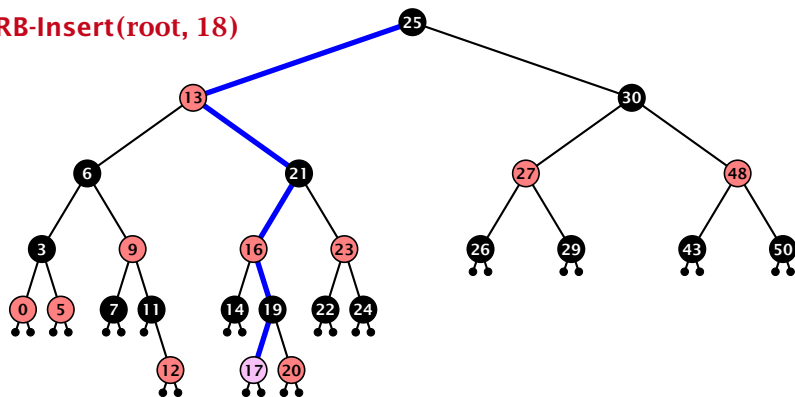


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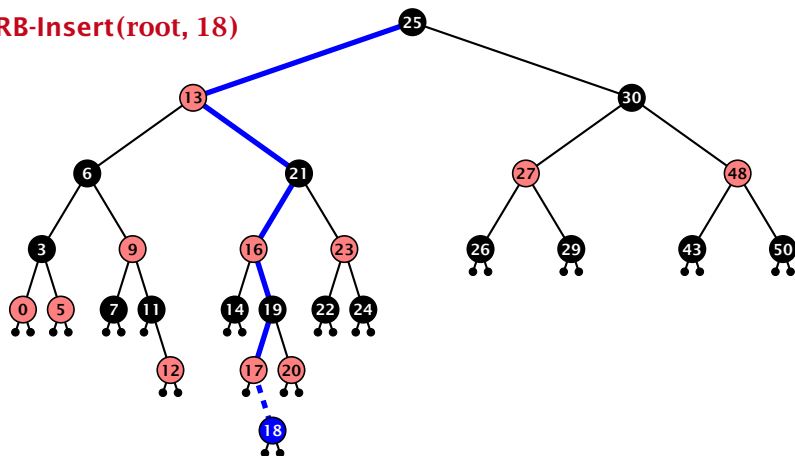


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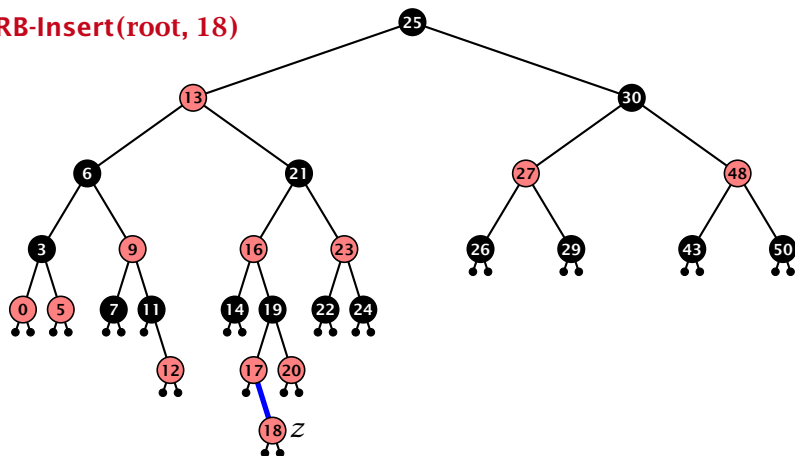
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  - ▶ either both of them are red (most important case)

# Red Black Trees: Insert

## Invariant of the fix-up algorithm:

- ▶  $z$  is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at  $z$  and  $\text{parent}[z]$ 
  - ▶ either both of them are red  
(most important case)
  - ▶ or the parent does not exist  
(violation since root must be black)

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(violation since root must be black)

If  $z$  has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
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12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```



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```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
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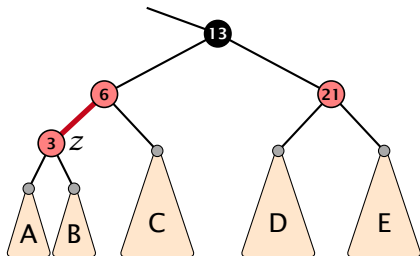
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1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
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3:     uncle  $\leftarrow$  right[grandparent[ $z$ ]]
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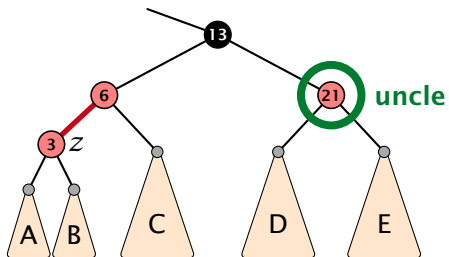
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10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red; 2b:  $z$  left child
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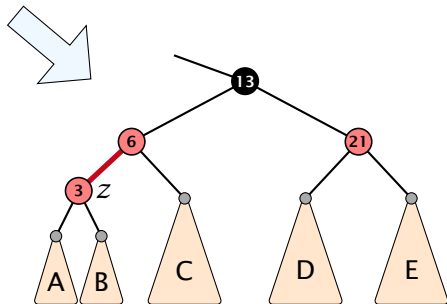
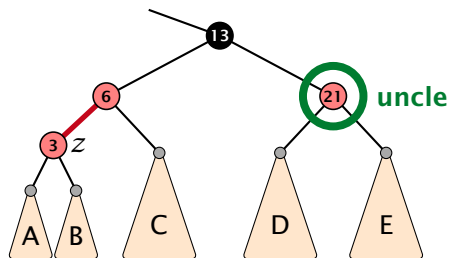
## Case 1: Red Uncle



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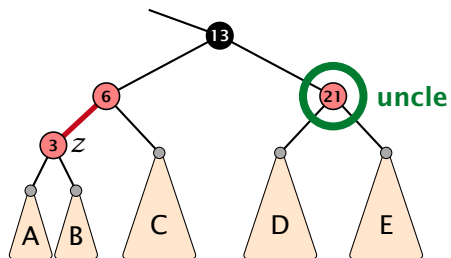


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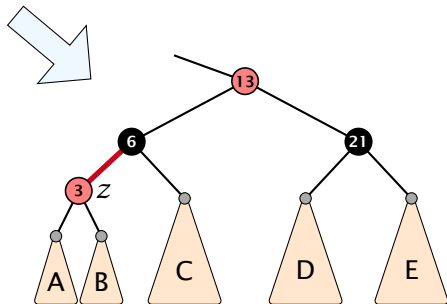




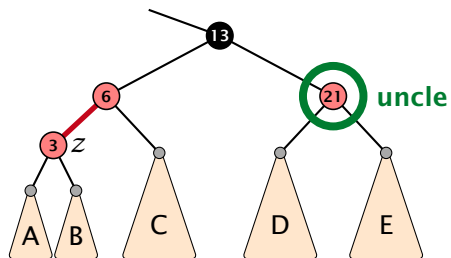
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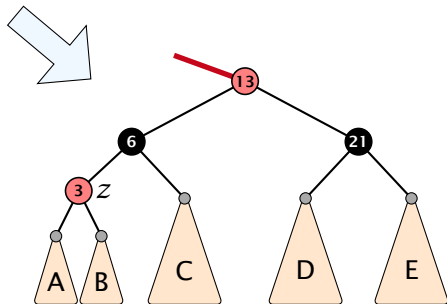
1. recolour



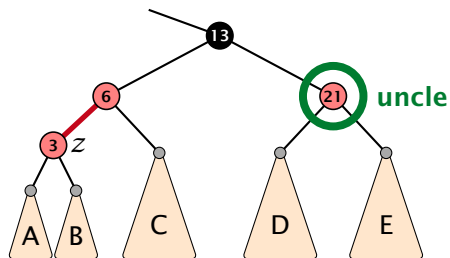
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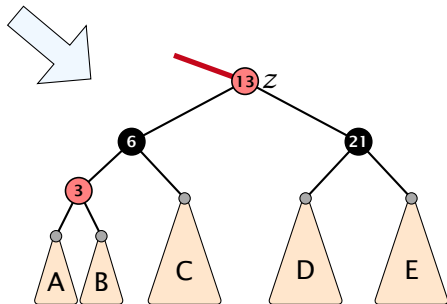
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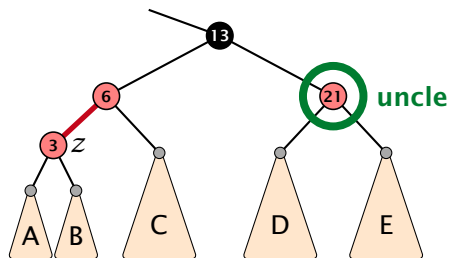
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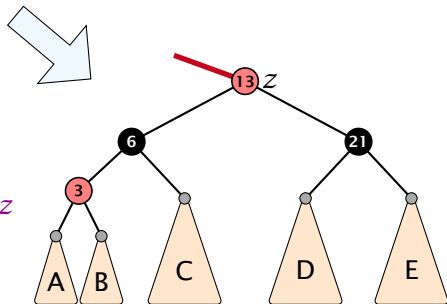
1. recolour
2. move  $z$  to grand-parent



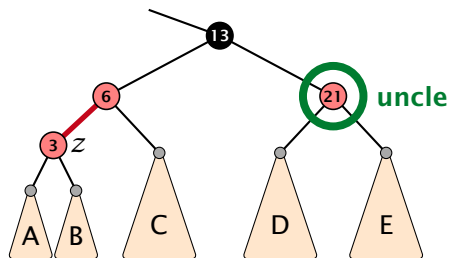
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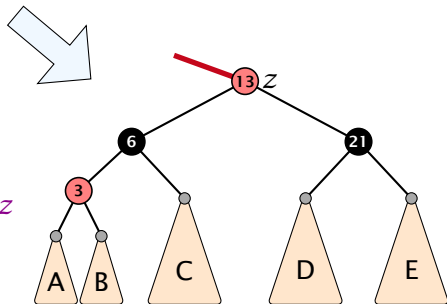
1. recolor
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$



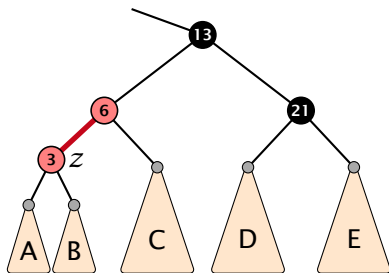
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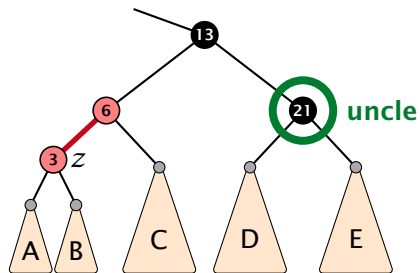
1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$
4. you made progress



## Case 2b: Black uncle and z is left child

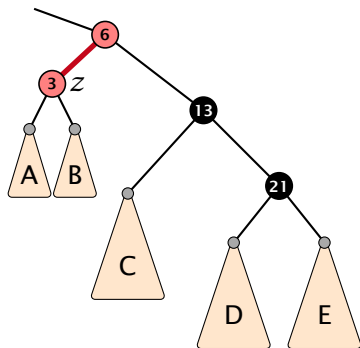
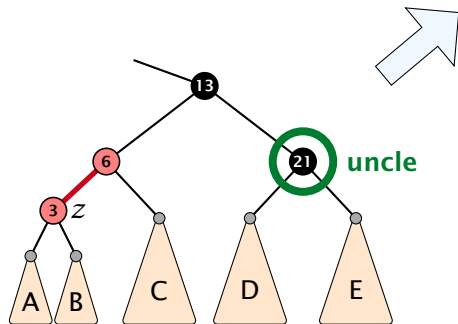


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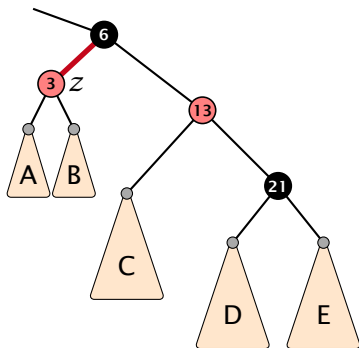
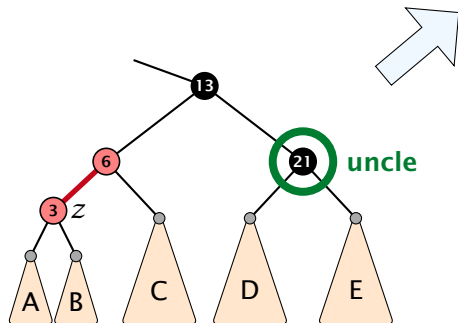
1. rotate around grandparent





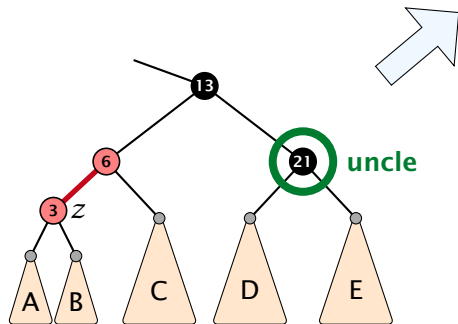
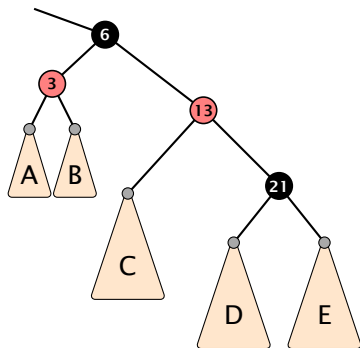
## Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

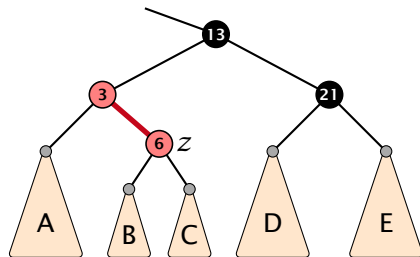


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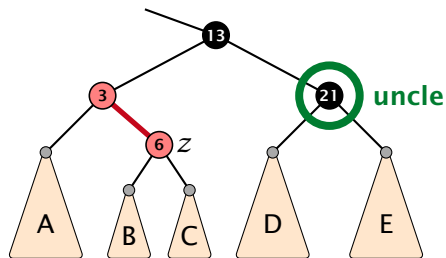
1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



## Case 2a: Black uncle and z is right child

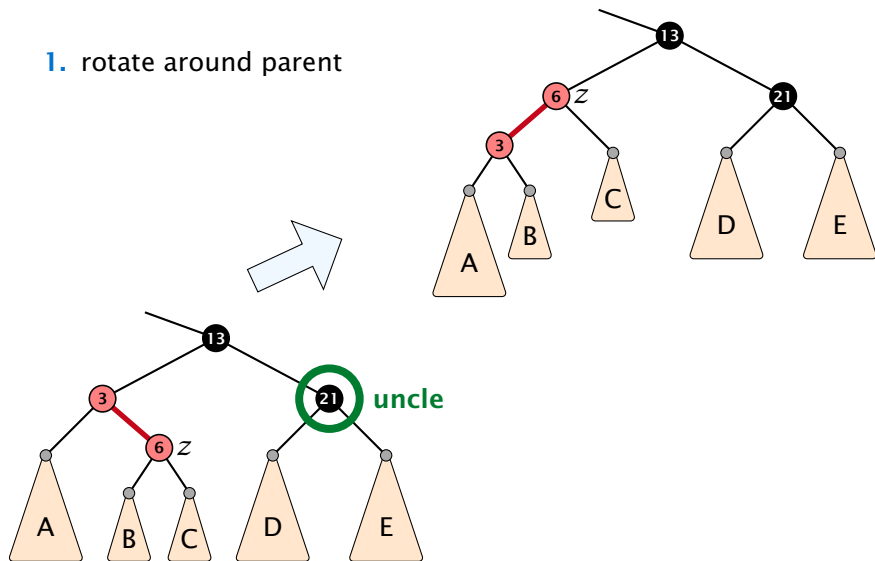


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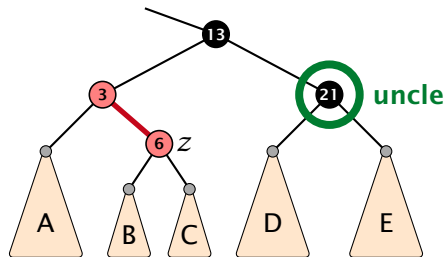
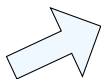
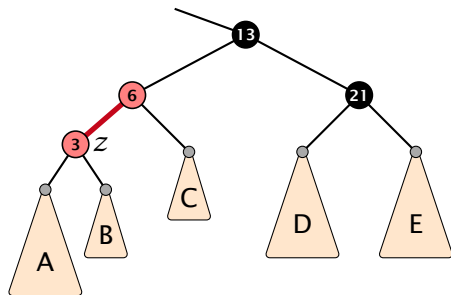
## Case 2a: Black uncle and z is right child

1. rotate around parent



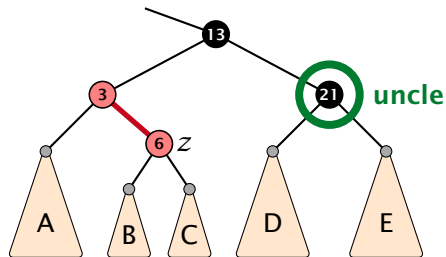
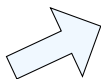
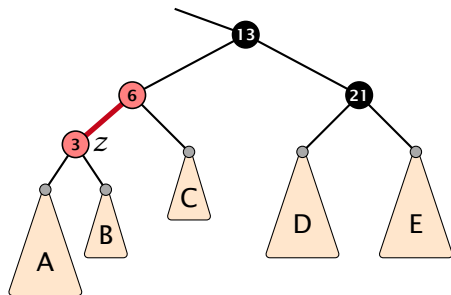
## Case 2a: Black uncle and z is right child

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3. you have Case 2b.



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## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.



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Performing Case 1 at most  $\mathcal{O}(\log n)$  times and every other case at most once, we get a red-black tree. Hence  $\mathcal{O}(\log n)$  re-colorings and at most 2 rotations.

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- ▶ If you delete the root, the root may now be red.

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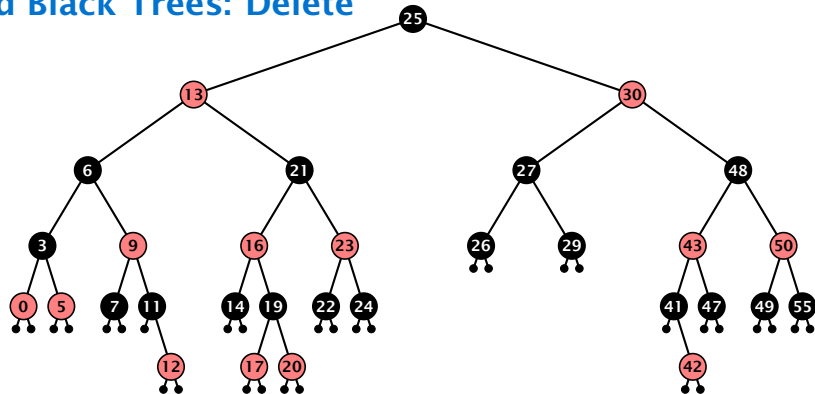
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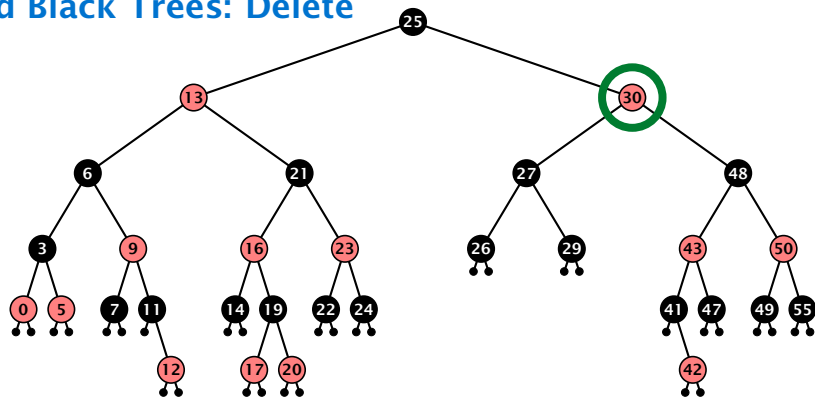
If it was black there may be the following problems.

- ▶ Parent and child of  $x$  were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of  $x$  to a descendant leaf of  $x$  changes the number of black nodes. Black height property might be violated.

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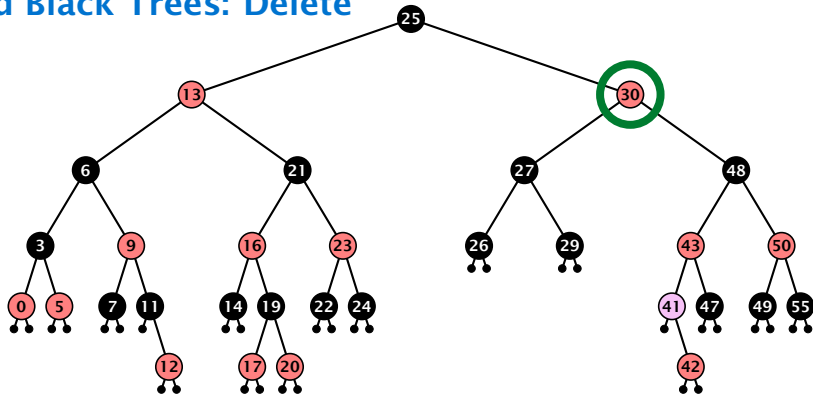


### Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

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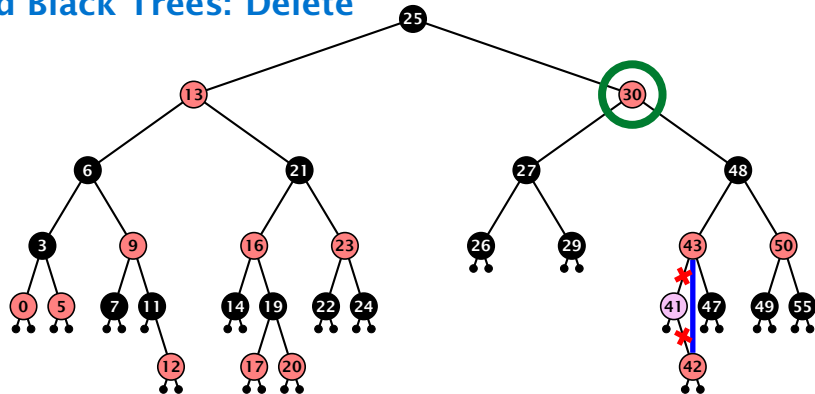


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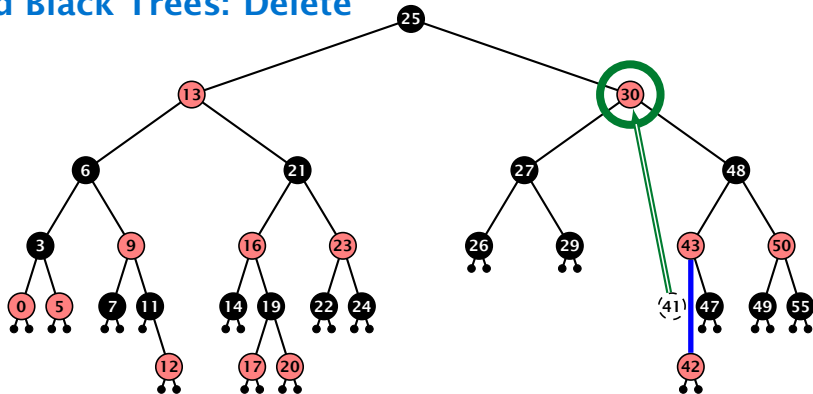


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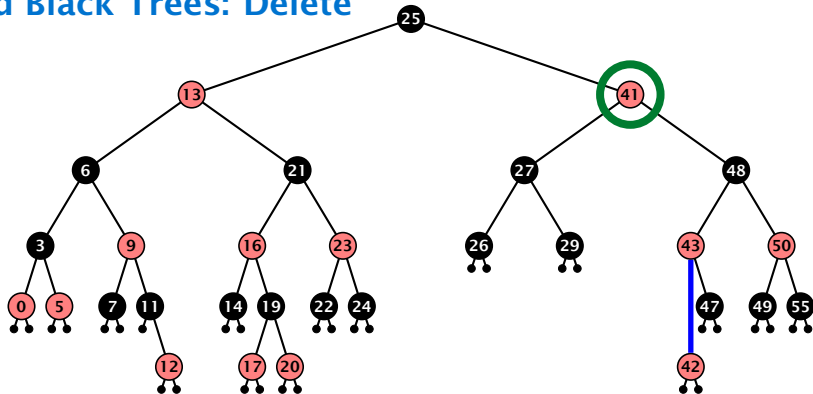


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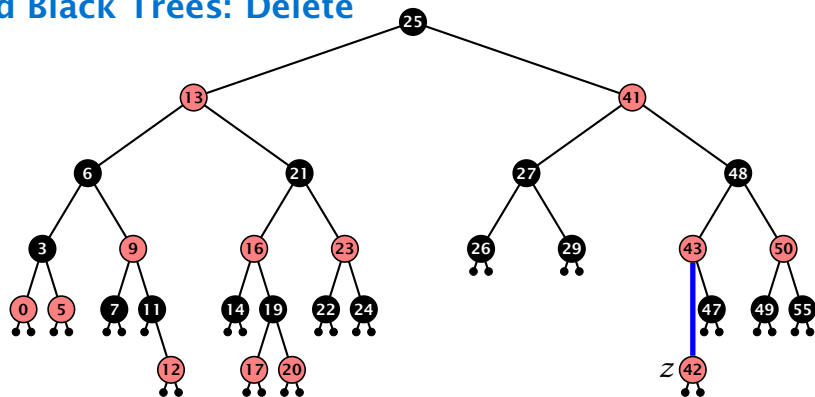
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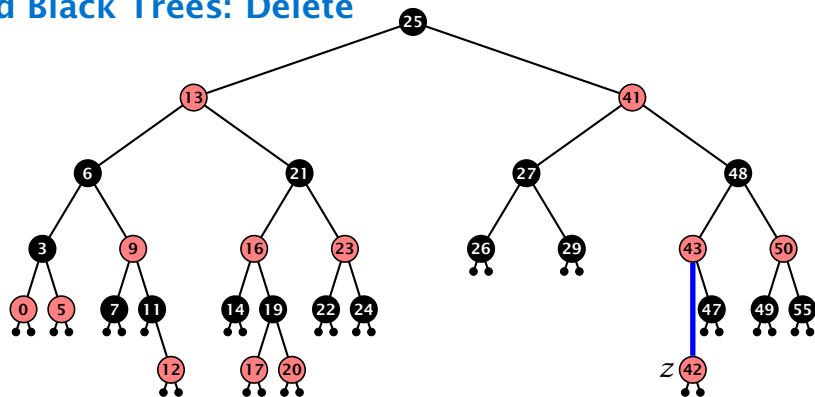
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Delete:

- ▶ deleting black node messes up black-height property

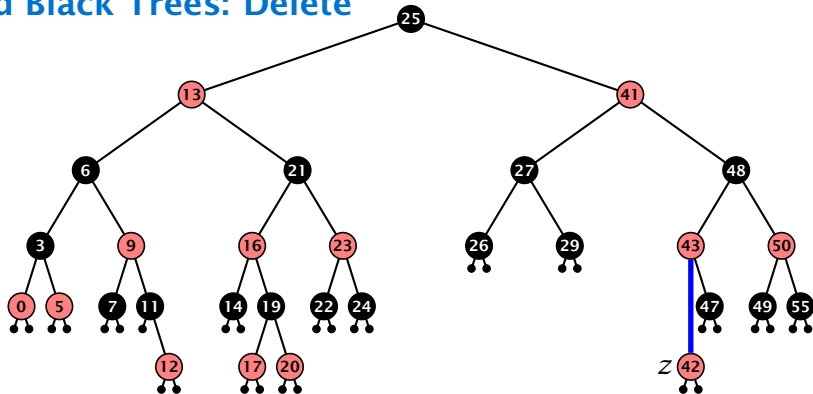
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## Red Black Trees: Delete



### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine
- ▶ the problem is if  $z$  is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

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## Invariant of the fix-up algorithm

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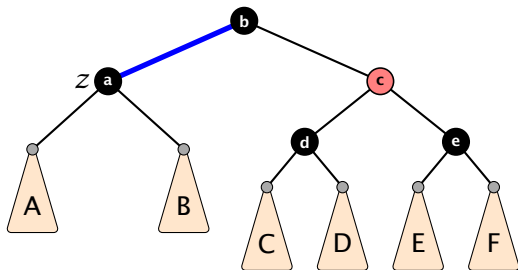
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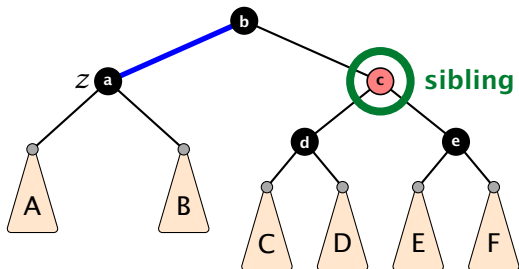
- ▶ the node  $z$  is black
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**Goal:** make rotations in such a way that you at some point can remove the fake black unit from the edge.

## Case 1: Sibling of $z$ is red

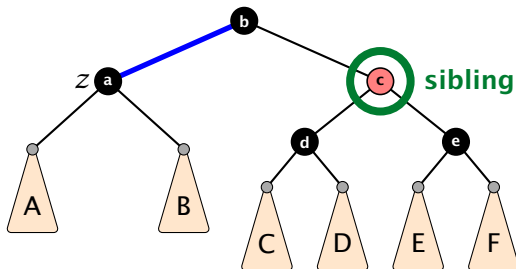


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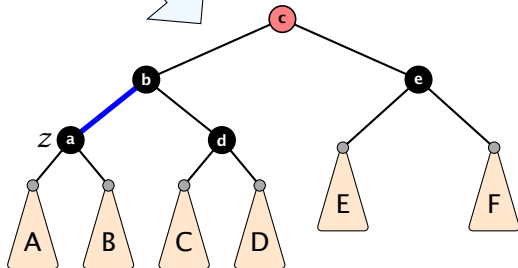




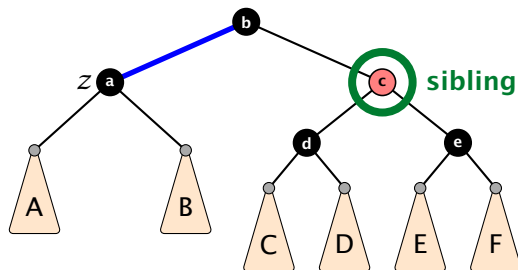
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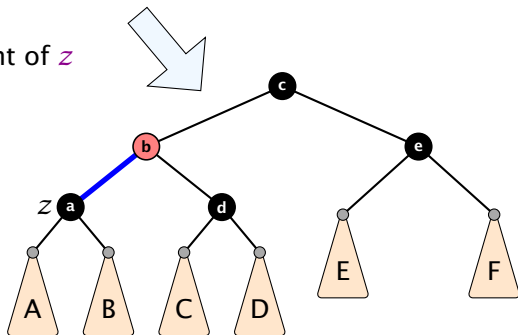
1. left-rotate around parent of  $z$



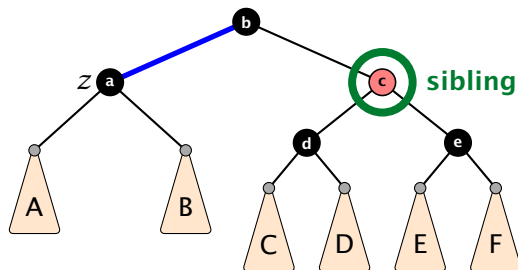
## Case 1: Sibling of $z$ is red



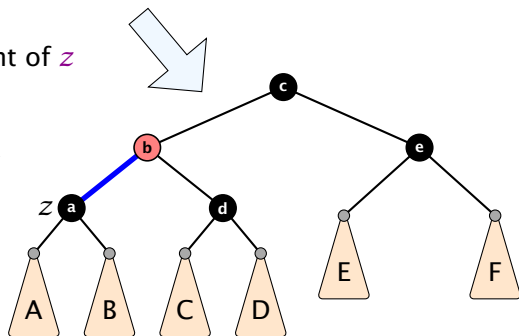
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$



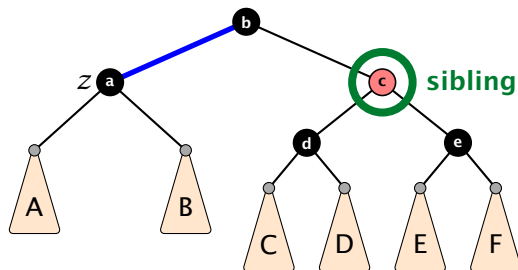
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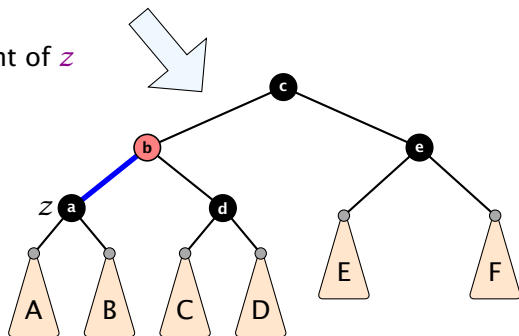
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)



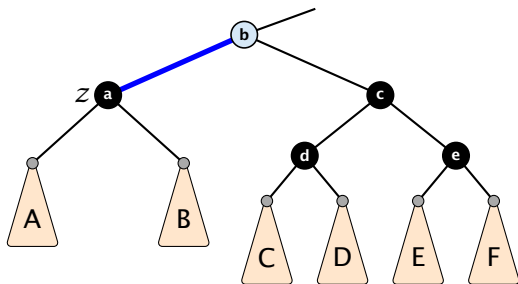
## Case 1: Sibling of $z$ is red



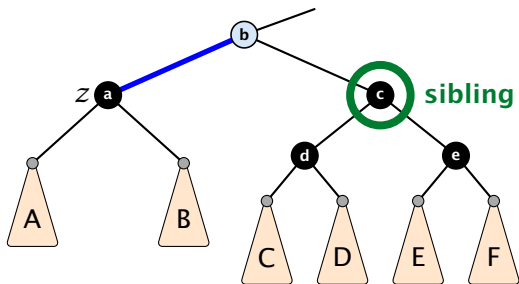
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)
4. Case 2 (special), or Case 3, or Case 4



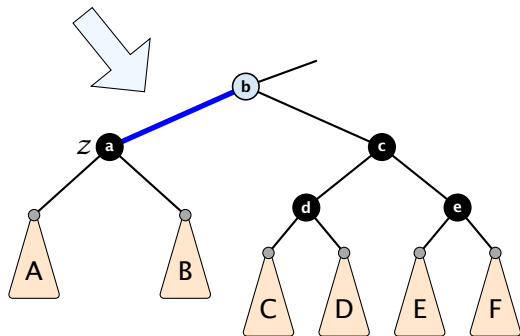
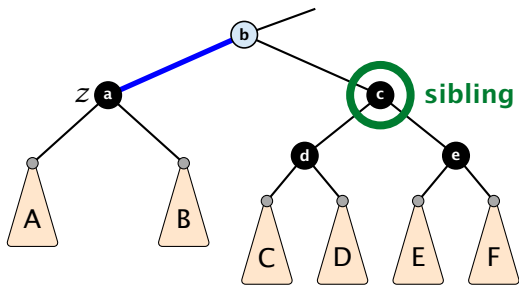
## Case 2: Sibling is black with two black children



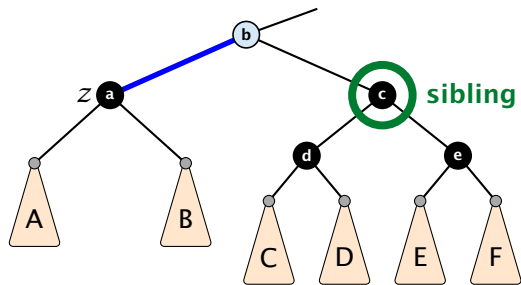
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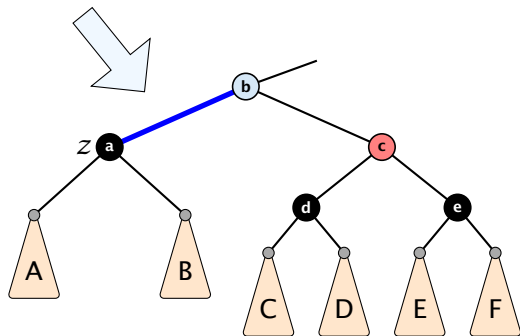
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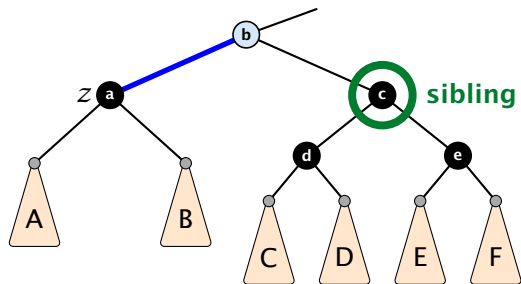


1. re-color node  $c$

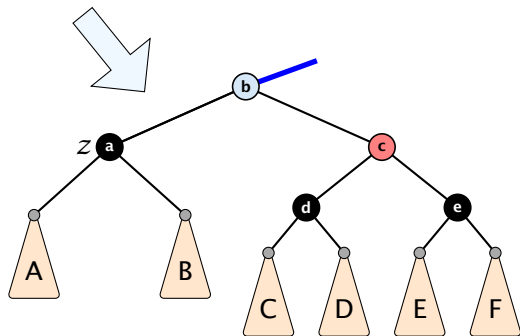




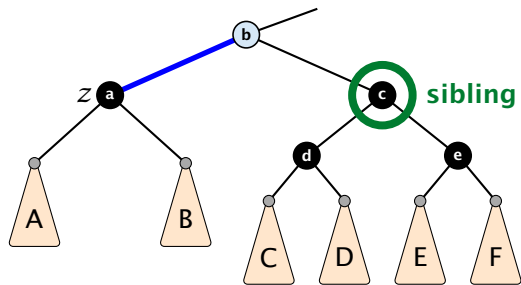
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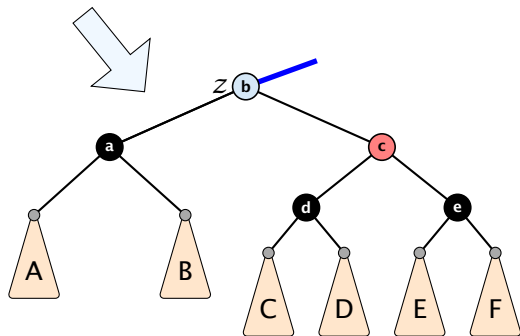
1. re-color node *c*
2. move fake black unit upwards



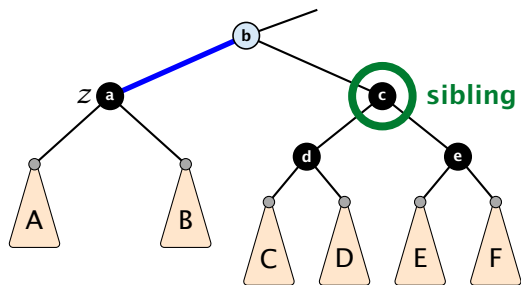
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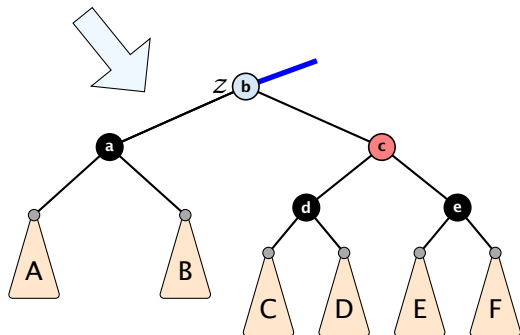
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards



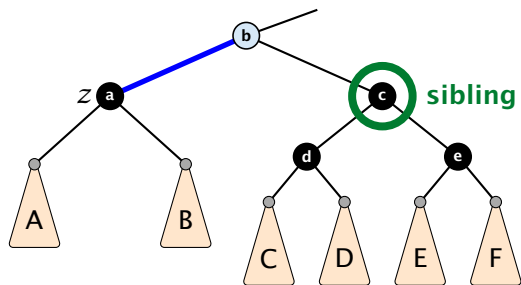
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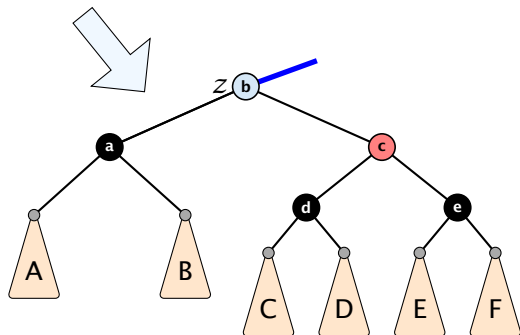
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress



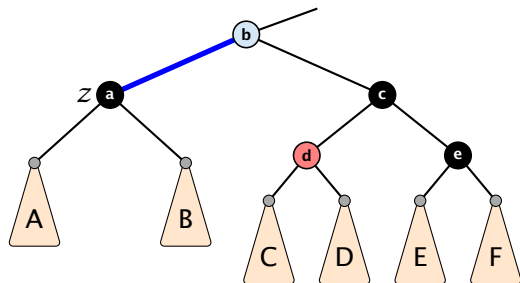
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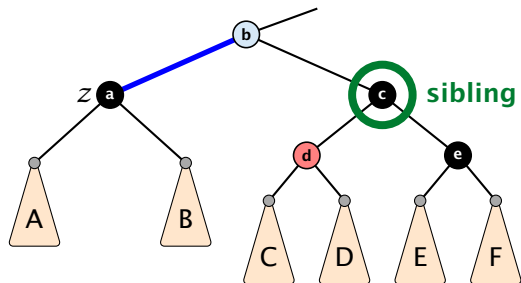
1. re-color node  $c$
2. move fake black unit upwards
3. move  $z$  upwards
4. we made progress
5. if  $b$  is red we color it black and are done



## Case 3: Sibling black with one black child to the right

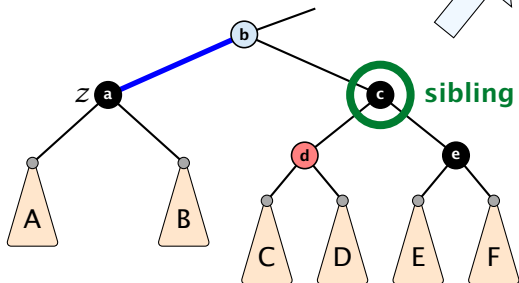
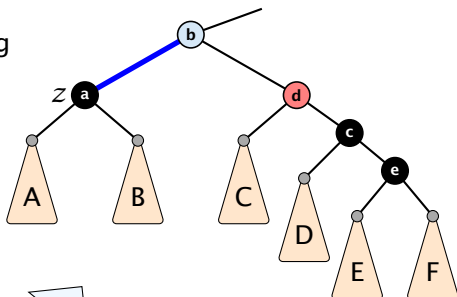


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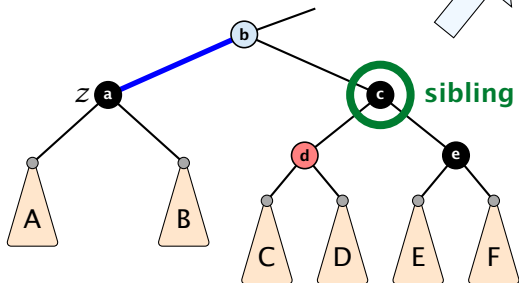
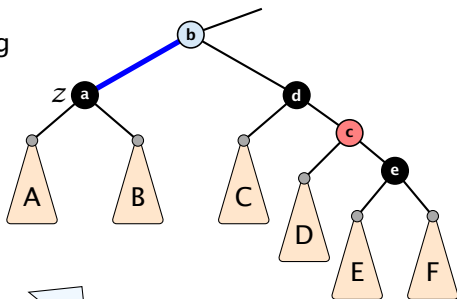
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1. do a right-rotation at sibling



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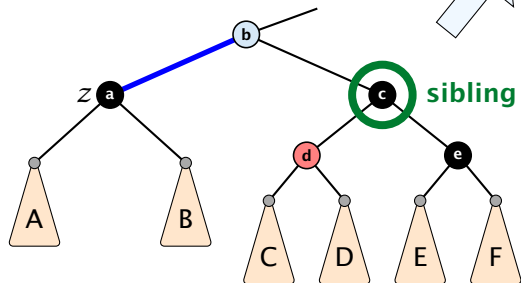
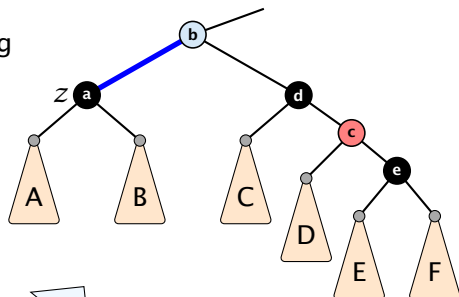
1. do a right-rotation at sibling
2. recolor  $c$  and  $d$



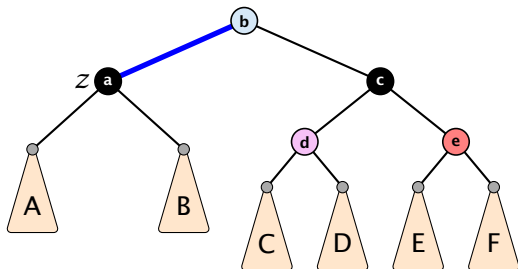


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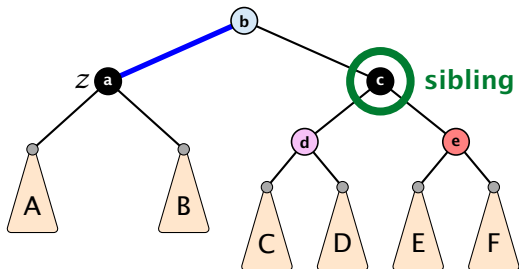
1. do a right-rotation at sibling
2. recolor  $c$  and  $d$
3. new sibling is black with red right child (Case 4)



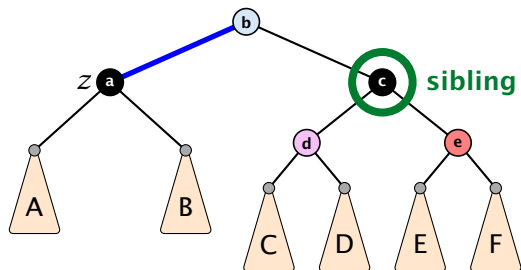
## Case 4: Sibling is black with red right child



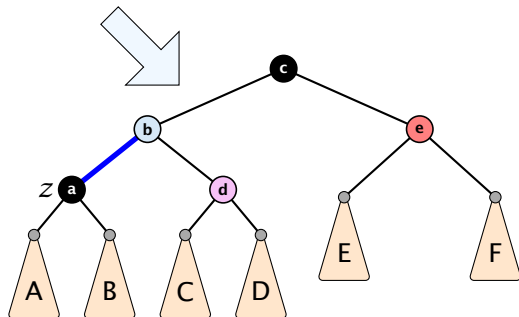
## Case 4: Sibling is black with red right child



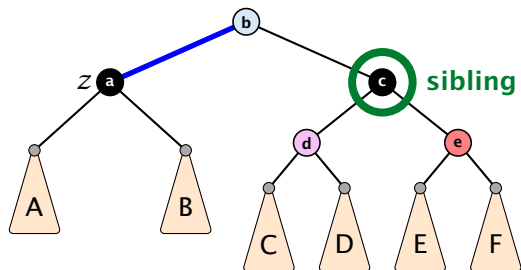
## Case 4: Sibling is black with red right child



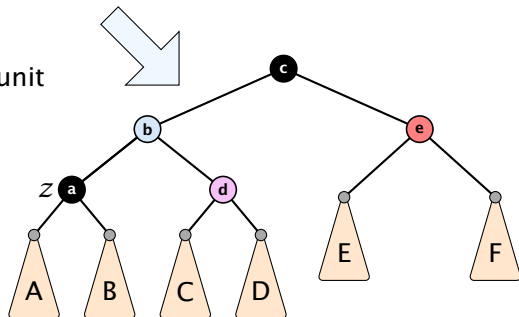
1. left-rotate around *b*



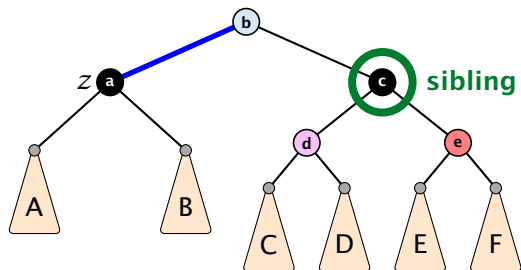
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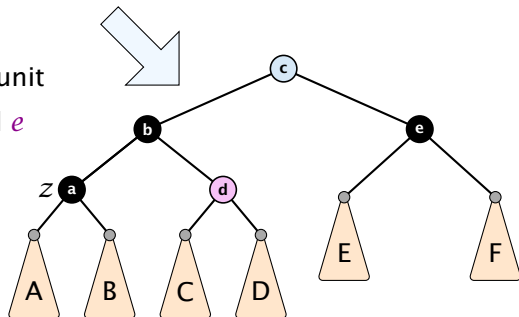
1. left-rotate around *b*
2. remove the fake black unit



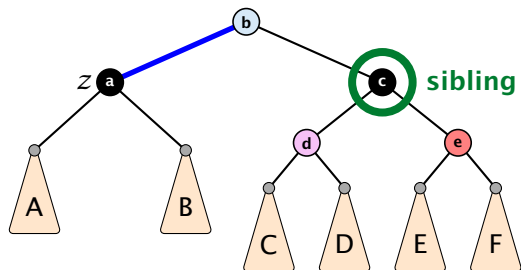
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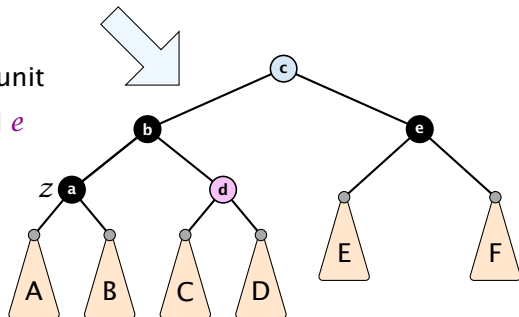
1. left-rotate around **b**
2. remove the fake black unit
3. recolor nodes **b**, **c**, and **e**



## Case 4: Sibling is black with red right child



1. left-rotate around  $b$
2. remove the fake black unit
3. recolor nodes  $b$ ,  $c$ , and  $e$
4. you have a valid red black tree



## Running time:

- ▶ only Case 2 can repeat; but only  $h$  many steps, where  $h$  is the height of the tree



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Case 1 → Case 3 → Case 4 → red black tree  
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Performing Case 2 at most  $\mathcal{O}(\log n)$  times and every other step at most once, we get a red black tree. Hence,  $\mathcal{O}(\log n)$  re-colorings and at most 3 rotations.

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repeated accesses are faster
- only amortized guarantee
- read-operations change the tree

# Splay Trees

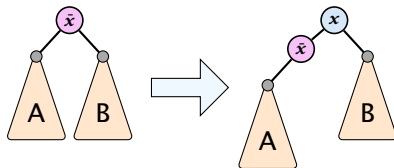
## **find( $x$ )**

- ▶ search for  $x$  according to a search tree
- ▶ let  $\tilde{x}$  be last element on search-path
- ▶  $\text{splay}(\tilde{x})$

# Splay Trees

## insert( $x$ )

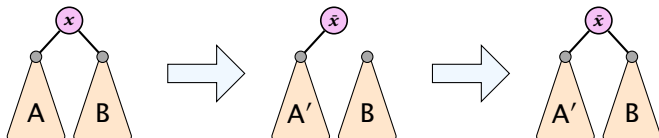
- ▶ search for  $x$ ;  $\bar{x}$  is last visited element during search (successor or predecessor of  $x$ )
- ▶ splay( $\bar{x}$ ) moves  $\bar{x}$  to the root
- ▶ insert  $x$  as new root



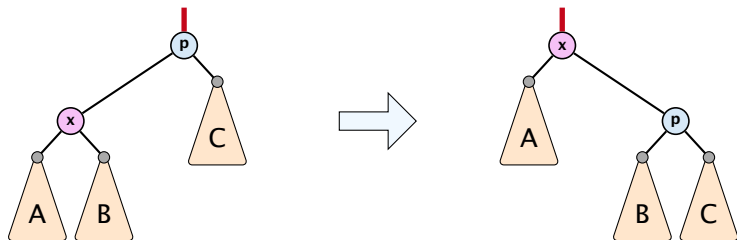
# Splay Trees

## delete( $x$ )

- ▶ search for  $x$ ; splay( $x$ ); remove  $x$
- ▶ search largest element  $\bar{x}$  in  $A$
- ▶ splay( $\bar{x}$ ) (on subtree  $A$ )
- ▶ connect root of  $B$  as right child of  $\bar{x}$



# Move to Root

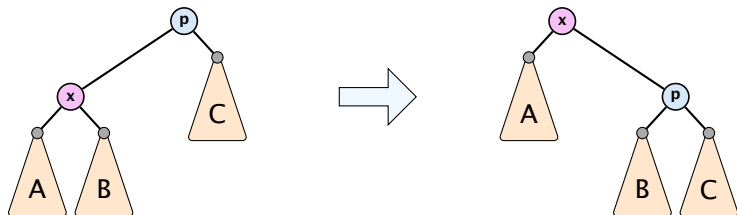


## How to bring element to root?

- ▶ one (bad) option: `moveToRoot( $x$ )`
- ▶ iteratively do rotation around parent of  $x$  until  $x$  is root
- ▶ if  $x$  is left child do right rotation otw. left rotation



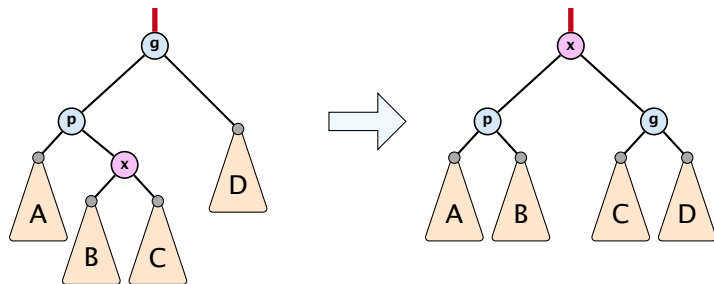
## Splay: Zig Case



**better option splay( $x$ ):**

- ▶ zig case: if  $x$  is child of root do left rotation or right rotation around parent

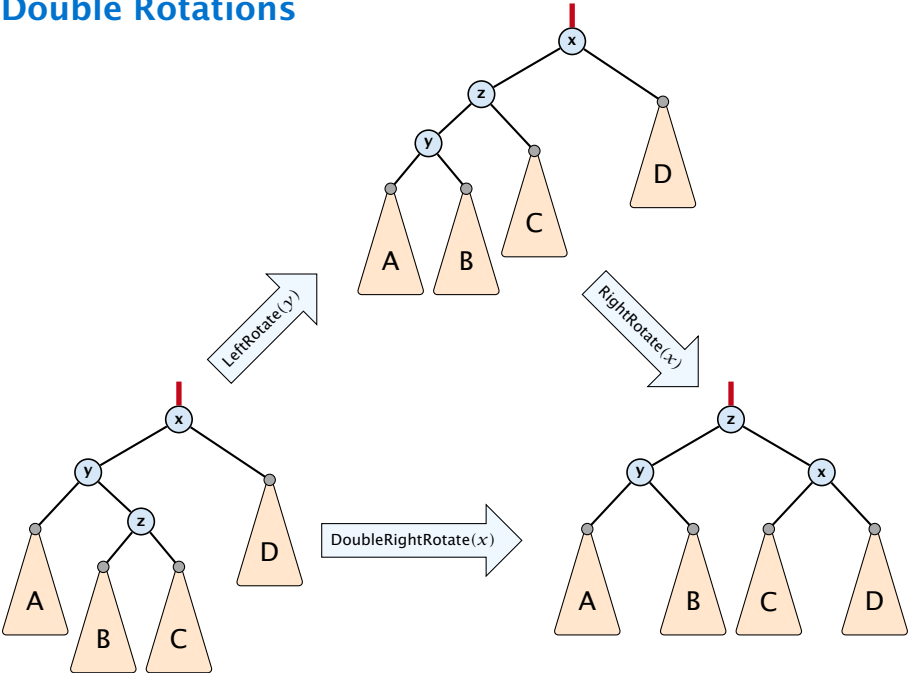
## Splay: Zigzag Case



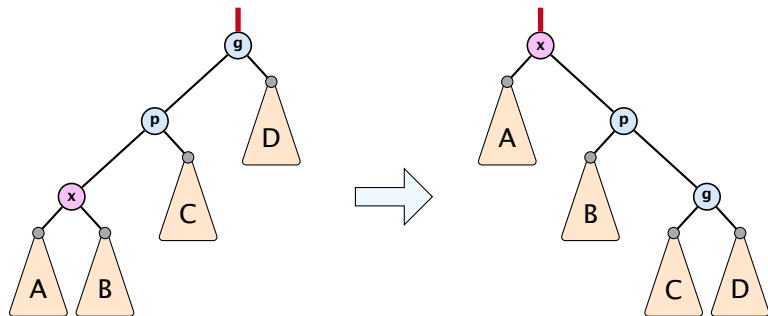
### better option $\text{splay}(x)$ :

- ▶ zigzag case: if  $x$  is right child and parent of  $x$  is left child (or  $x$  left child parent of  $x$  right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

# Double Rotations



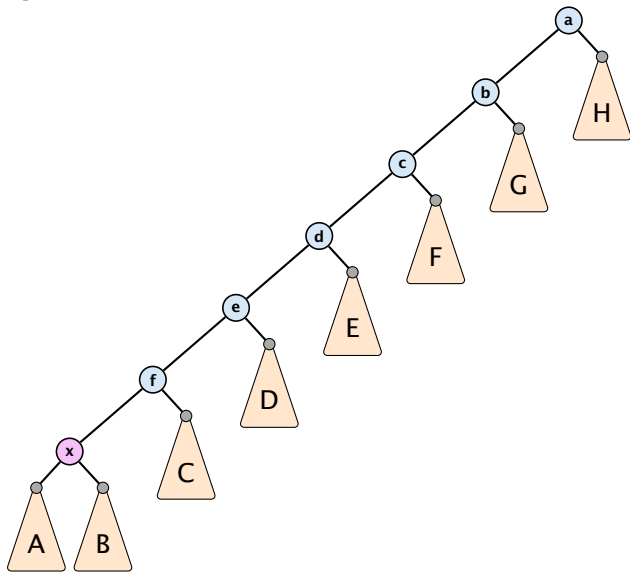
## Splay: Zigzig Case



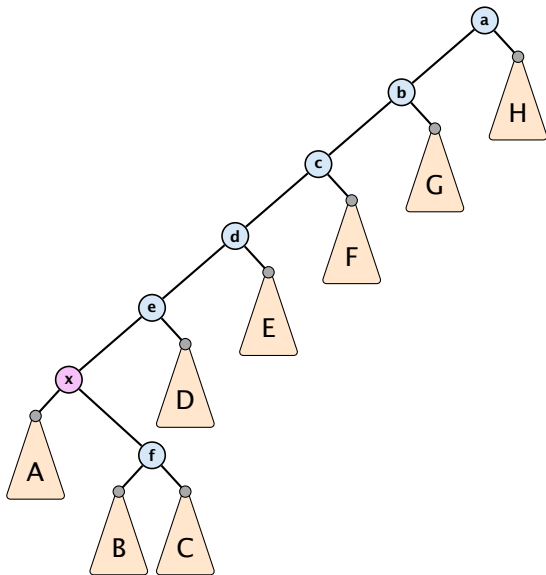
### better option $\text{splay}(x)$ :

- ▶ zigzig case: if  $x$  is left child and parent of  $x$  is left child (or  $x$  right child, parent of  $x$  right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

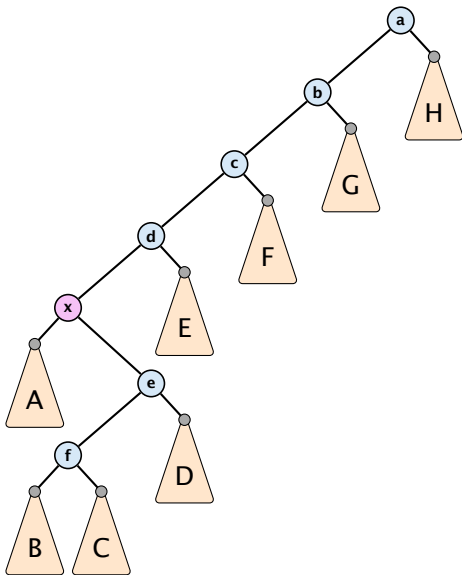
# Splay vs. Move to Root



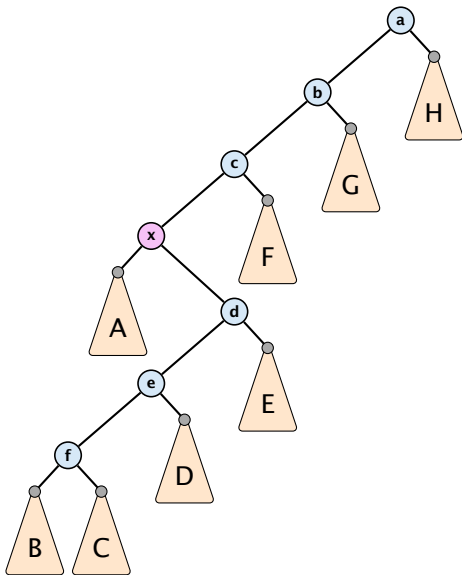
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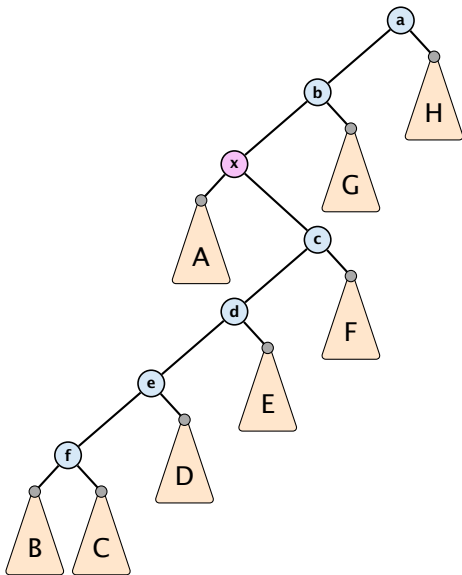


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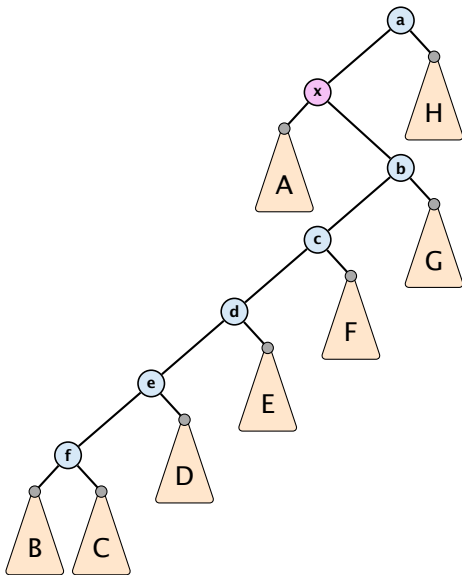




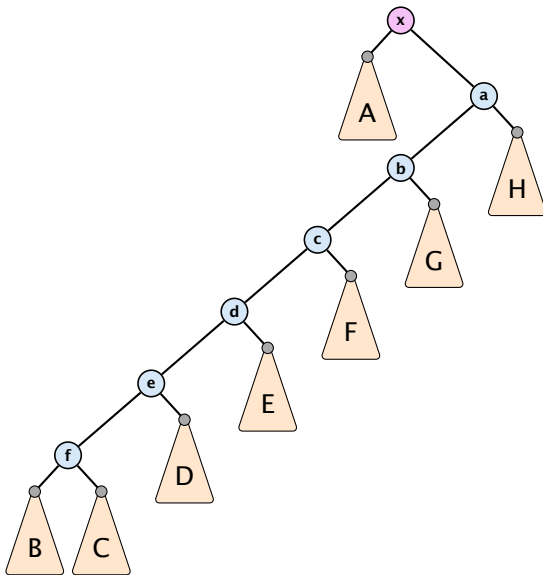
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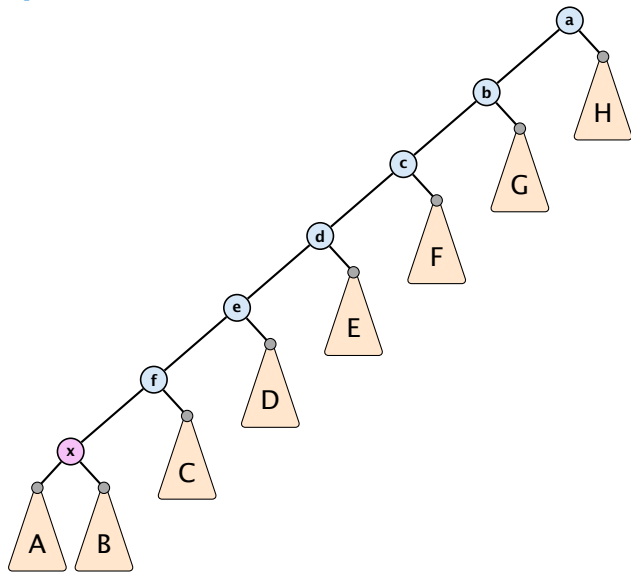
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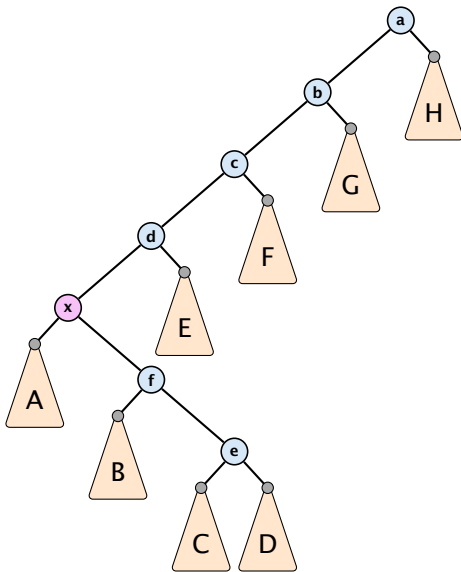
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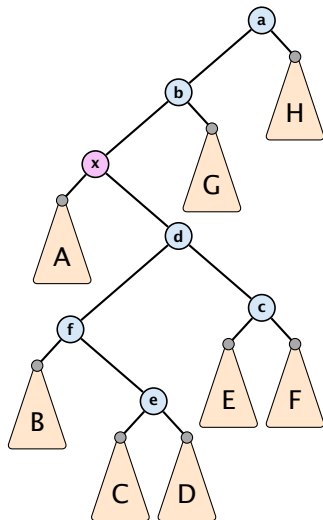
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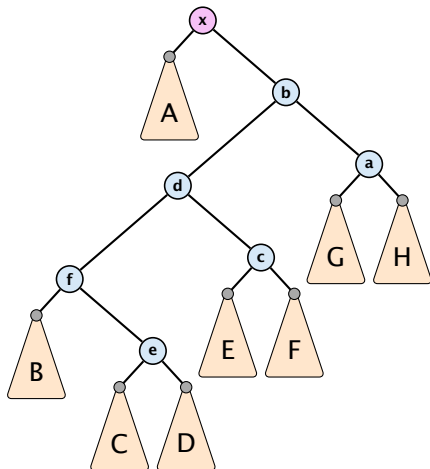
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# Static Optimality

Suppose we have a sequence of  $m$  find-operations.  $\text{find}(x)$  appears  $h_x$  times in this sequence.

The cost of a **static** search tree  $T$  is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is  $\mathcal{O}(\text{cost}(T_{\min}))$ , where  $T_{\min}$  is an **optimal static search tree**.



# Dynamic Optimality

Let  $S$  be a sequence with  $m$  find-operations.

Let  $A$  be a data-structure based on a search tree:

- ▶ the cost for accessing element  $x$  is  $1 + \text{depth}(x)$ ;
- ▶ after accessing  $x$  the tree may be re-arranged through rotations;

## Conjecture:

A splay tree that only contains elements from  $S$  has cost  $\mathcal{O}(\text{cost}(A, S))$ , for processing  $S$ .

## Lemma 16

*Splay Trees have an **amortized** running time of  $\mathcal{O}(\log n)$  for all operations.*

# Amortized Analysis

## Definition 17

A data structure with operations  $\text{op}_1(), \dots, \text{op}_k()$  has amortized running times  $t_1, \dots, t_k$  for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most  $n$  elements, and let  $k_i$  denote the number of occurrences of  $\text{op}_i()$  within this sequence. Then the actual running time must be at most  $\sum_i k_i \cdot t_i(n)$ .

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- ▶ Show that  $\Phi(D_i) \geq \Phi(D_0)$ .

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0)$$

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- ▶  $\Phi(D_i)$  is the potential after the  $i$ -th operation.
- ▶ Amortized cost of the  $i$ -th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

- ▶ Show that  $\Phi(D_i) \geq \Phi(D_0)$ .

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

# Example: Stack

## Stack

- ▶  $S.$  push()
- ▶  $S.$  pop()
- ▶  $S.$  multipop( $k$ ): removes  $k$  items from the stack. If the stack currently contains less than  $k$  items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

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## Actual cost:

- ▶  $S.$  push(): cost 1.
- ▶  $S.$  pop(): cost 1.
- ▶  $S.$  multipop( $k$ ): cost  $\min\{\text{size}, k\} = k$ .

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$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \leq 0 .$$

- ▶  **$S$ . multipop( $k$ ):** cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 .$$



## Example: Binary Counter

### **Incrementing a binary counter:**

Consider a computational model where each bit-operation costs one time-unit.

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### Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ Increment: cost is  $k + 1$ , where  $k$  is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has  $k = 1$ ).

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Choose potential function  $\Phi(x) = k$ , where  $k$  denotes the number of ones in the binary representation of  $x$ .

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- ▶ **Increment:** Let  $k$  denotes the number of consecutive ones in the least significant bit-positions. An increment involves  $k$   $(1 \rightarrow 0)$ -operations, and one  $(0 \rightarrow 1)$ -operation.

Hence, the amortized cost is  $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$ .



# Splay Trees

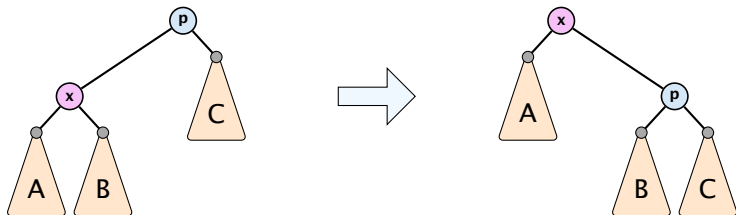
## potential function for splay trees:

- ▶ size  $s(x) = |T_x|$
- ▶ rank  $r(x) = \log_2(s(x))$
- ▶  $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

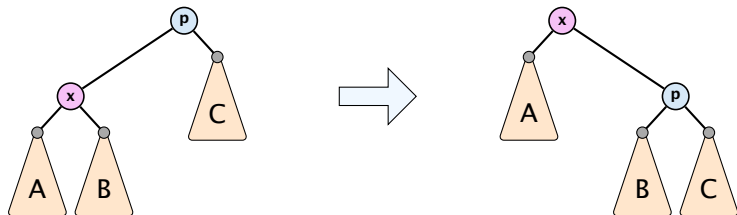
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

## Splay: Zig Case



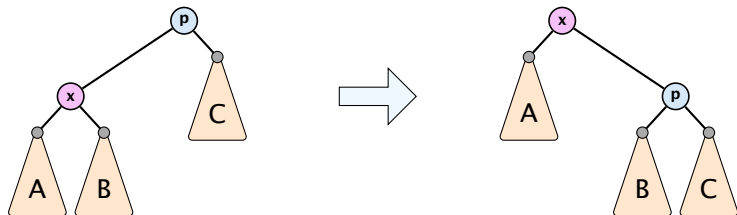
$$\Delta\Phi =$$

## Splay: Zig Case



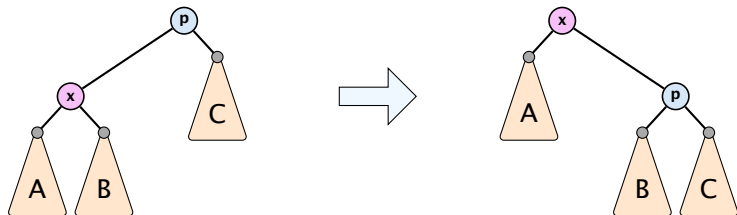
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

## Splay: Zig Case



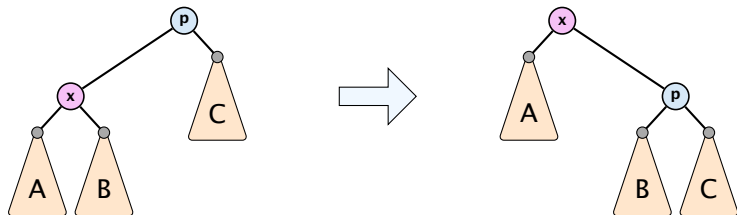
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

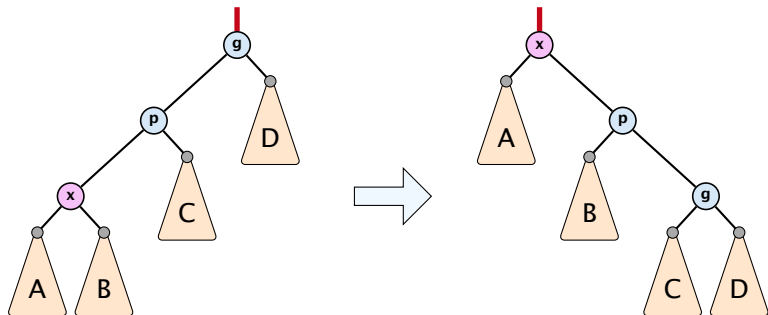
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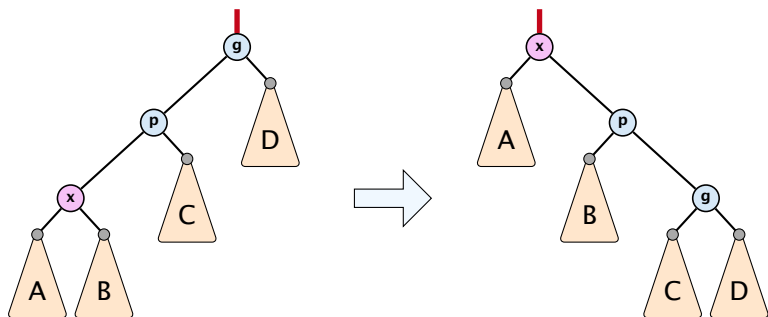
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

## Splay: Zigzig Case



$$\Delta\Phi =$$

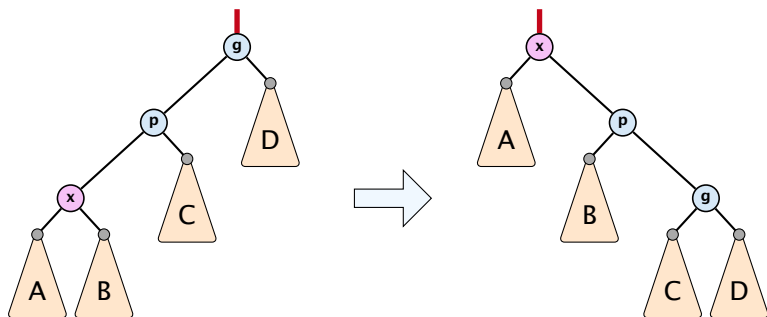
## Splay: Zigzig Case



$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

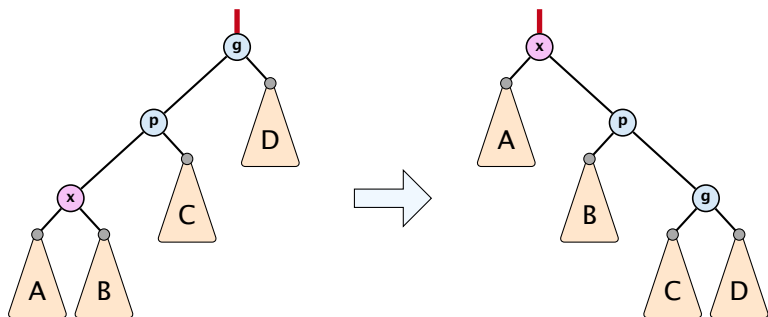


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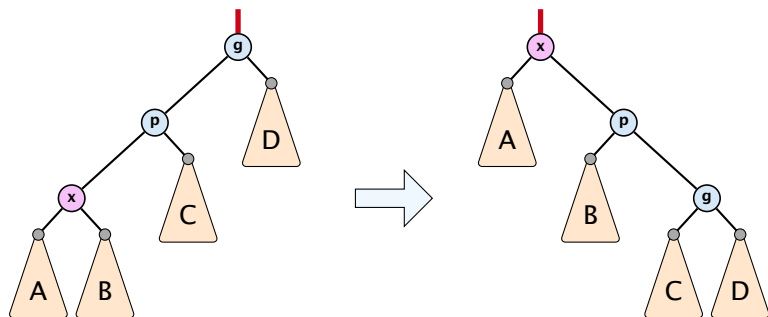
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

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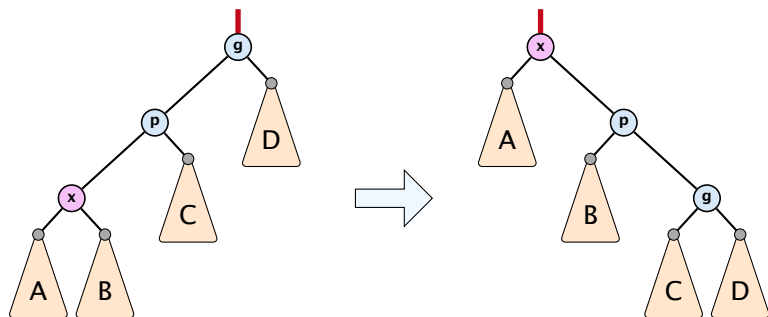
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x)\end{aligned}$$

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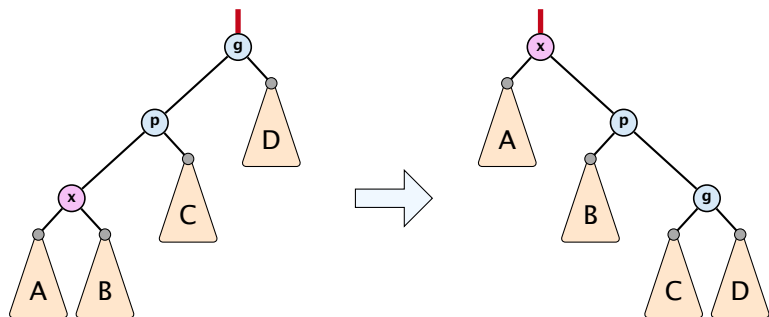
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\end{aligned}$$

## Splay: Zigzig Case



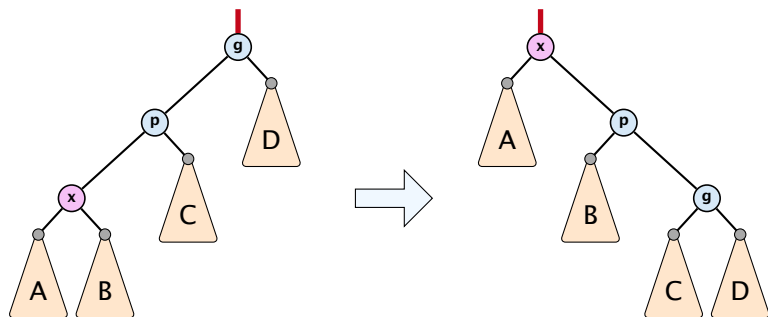
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzig Case



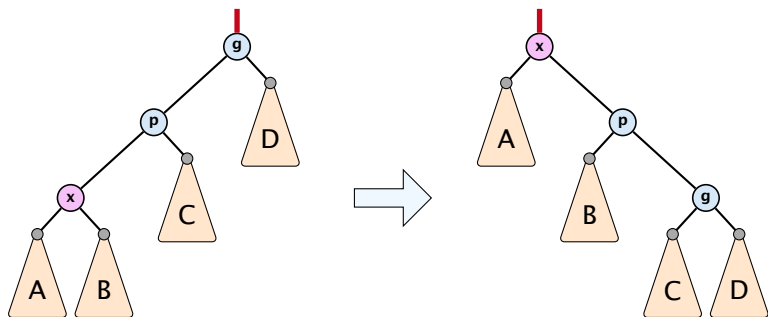
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x))\end{aligned}$$

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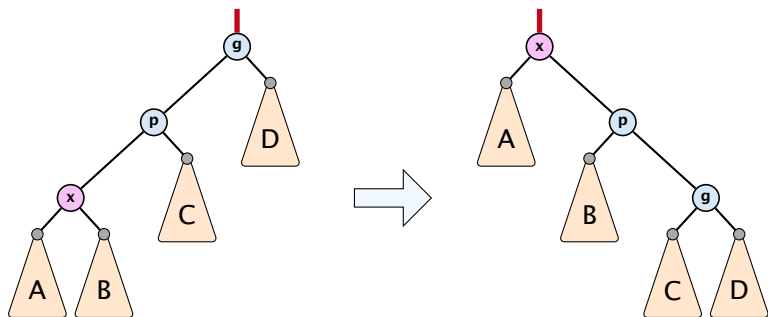
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## Splay: Zigzig Case



$$\frac{1}{2}(r(x) + r'(g) - 2r'(x))$$

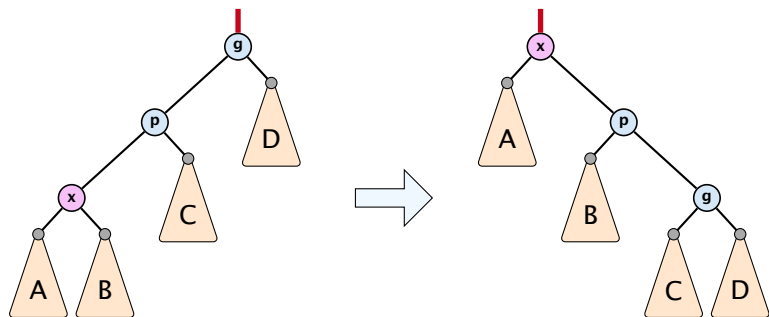
## Splay: Zigzig Case



$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

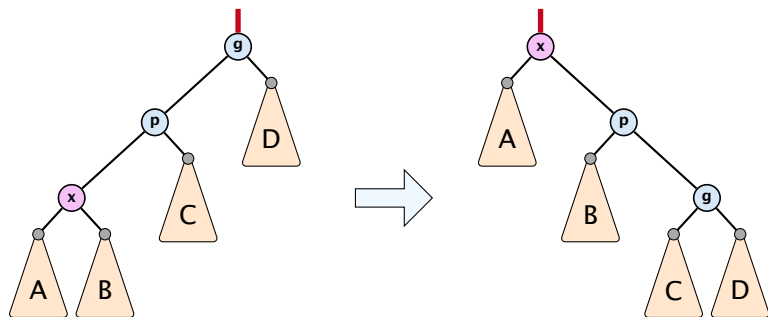


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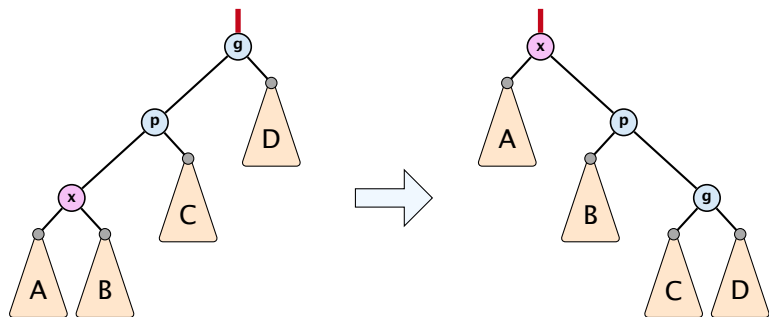
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \end{aligned}$$

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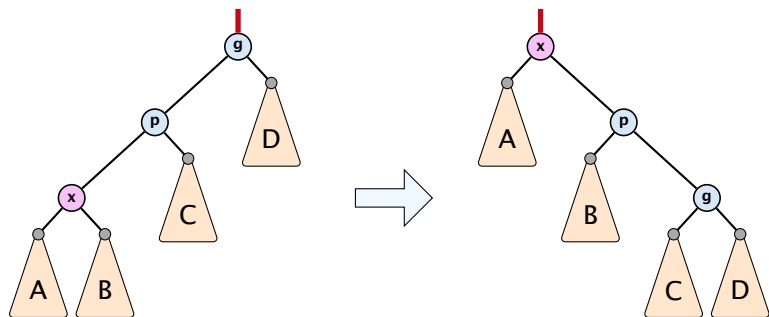
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \end{aligned}$$

## Splay: Zigzig Case



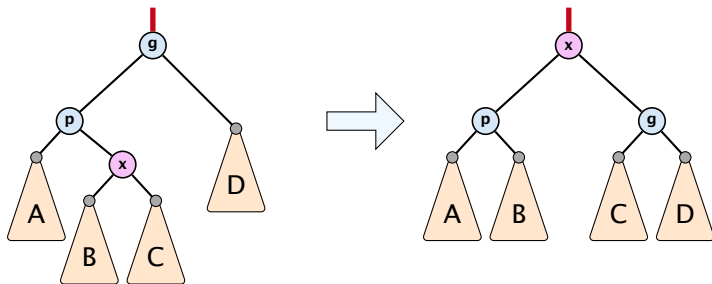
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) \end{aligned}$$

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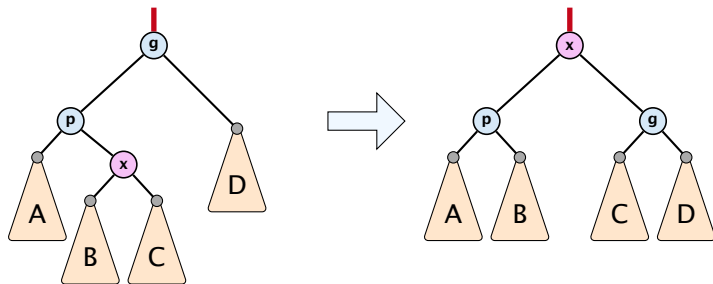
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## Splay: Zigzag Case



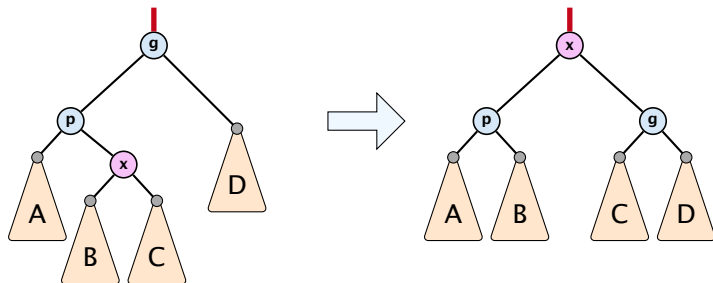
$\Delta\Phi =$

## Splay: Zigzag Case



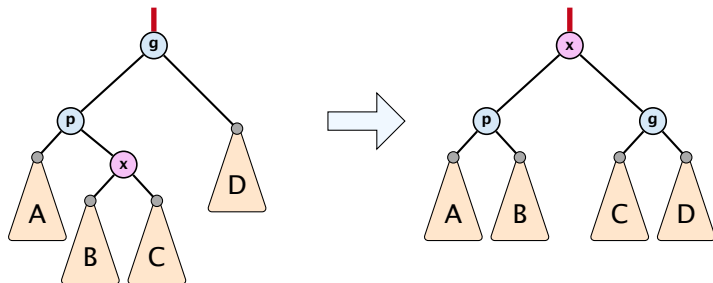
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## Splay: Zigzag Case



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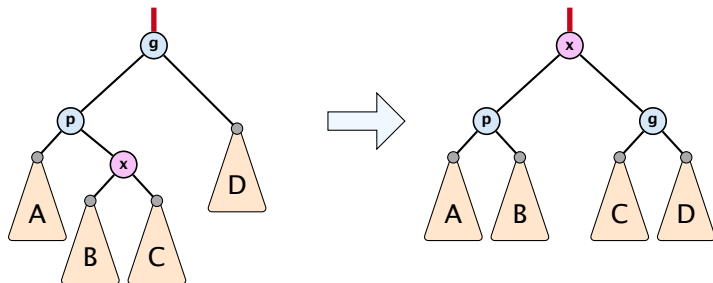
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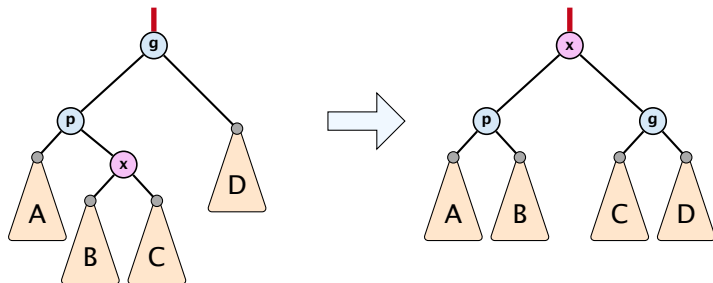


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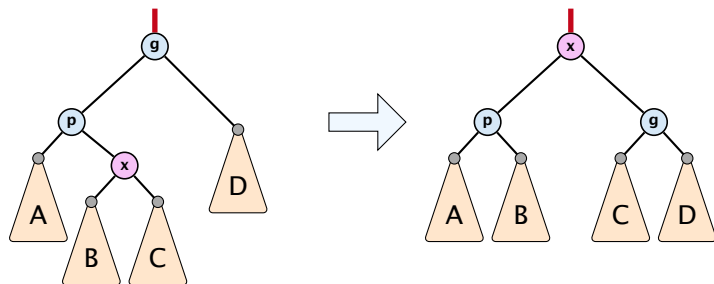
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## Splay: Zigzag Case



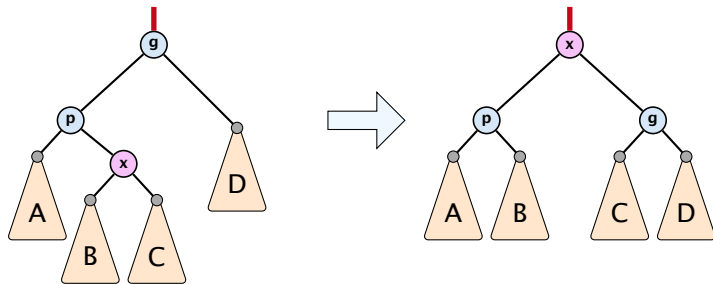
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## Splay: Zigzag Case



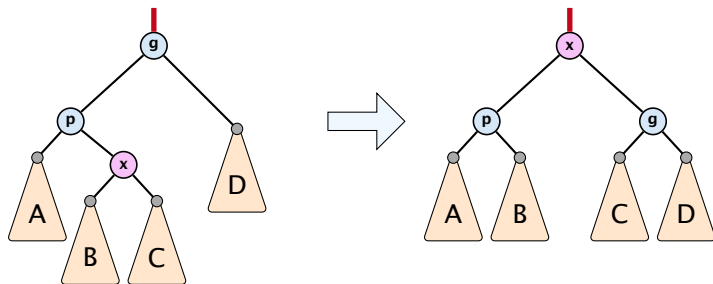
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \quad \Rightarrow \text{COST}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



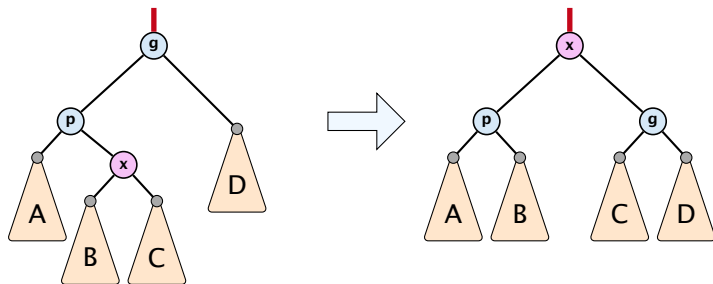
$$\frac{1}{2}(r'(p) + r'(g) - 2r'(x))$$

## Splay: Zigzag Case



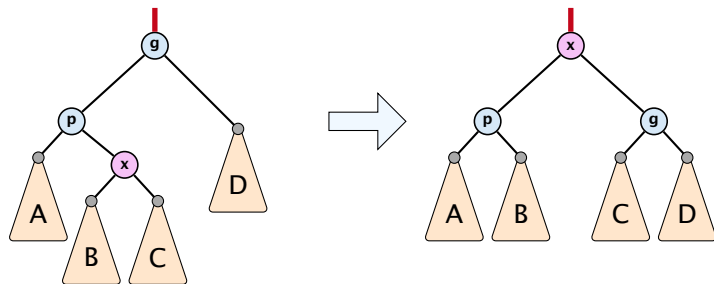
$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

## Splay: Zigzag Case



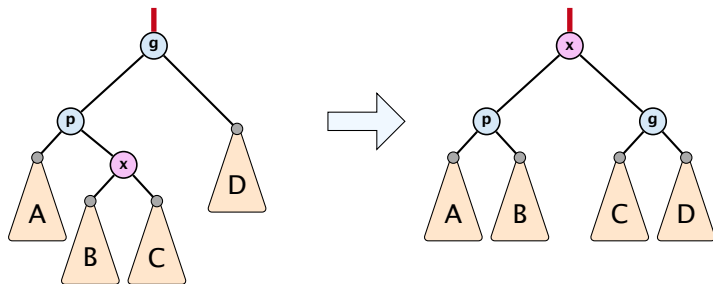
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## Splay: Zigzag Case



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## Splay: Zigzag Case



$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$



Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
- ▶ **Search( $k$ )**: search for element with key  $k$ .
- ▶ **Delete( $x$ )**: delete element referenced by pointer  $x$ .
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**Augment an existing data-structure instead of developing a new one.**

## 7.4 Augmenting Data Structures

### How to augment a data-structure

1. choose an underlying data-structure

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3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

## 7.4 Augmenting Data Structures

### How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
4. develop the new operations

## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

1. We choose a red-black tree as the underlying data-structure.



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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

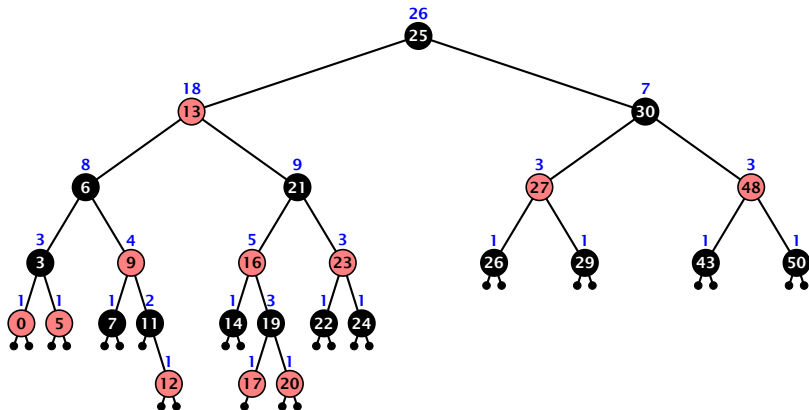
4. How does find-by-rank work?

Find-by-rank( $k$ ) := Select( $root, k$ ) with

**Algorithm 1** Select( $x, i$ )

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

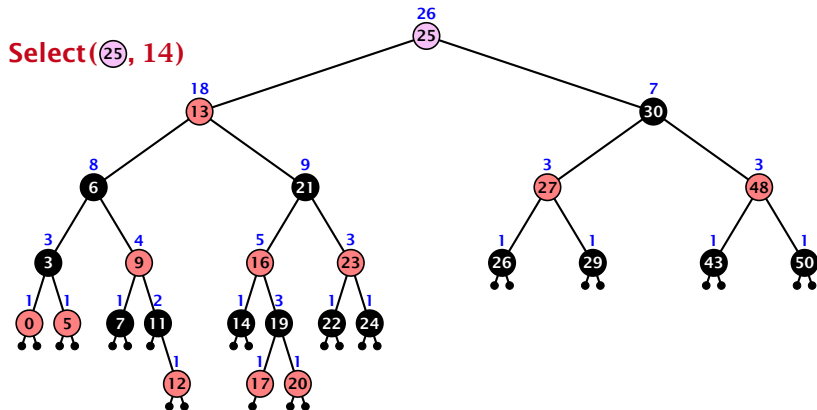
## Select( $x, i$ )



### Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

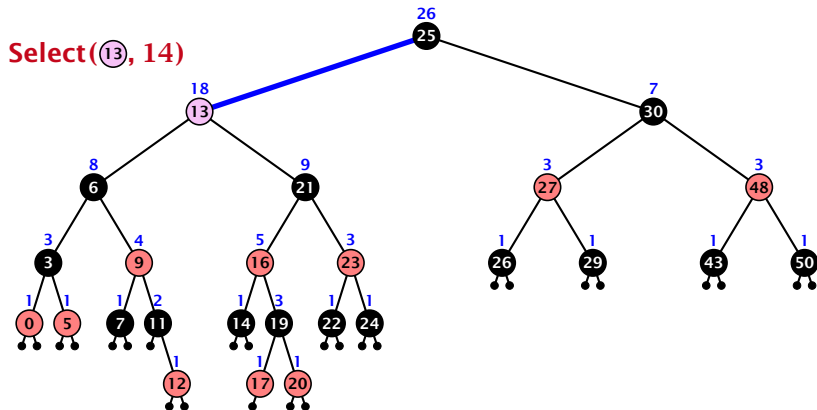
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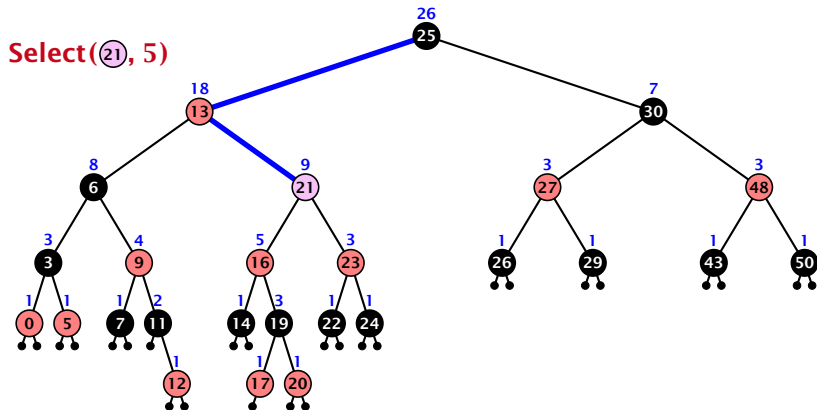
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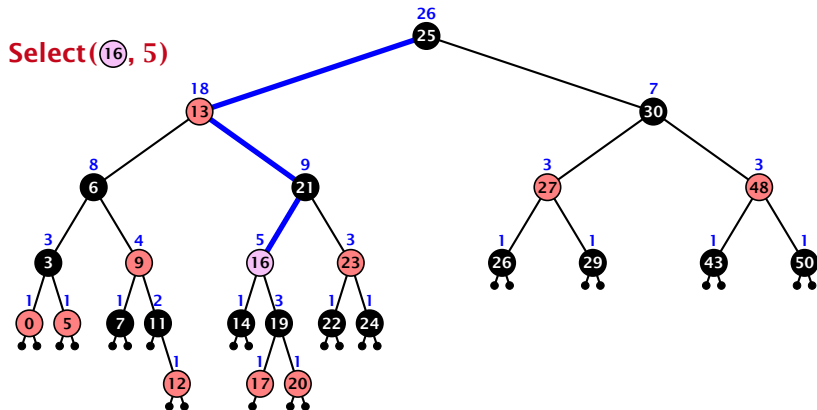
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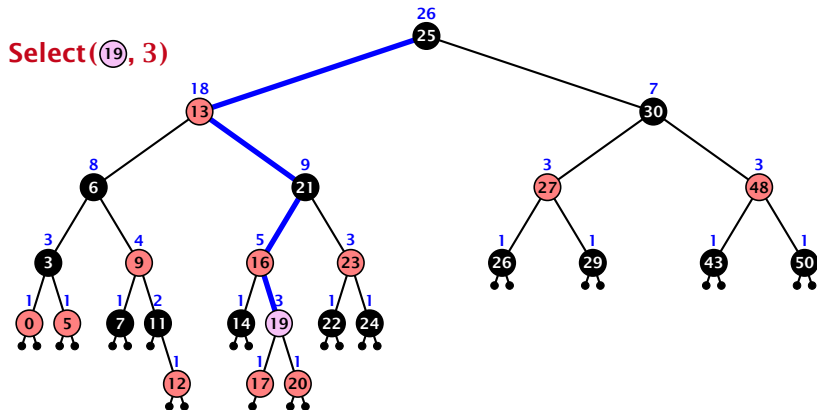


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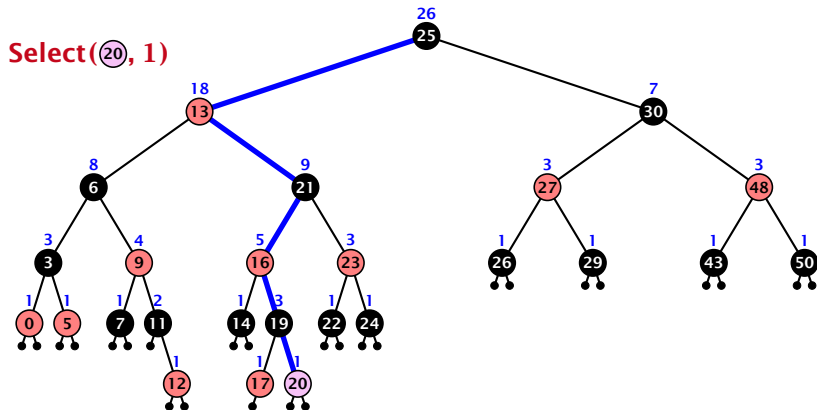
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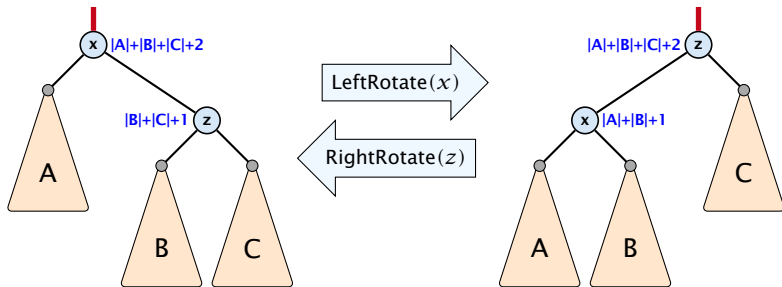
**Search( $k$ ):** Nothing to do.

**Insert( $x$ ):** When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

**Delete( $x$ ):** Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

# Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes  $x$  and  $z$  are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

## 7.5 Skip Lists

**Why do we not use a list for implementing the ADT Dynamic Set?**



## 7.5 Skip Lists

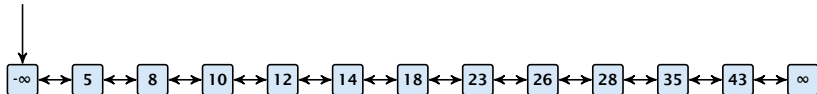
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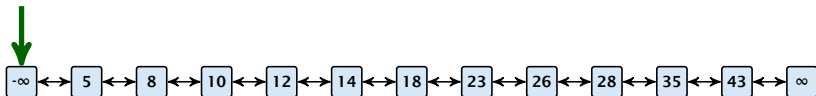
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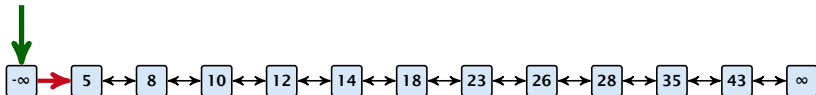
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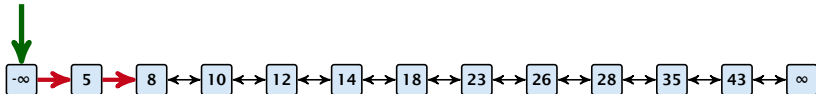
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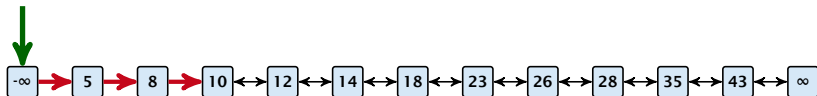
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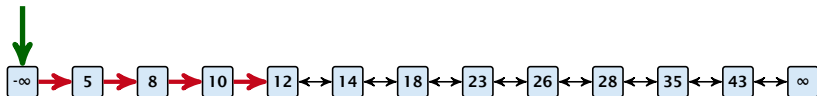
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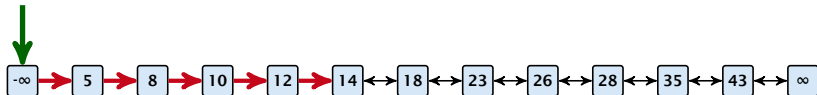
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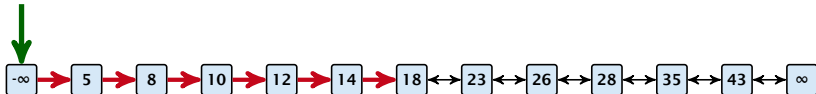




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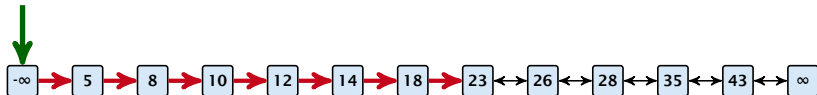
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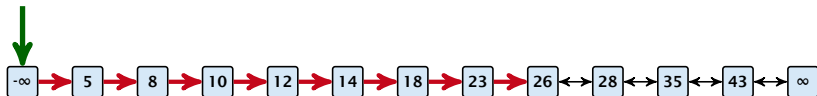
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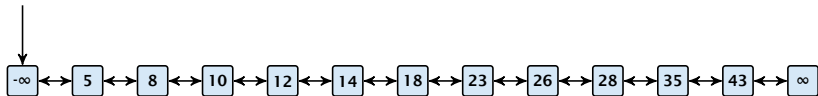
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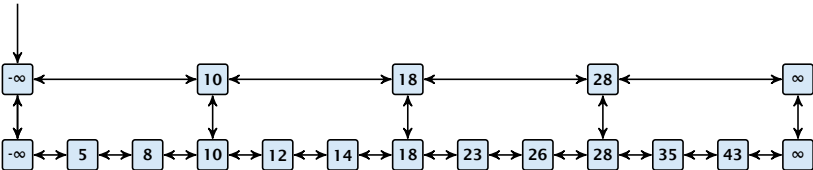
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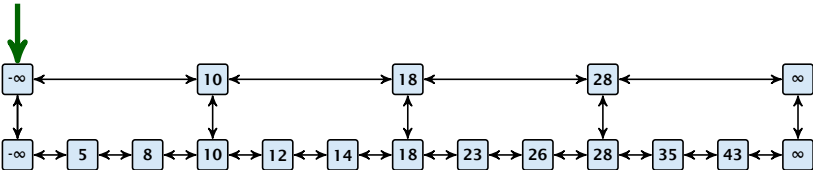
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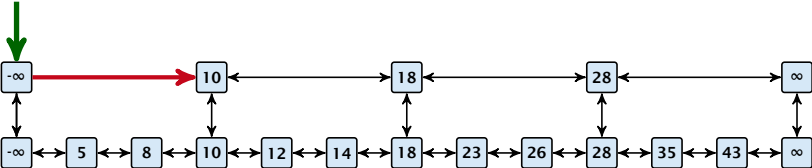




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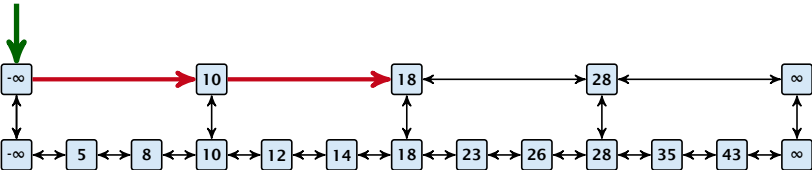
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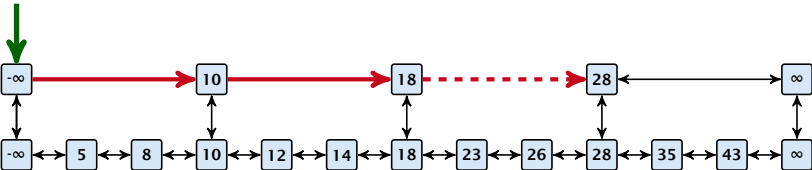
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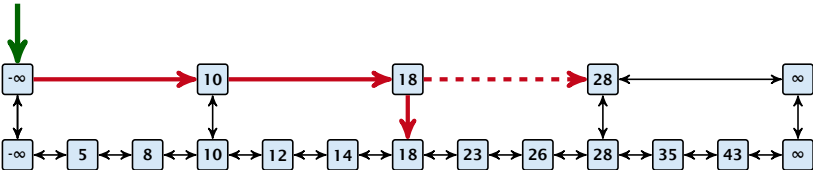
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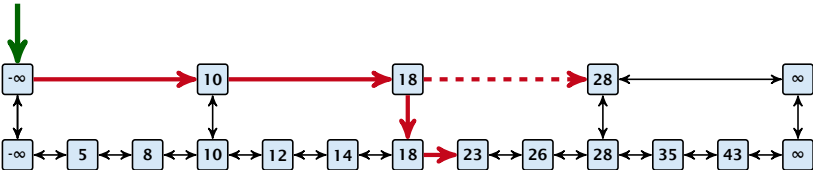
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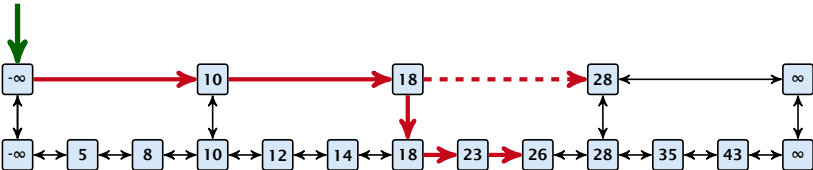
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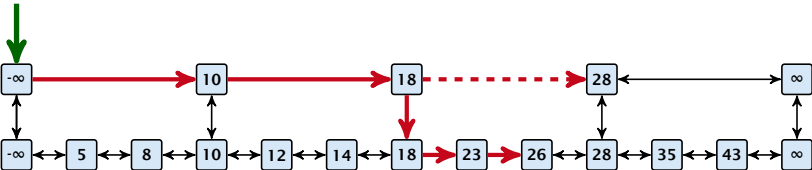
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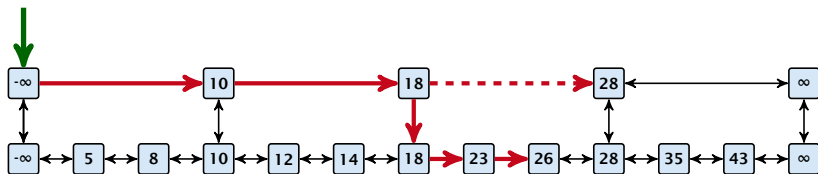


Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

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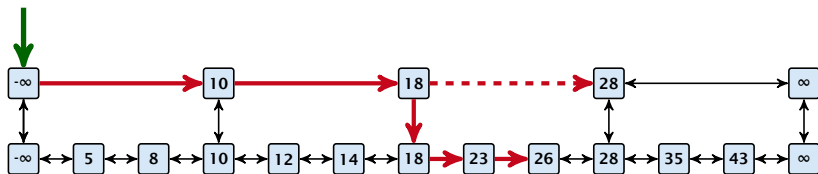
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Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

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- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.



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Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

## 7.5 Skip Lists

**How to do insert and delete?**



## 7.5 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

## 7.5 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

**Use randomization instead!**

## 7.5 Skip Lists

**Insert:**

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### Insert:

- ▶ A search operation gives you the insert position for element  $x$  in every list.

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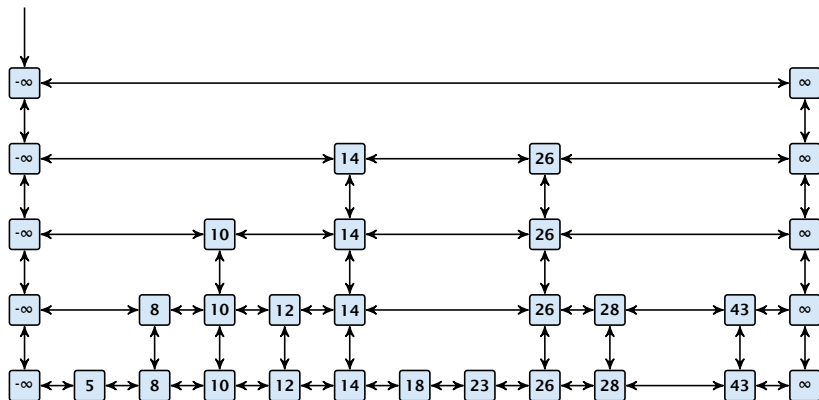
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**The time for both operations is dominated by the search time.**

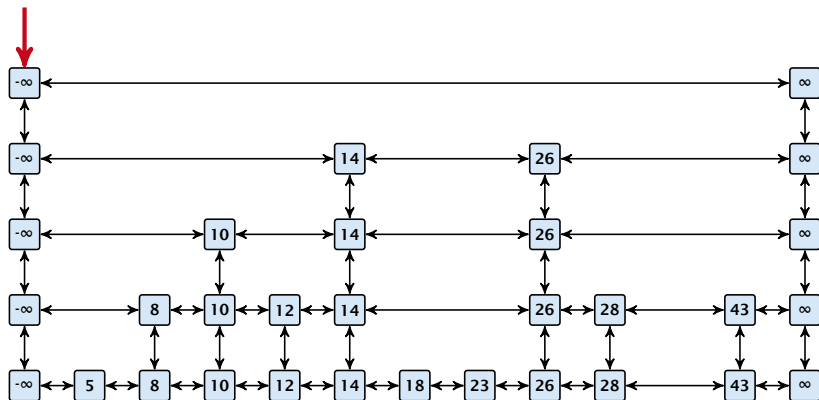
## 7.5 Skip Lists

Insert (35):



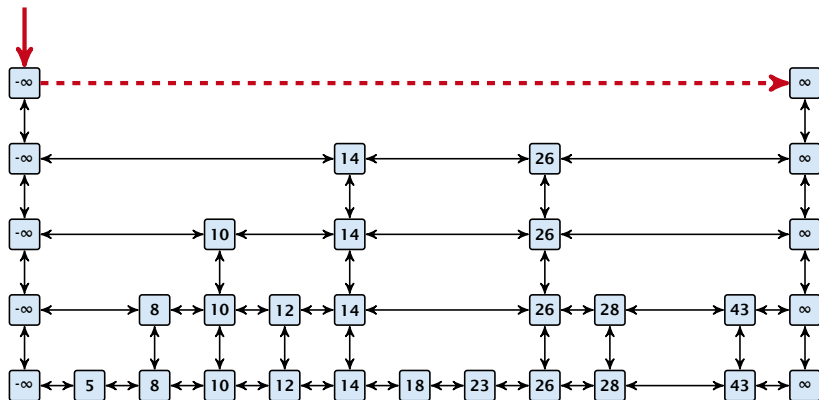
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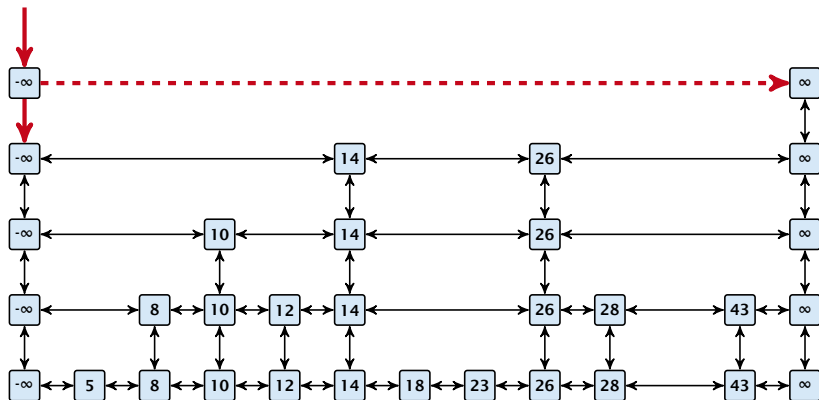
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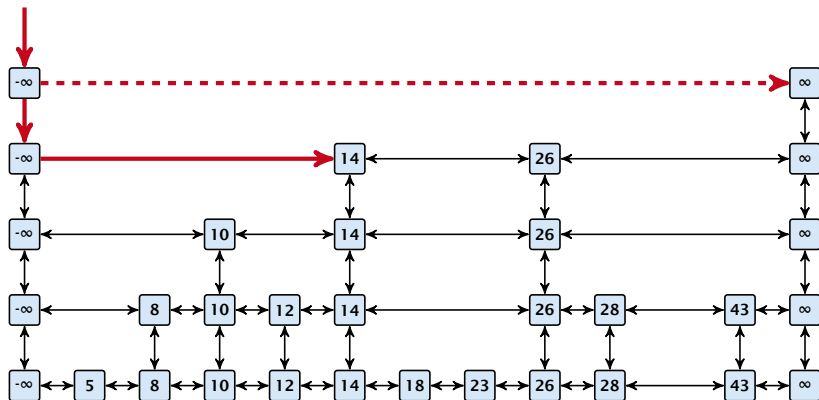
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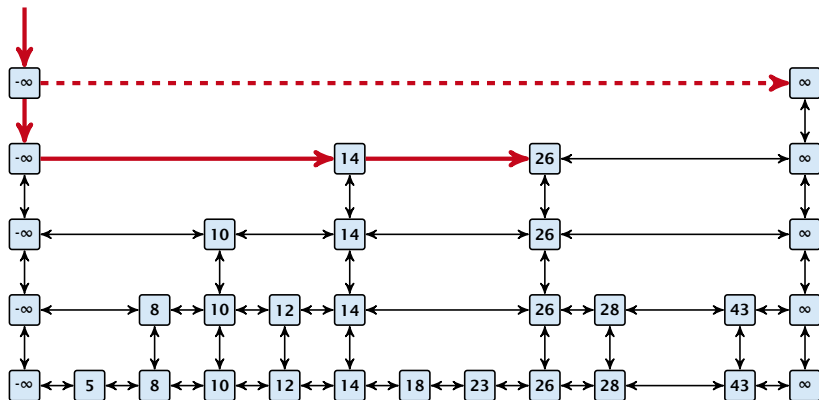
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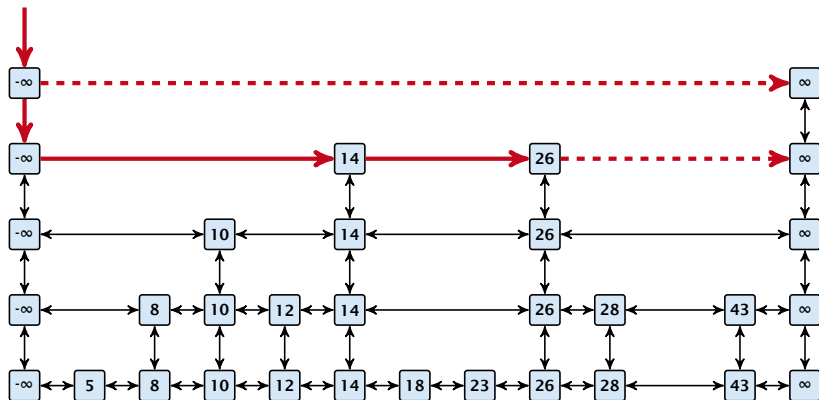
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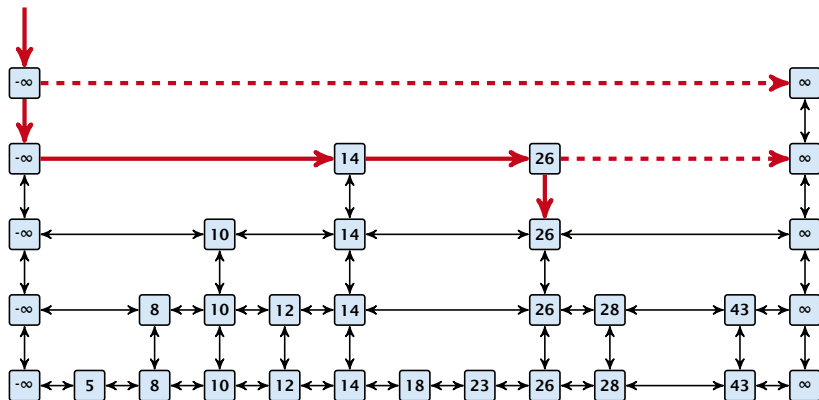
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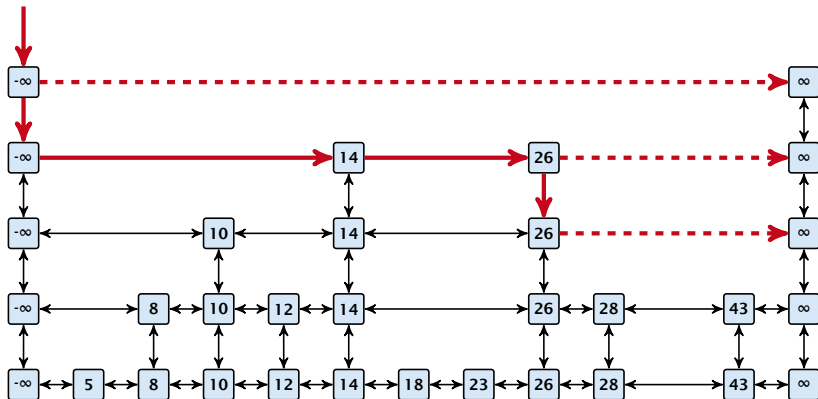
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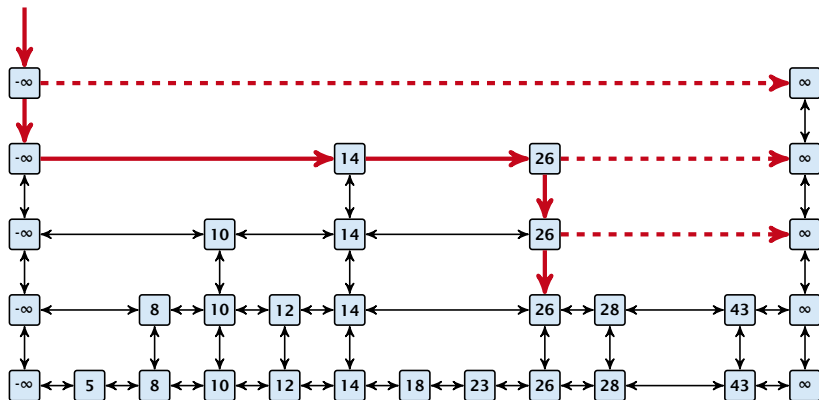
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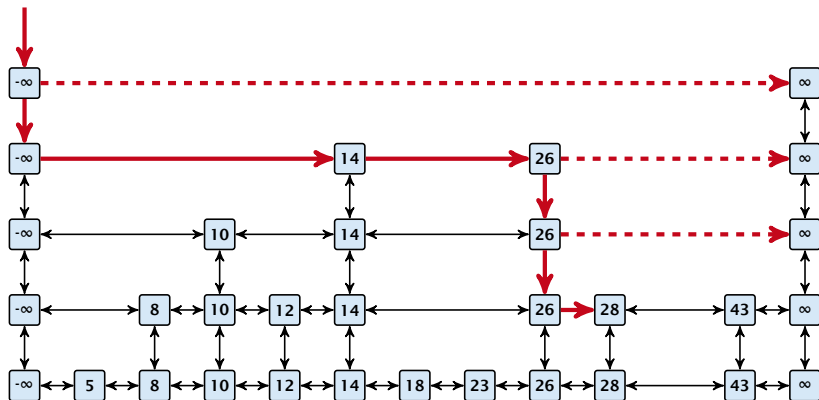
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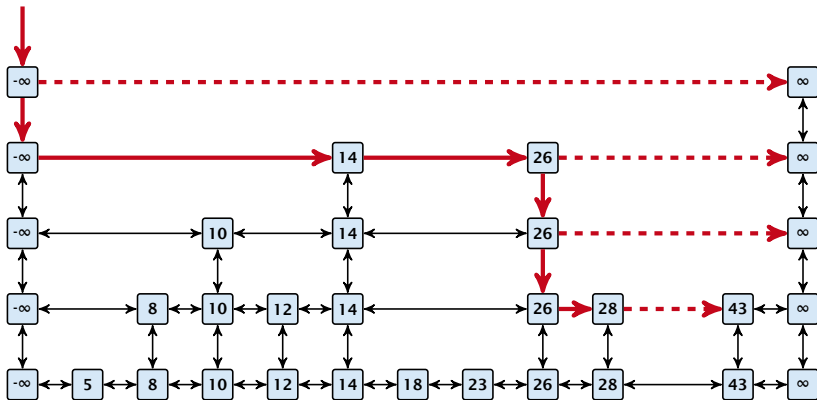
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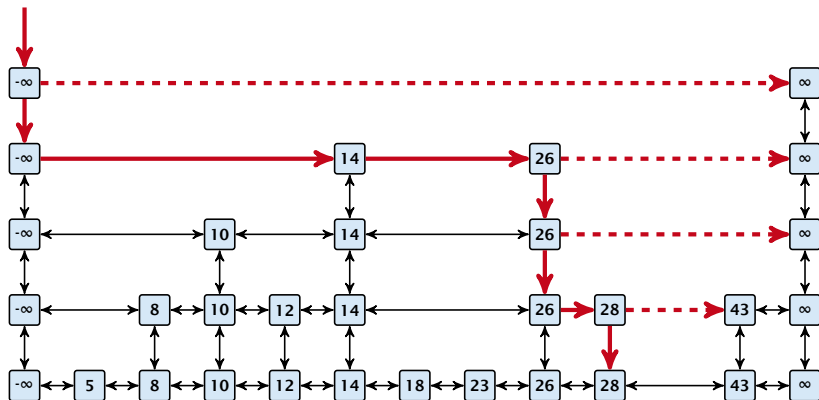
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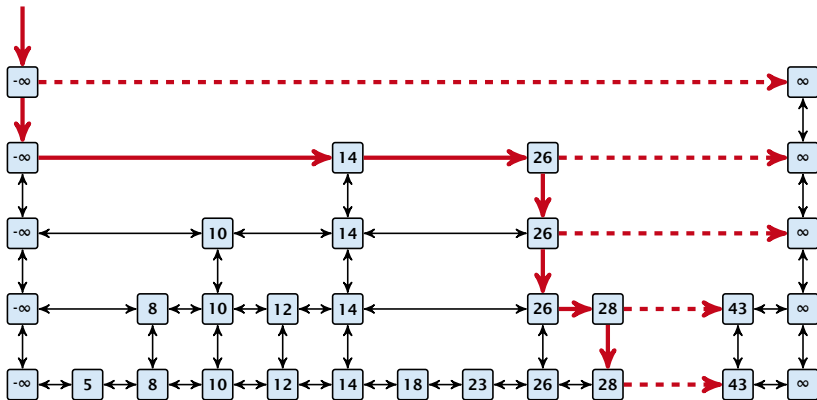
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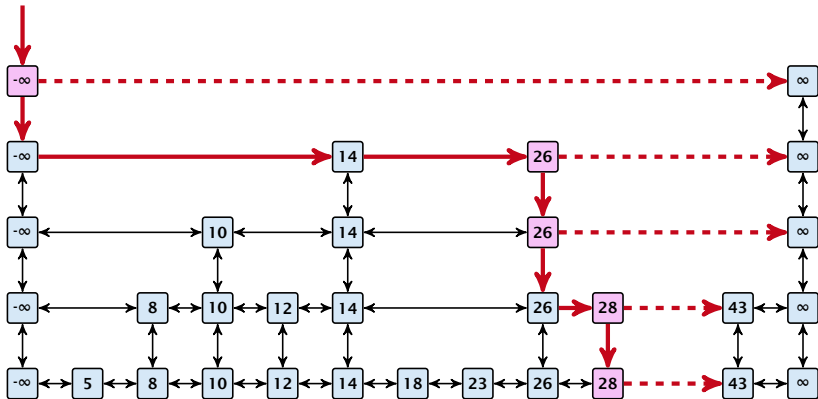
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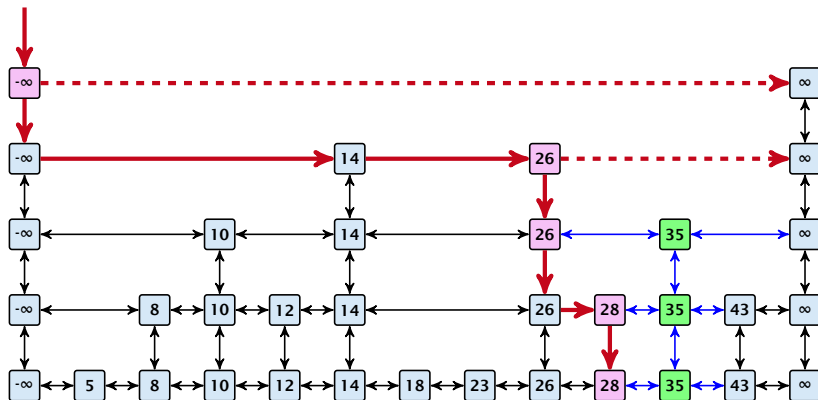
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# High Probability

## Definition 18 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

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Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .

# High Probability

Suppose there are **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

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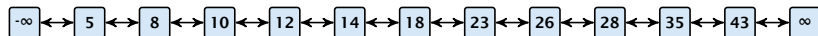
## 7.5 Skip Lists

### Lemma 19

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

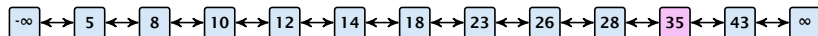
## 7.5 Skip Lists

Backward analysis:



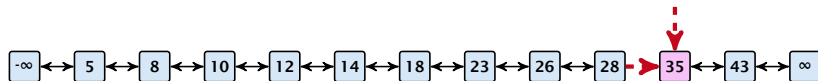
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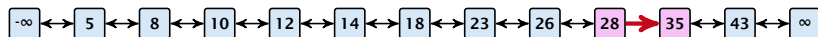
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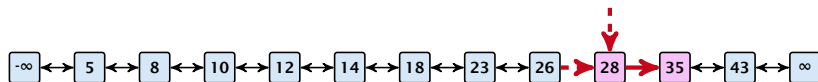
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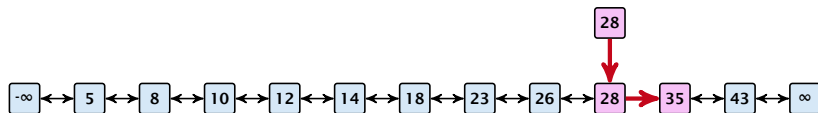
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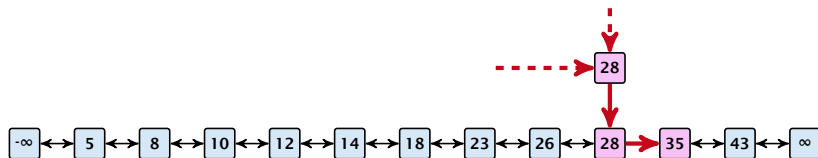
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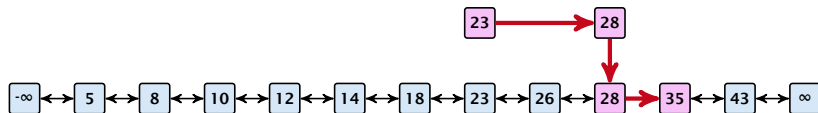
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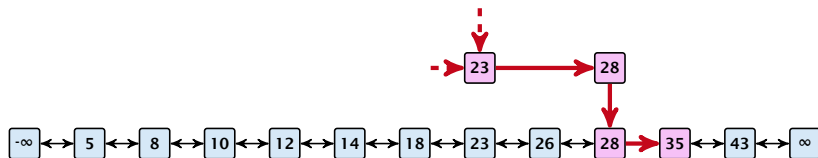
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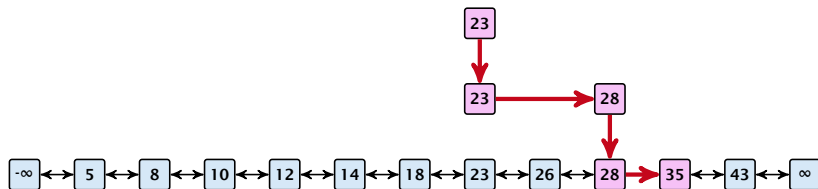
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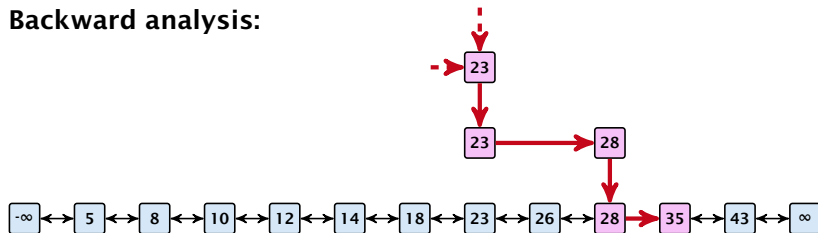
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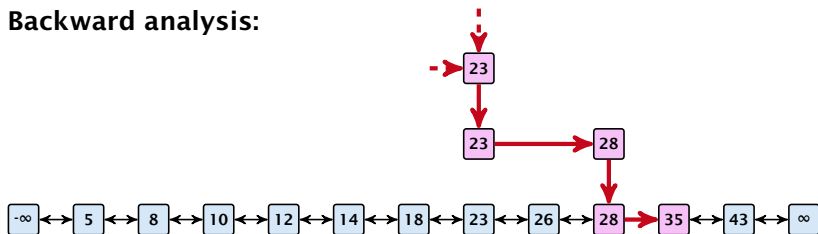
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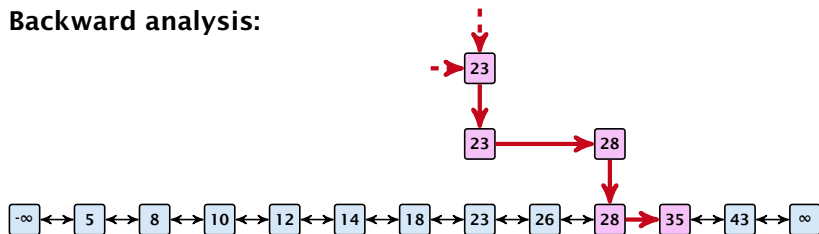
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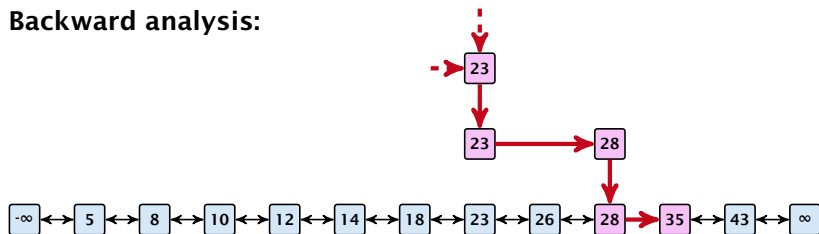
We show that w.h.p:

- ▶ A “long” search path must also go very high.



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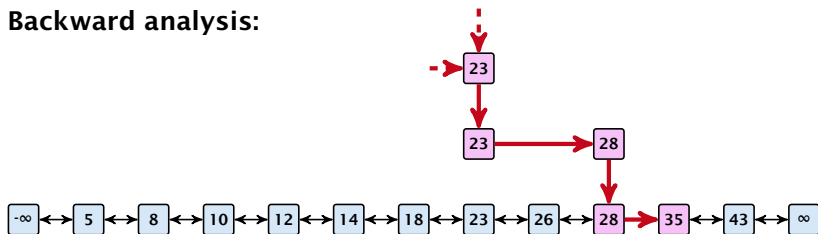
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From this it follows that w.h.p. there are no long paths.

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### Estimation for Binomial Coefficients

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In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.5 Skip Lists

$$\Pr[E_{z,k}]$$

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$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

## 7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

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for  $\alpha \geq 1$ .

## 7.5 Skip Lists

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Let  $A_{k+1}$  denote the event that the list  $L_{k+1}$  is non-empty. Then

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

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Hence,

$$\Pr[\text{search requires } z \text{ steps}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

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Let  $A_{k+1}$  denote the event that the list  $L_{k+1}$  is non-empty. Then

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most  $z$  steps, w. h. p.



## 7.6 van Emde Boas Trees

### Dynamic Set Data Structure $S$ :

- ▶  $S.insert(x)$
- ▶  $S.delete(x)$
- ▶  $S.search(x)$
- ▶  $S.min()$
- ▶  $S.max()$
- ▶  $S.succ(x)$
- ▶  $S.pred(x)$

## 7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

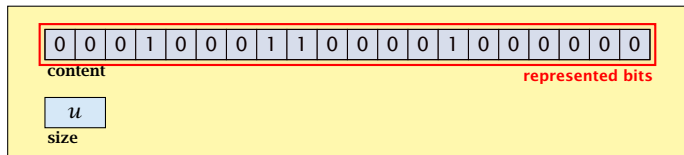
- ▶  **$S$ . insert( $x$ ):** Inserts  $x$  into  $S$ .
- ▶  **$S$ . delete( $x$ ):** Deletes  $x$  from  $S$ . Usually assumes that  $x \in S$ .
- ▶  **$S$ . member( $x$ ):** Returns 1 if  $x \in S$  and 0 otherwise.
- ▶  **$S$ . min():** Returns the value of the minimum element in  $S$ .
- ▶  **$S$ . max():** Returns the value of the maximum element in  $S$ .
- ▶  **$S$ . succ( $x$ ):** Returns successor of  $x$  in  $S$ . Returns **null** if  $x$  is maximum or larger than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .
- ▶  **$S$ . pred( $x$ ):** Returns the predecessor of  $x$  in  $S$ . Returns **null** if  $x$  is minimum or smaller than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .

## 7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from  $\{0, 1, \dots, u - 1\}$ , where  $u$  denotes the size of the universe.

# Implementation 1: Array



one array of  $u$  bits

Use an array that encodes the indicator function of the dynamic set.

# Implementation 1: Array

**Algorithm 1** `array.insert( $x$ )`

1: `content[ $x$ ] ← 1;`

**Algorithm 2** `array.delete( $x$ )`

1: `content[ $x$ ] ← 0;`

**Algorithm 3** `array.member( $x$ )`

1: **return** `content[ $x$ ];`

- ▶ Note that we assume that  $x$  is valid, i.e., it falls within the array boundaries.
- ▶ Obviously(?) the running time is constant.

## Implementation 1: Array

### Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

# Implementation 1: Array

## Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

## Algorithm 5 `array.min()`

```
1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
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# Implementation 1: Array

## Algorithm 4 `array.max()`

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1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
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## Algorithm 5 `array.min()`

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1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.



## Implementation 1: Array

### Algorithm 6 `array.succ(x)`

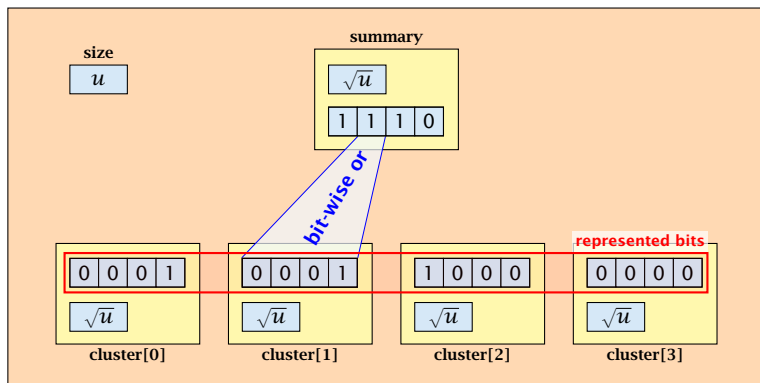
```
1: for ( $i = x + 1$ ;  $i < \text{size}$ ;  $i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

### Algorithm 7 `array.pred(x)`

```
1: for ( $i = x - 1$ ;  $i \geq 0$ ;  $i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.

## Implementation 2: Summary Array



- ▶  $\sqrt{u}$  cluster-arrays of  $\sqrt{u}$  bits.
- ▶ One summary-array of  $\sqrt{u}$  bits. The  $i$ -th bit in the summary array stores the bit-wise or of the bits in the  $i$ -th cluster.

# Implementation 2: Summary Array

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The bit for a key  $x$  is contained in cluster number  $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$ .

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Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\lfloor \frac{x}{\sqrt{u}} \rfloor$ .

Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

For simplicity we assume that  $u = 2^{2k}$  for some  $k \geq 1$ . Then we can compute the cluster-number for an entry  $x$  as  $\text{high}(x)$  (the upper half of the dual representation of  $x$ ) and the position of  $x$  within its cluster as  $\text{low}(x)$  (the lower half of the dual representation).

## Implementation 2: Summary Array

**Algorithm 8**  $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

### Algorithm 9 $\text{insert}(x)$

1:  $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x));$

2:  $\text{summary}.\text{insert}(\text{high}(x));$



## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

```
1: return cluster[high(x)].member(low(x));
```

### Algorithm 9 $\text{insert}(x)$

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

- ▶ The running times are constant, because the corresponding array-functions have constant running times.

## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

- 1: cluster[high( $x$ )].delete(low( $x$ ));
- 2: **if** cluster[high( $x$ )].min() = null **then**
- 3:     summary.delete(high( $x$ ));

## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

- ▶ The running time is dominated by the cost of a minimum computation on an array of size  $\sqrt{u}$ . Hence,  $\mathcal{O}(\sqrt{u})$ .

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

- 1:  $\text{maxcluster} \leftarrow \text{summary.max}();$
- 2: **if**  $\text{maxcluster} = \text{null}$  **return**  $\text{null};$
- 3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$
- 4: **return**  $\text{maxcluster} \circ \text{offs};$

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
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4: return  $\text{maxcluster} \circ \text{offs};$ 
```

### Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs};$ 
```

### Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

The operator  $\circ$  stands for the concatenation of two bitstrings.

This means if  $x = 0111_2$  and  $y = 0001_2$  then  $x \circ y = 01110001_2$ .

- ▶ Running time is roughly  $2\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 2: Summary Array

### Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

## Implementation 2: Summary Array

### Algorithm 13 $\text{succ}(x)$

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2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
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4: if  $\text{succcluster} \neq \text{null}$  then
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6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.



## Implementation 2: Summary Array

### Algorithm 14 $\text{pred}(x)$

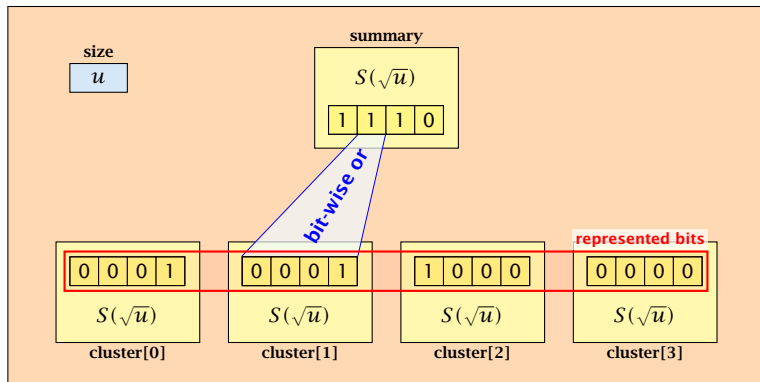
```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$ ;
4: if  $\text{predcluster} \neq \text{null}$  then
5:    $\text{offs} \leftarrow \text{cluster}[\text{predcluster}].\text{max}()$ ;
6:   return  $\text{predcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$  is a dynamic set data-structure representing  $u$  bits:



## Implementation 3: Recursion

We assume that  $u = 2^{2^k}$  for some  $k$ .

The data-structure  $S(2)$  is defined as an array of 2-bits (end of the recursion).

# Implementation 3: Recursion

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The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an  $S(4)$  will contain  $S(2)$ 's as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure  $S(4)$  is **not** a recursive call as it will call the function `array.min()`.

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This means that the non-recursive case is been dealt with while initializing the data-structure.

## Implementation 3: Recursion

**Algorithm 15**  $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$



## Implementation 3: Recursion

### Algorithm 16 insert( $x$ )

```
1: cluster[high( $x$ )].insert(low( $x$ ));  
2: summary.insert(high( $x$ ));
```

►  $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 17 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

►  $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 18 $\text{min}()$

```
1: mincluster  $\leftarrow$  summary.min();  
2: if mincluster = null return null;  
3: offs  $\leftarrow$  cluster[mincluster].min();  
4: return mincluster  $\circ$  offs;
```

►  $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 19 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

►  $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

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$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ .

## Implementation 3: Recursion

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\log \ell)$ , and hence  $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$ .



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The same holds for  $T_{\text{max}}(\mathbf{u})$  and  $T_{\text{min}}(\mathbf{u})$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

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$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell \end{aligned}$$

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Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

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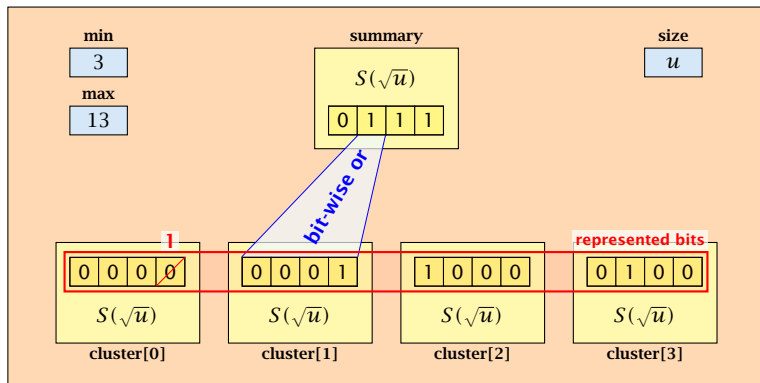
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$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X\left(\frac{\ell}{2}\right) + c\ell . \end{aligned}$$

Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

The same holds for  $T_{\text{pred}}(u)$  and  $T_{\text{succ}}(u)$ .

# Implementation 4: van Emde Boas Trees



- ▶ The bit referenced by **min** is **not** set within sub-datastructures.
- ▶ The bit referenced by **max** is set within sub-datastructures (if  $\text{max} \neq \text{min}$ ).

# Implementation 4: van Emde Boas Trees

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# Implementation 4: van Emde Boas Trees

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# Implementation 4: van Emde Boas Trees

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- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.

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## Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.
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- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting **min = max =  $x$** .
- ▶ We can delete from a data-structure that just contains one element in constant time by setting **min = max = null**.

## Implementation 4: van Emde Boas Trees

**Algorithm 20** max()

---

1: **return** max;

**Algorithm 21** min()

---

1: **return** min;

- ▶ Constant time.

## Implementation 4: van Emde Boas Trees

### Algorithm 22 `member(x)`

```
1: if  $x = \min$  then return 1; // TRUE  
2: return cluster[high(x)].member(low(x));
```

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Rightarrow T(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 23 $\text{succ}(x)$

```
1: if  $\text{min} \neq \text{null} \wedge x < \text{min}$  then return  $\text{min}$ ;  
2:  $\text{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\text{max}()$ ;  
3: if  $\text{maxincluster} \neq \text{null} \wedge \text{low}(x) < \text{maxincluster}$  then  
4:    $\text{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ ;  
5:   return  $\text{high}(x) \circ \text{offs}$ ;  
6: else  
7:    $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;  
8:   if  $\text{succcluster} = \text{null}$  then return  $\text{null}$ ;  
9:    $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;  
10:  return  $\text{succcluster} \circ \text{offs}$ ;
```

►  $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 35 insert( $x$ )

```
1: if min = null then
2:     min =  $x$ ; max =  $x$ ;
3: else
4:     if  $x < \text{min}$  then exchange  $x$  and min;
5:     if  $x > \text{max}$  then max =  $x$ ;
6:     if cluster[high( $x$ )].min = null; then
7:         summary.insert(high( $x$ ));
8:         cluster[high( $x$ )].insert(low( $x$ ));
9:     else
10:        cluster[high( $x$ )].insert(low( $x$ ));
```

►  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Rightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

Note that the recursive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$ .



## Implementation 4: van Emde Boas Trees

- ▶ **Assumes that  $x$  is contained in the structure.**

**Algorithm 36** delete( $x$ )

```
1: if min = max then  
2:     min = max = null;  
3: else  
4:     if  $x$  = min then  
5:         firstcluster  $\leftarrow$  summary.min();  
6:         offs  $\leftarrow$  cluster[firstcluster].min();  
7:          $x \leftarrow$  firstcluster  $\circ$  offs;  
8:         min  $\leftarrow$   $x$ ;  
9:     cluster[high( $x$ )].delete(low( $x$ ));  
                                     continued...
```

## Implementation 4: van Emde Boas Trees

- ▶ **Assumes that  $x$  is contained in the structure.**

**Algorithm 36** delete( $x$ )

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x = \text{min}$  then find new minimum
5:         firstcluster  $\leftarrow$  summary.min();
6:         offs  $\leftarrow$  cluster[firstcluster].min();
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
continued...
```

## Implementation 4: van Emde Boas Trees

- ▶ **Assumes that  $x$  is contained in the structure.**

**Algorithm 36** delete( $x$ )

```
1: if min = max then  
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3: else  
4:     if  $x$  = min then  
5:         firstcluster  $\leftarrow$  summary.min();  
6:         offs  $\leftarrow$  cluster[firstcluster].min();  
7:          $x \leftarrow$  firstcluster  $\circ$  offs;  
8:         min  $\leftarrow$   $x$ ;  
9:     cluster[high( $x$ )].delete(low( $x$ )); delete
```

continued...

## Implementation 4: van Emde Boas Trees

### Algorithm 36 delete( $x$ )

...continued

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:   if  $x$  = max then
13:       summax  $\leftarrow$  summary.max();
14:       if summax = null then max  $\leftarrow$  min;
15:       else
16:           offs  $\leftarrow$  cluster[summax].max();
17:           max  $\leftarrow$  summax  $\circ$  offs
18:   else
19:       if  $x$  = max then
20:           offs  $\leftarrow$  cluster[high( $x$ )].max();
21:           max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```

## Implementation 4: van Emde Boas Trees

### Algorithm 36 delete( $x$ )

...continued

fix maximum

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
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13:           summax  $\leftarrow$  summary.max();
14:           if summax = null then max  $\leftarrow$  min;
15:           else
16:               offs  $\leftarrow$  cluster[summax].max();
17:               max  $\leftarrow$  summax  $\circ$  offs
18:       else
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20:               offs  $\leftarrow$  cluster[high( $x$ )].max();
21:               max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```

## Implementation 4: van Emde Boas Trees

Note that only one of the possible recursive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where  $x$  was deleted is now empty. But this means that the call in Line 9 deleted the last element in  $\text{cluster}[\text{high}(x)]$ . Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c .$$

This gives  $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$ .

## 7.6 van Emde Boas Trees

### Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is  $S(u) = \mathcal{O}(u)$ . Exercise.

- ▶ Let the “real” recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; S(4) = c_2$$

- ▶ Replacing  $S(k)$  by  $R(k) := S(k)/c_2$  gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; R(4) = 1$$

where  $c = c_1/c_2 < 1$ .

- ▶ Now, we show  $R(k^2) \leq k^2 - 2$  for  $k^2 \geq 4$ .
  - ▶ Obviously, this holds for  $k^2 = 4$ .
  - ▶ For  $k^2 > 4$  we have

$$\begin{aligned} R(k^2) &= (1 + k)R(k) + ck \\ &\leq (1 + k)(k - 2) + k \leq k^2 - 2 \end{aligned}$$

- ▶ This shows that  $R(k)$  and, hence,  $S(k)$  grows linearly.



## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
- ▶  **$S.search(k)$** : Return a pointer to an element  $e$  with  $key[e] = k$  in  $S$  if it exists; otherwise return **null**.

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So far we have implemented the search for a key by carefully choosing split-elements.

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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.

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## 7.7 Hashing

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- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
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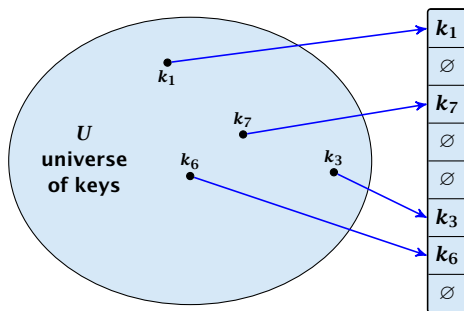
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### The hash-function $h$ should fulfill:

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- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

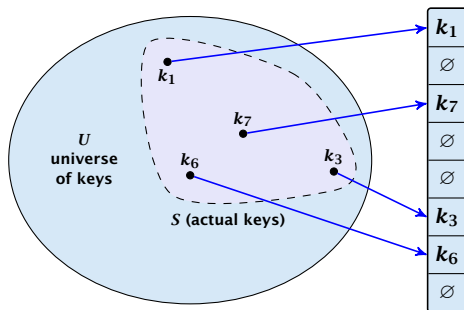
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

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## Lemma 20

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

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## Uniform hashing:

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

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Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

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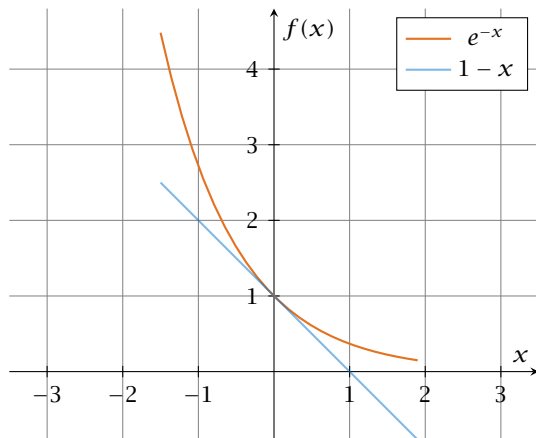
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Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
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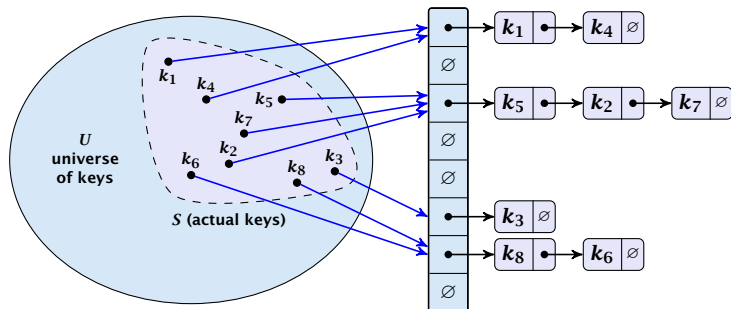
- ▶ **open addressing**, aka. closed hashing
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There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.





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We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$



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Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.



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Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

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**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes:  $h(k, i) = h(k) + ci \bmod n$ ).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

# Linear Probing

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## Lemma 21

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

# Quadratic Probing

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## Lemma 22

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

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## Lemma 23

Let  $D$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

$$D^- \approx \frac{1}{1 - \alpha}$$

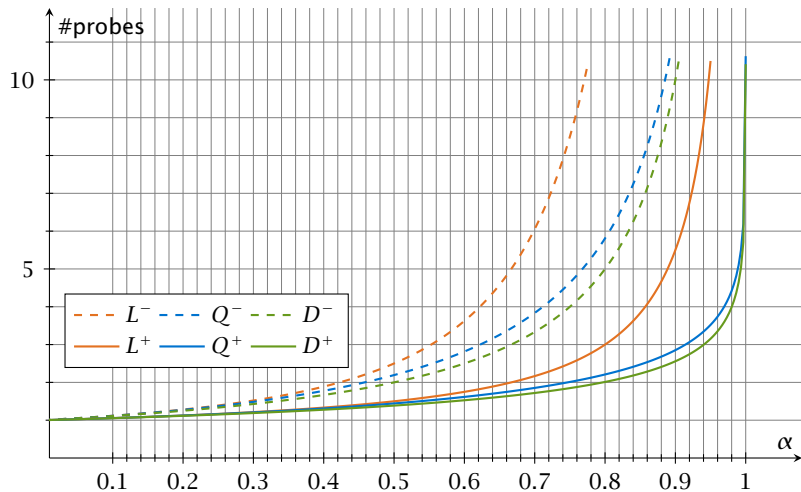


# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

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# Analysis of Idealized Open Address Hashing

$E[X]$

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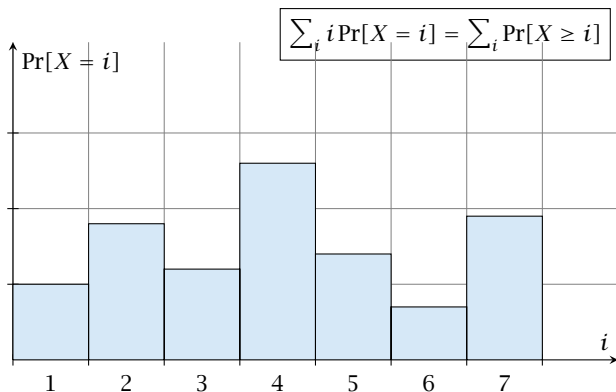


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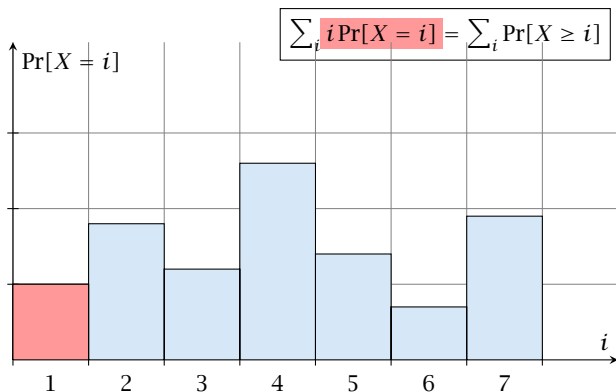
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing



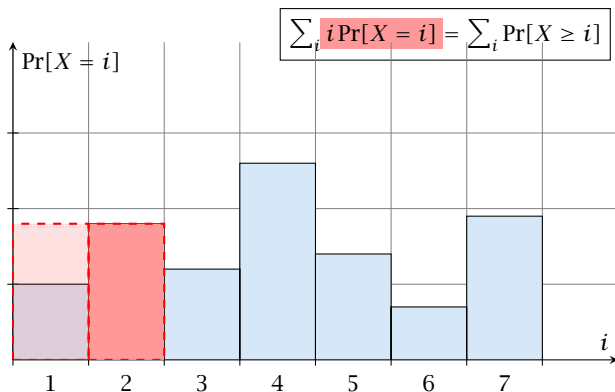
# Analysis of Idealized Open Address Hashing

$i = 1$



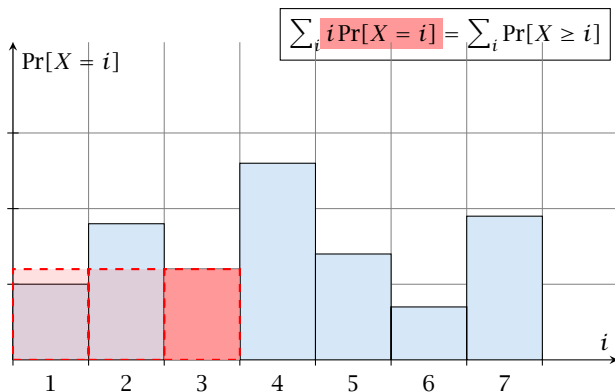
# Analysis of Idealized Open Address Hashing

$i = 2$



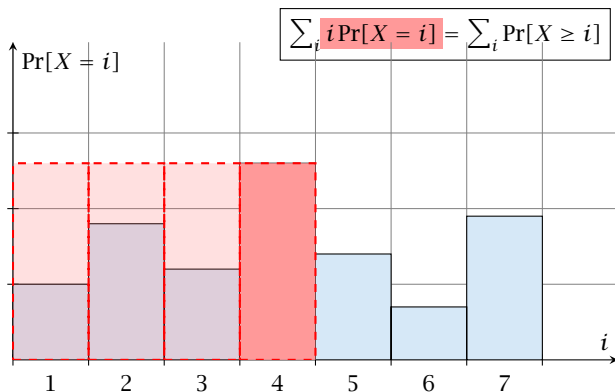
# Analysis of Idealized Open Address Hashing

$i = 3$



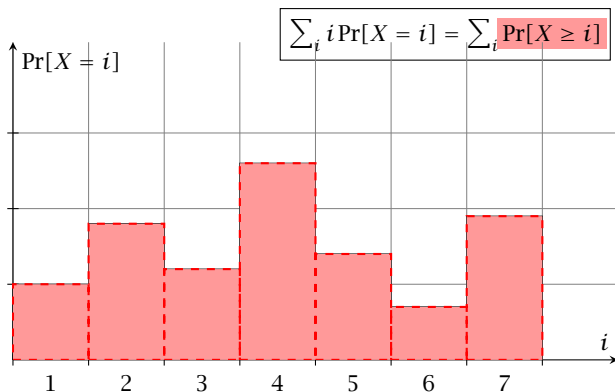
# Analysis of Idealized Open Address Hashing

$i = 4$



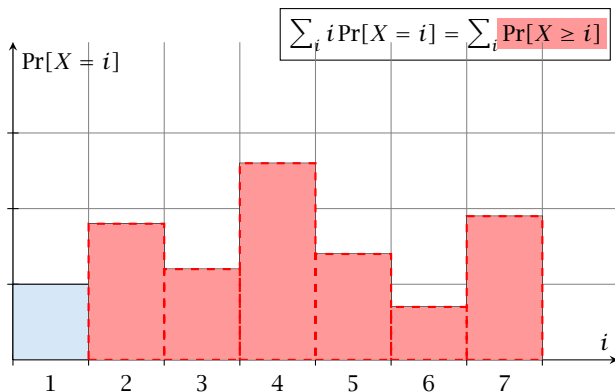
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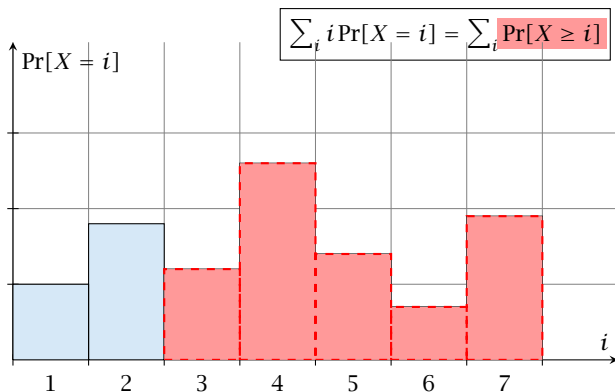
$i = 2$





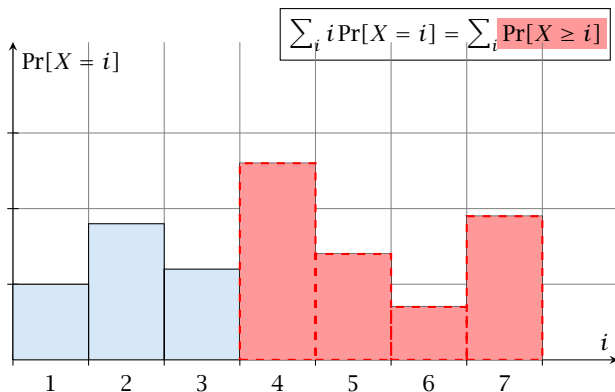
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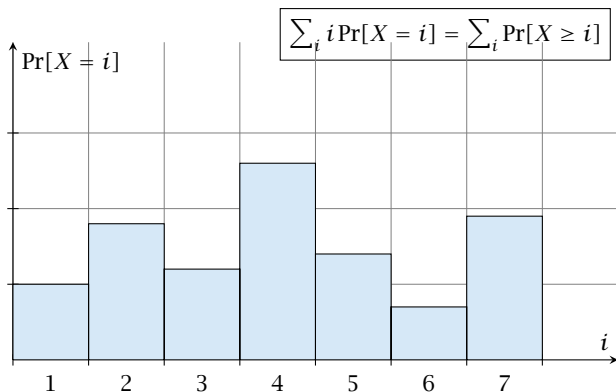


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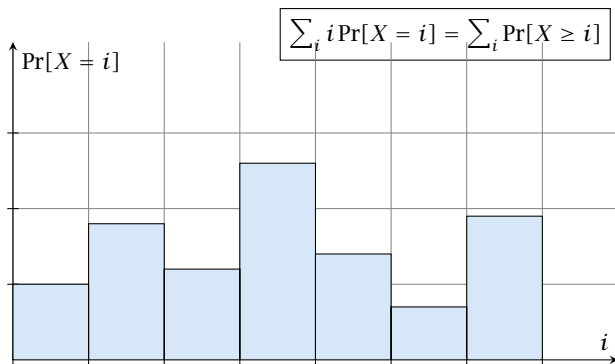
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# Analysis of Idealized Open Address Hashing



# Analysis of Idealized Open Address Hashing



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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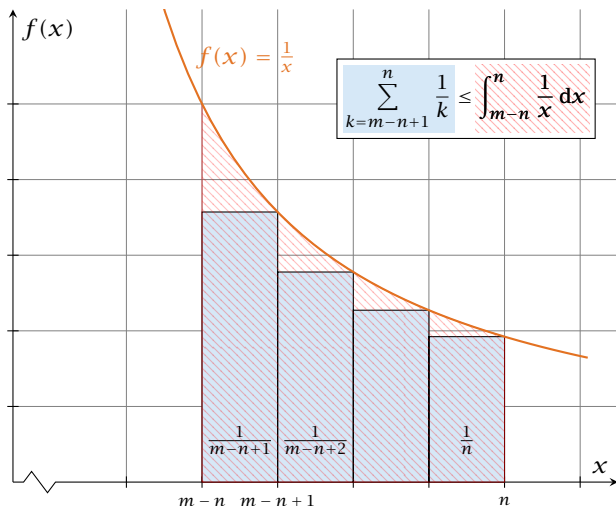
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## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

# Deletions in Hashtables

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- ▶ For open addressing this is difficult.



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- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

# Deletions for Linear Probing

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.



## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:    $\text{insert}(y)$ 
```

$p$  is the index into the table-cell that contains the object to be deleted.

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7:   insert( $y$ )
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Pointers into the hash-table become invalid.

# Universal Hashing

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Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

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However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

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Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 24

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .



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where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 25

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ ,  
i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

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This requirement clearly implies a universal hash-function.

## Definition 26

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

## Definition 27

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

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Let  $U := \{0, \dots, p - 1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p - 1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

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Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$



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$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

## Lemma 28

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

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Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

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If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

►  $ax + b \not\equiv ay + b \pmod{p}$

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where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

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$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

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This happens with probability at most  $\frac{1}{n}$ .



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It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 29

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

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For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

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The polynomial is defined by  $d+1$  distinct points.



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Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

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Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.



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Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

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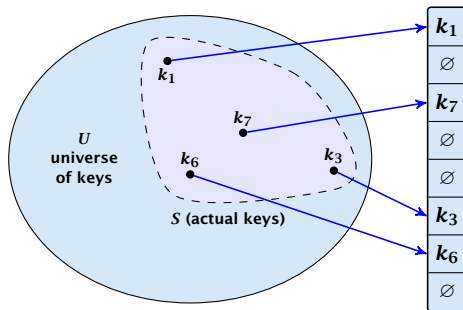
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This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.





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Can we get an upper bound on the **probability of having collisions**?

The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

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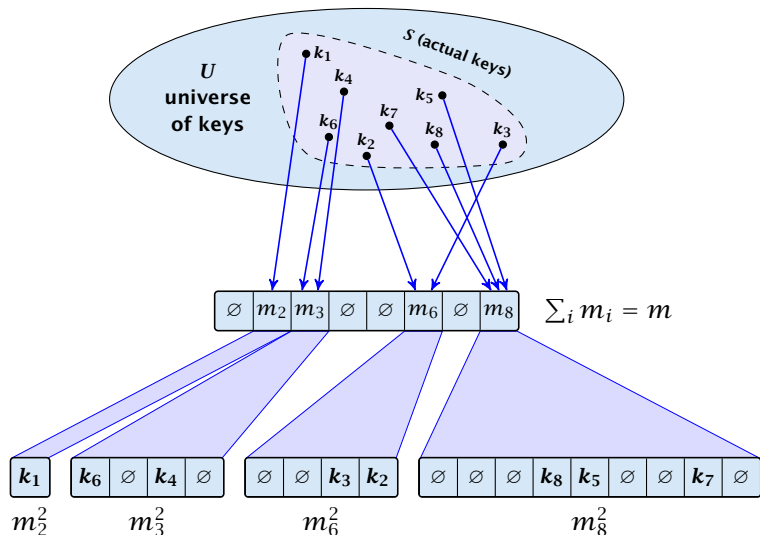
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Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

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The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ .  
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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

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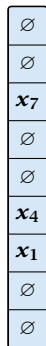
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- ▶ An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .
- ▶ A search clearly takes constant time if the above constraint is met.



# Cuckoo Hashing

Insert:



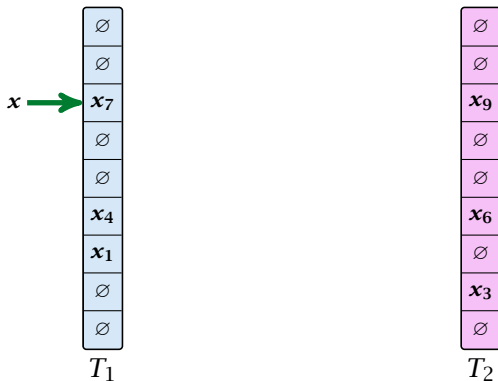
$T_1$



$T_2$

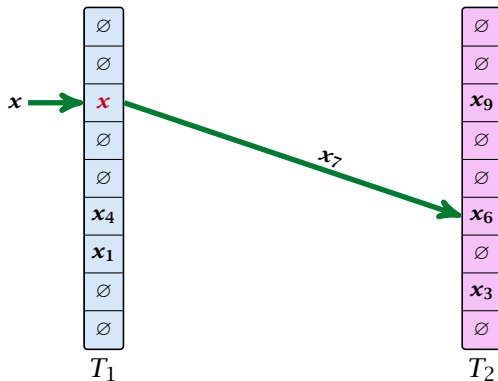
# Cuckoo Hashing

Insert:



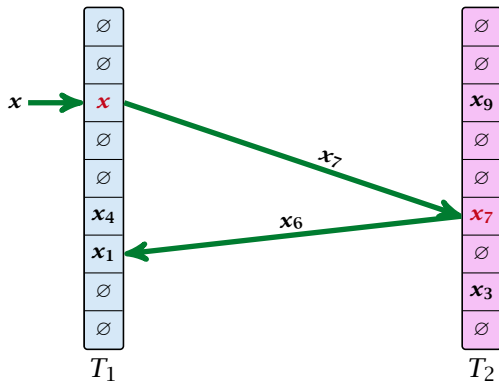
# Cuckoo Hashing

Insert:



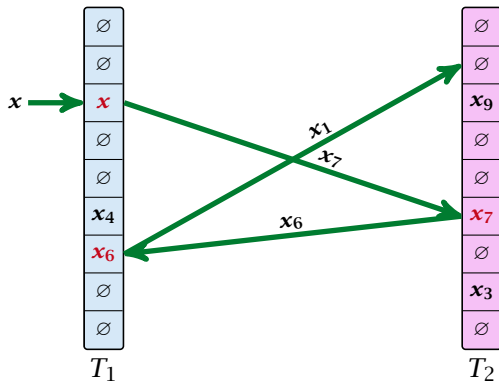
# Cuckoo Hashing

Insert:



# Cuckoo Hashing

Insert:



## Algorithm 38 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

# Cuckoo Hashing

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- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.



# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

# Cuckoo Hashing

**What is the expected time for an insert-operation?**

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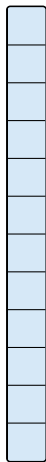
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

## What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert

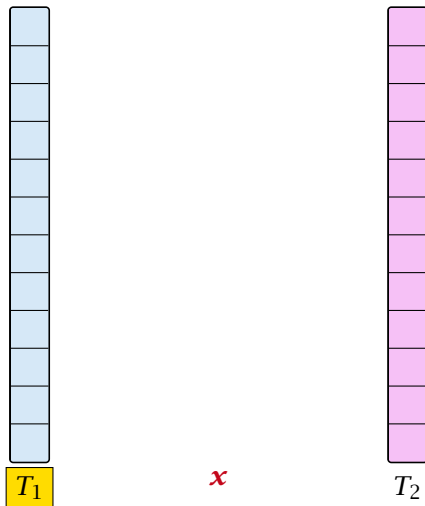


$T_1$

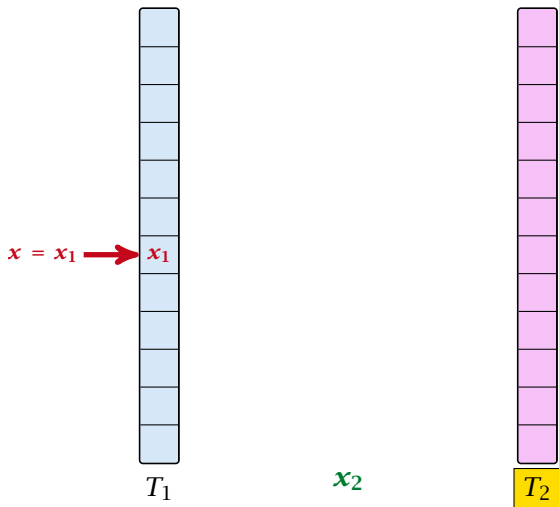


$T_2$

# Cuckoo Hashing: Insert

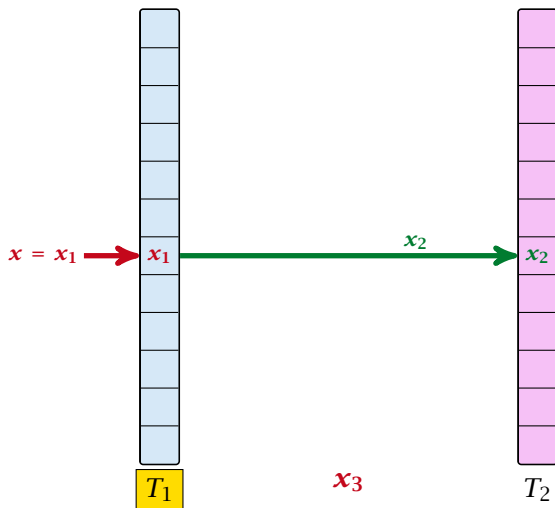


# Cuckoo Hashing: Insert

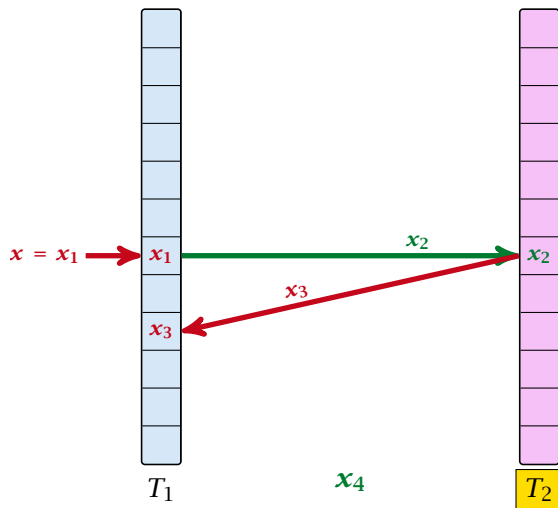




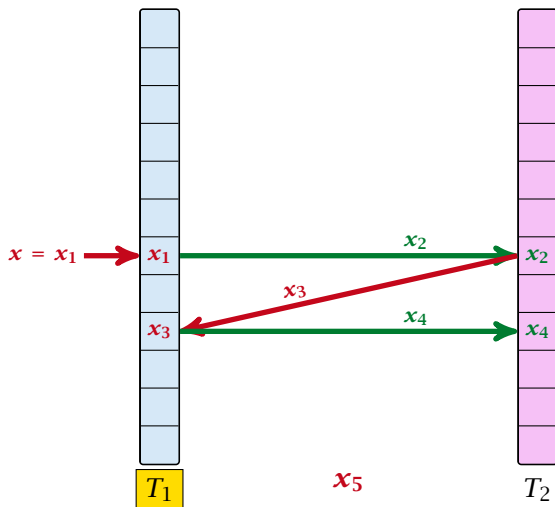
# Cuckoo Hashing: Insert



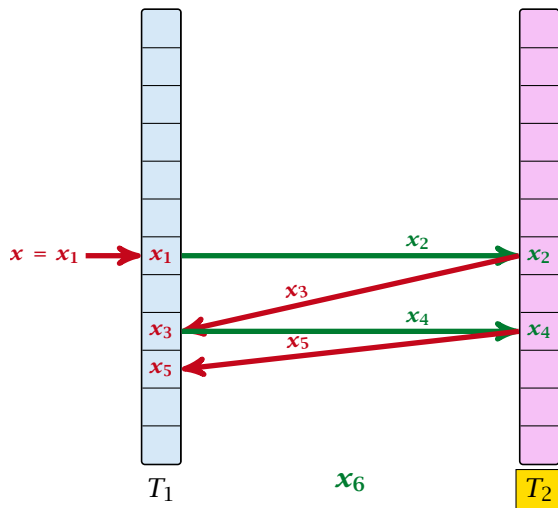
# Cuckoo Hashing: Insert



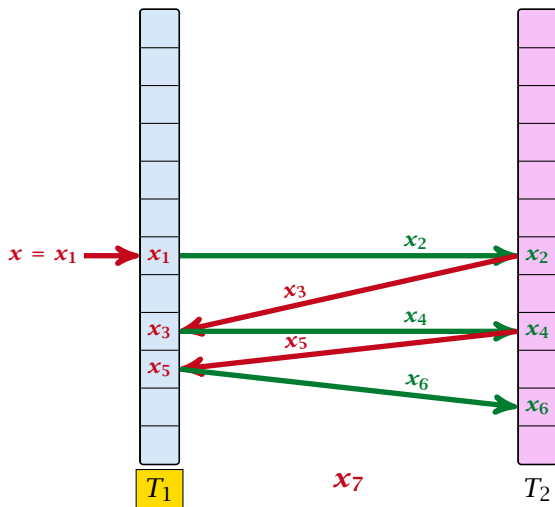
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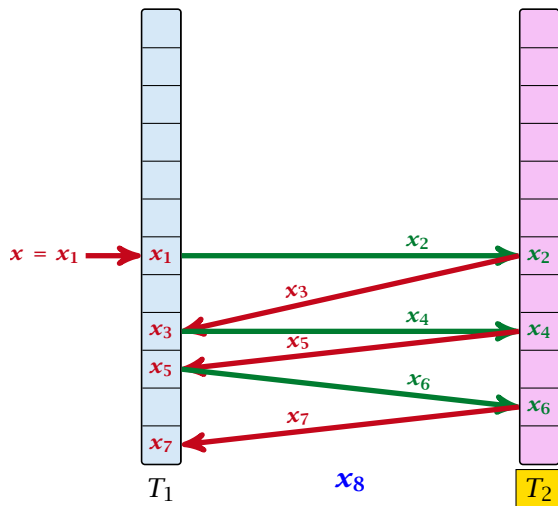
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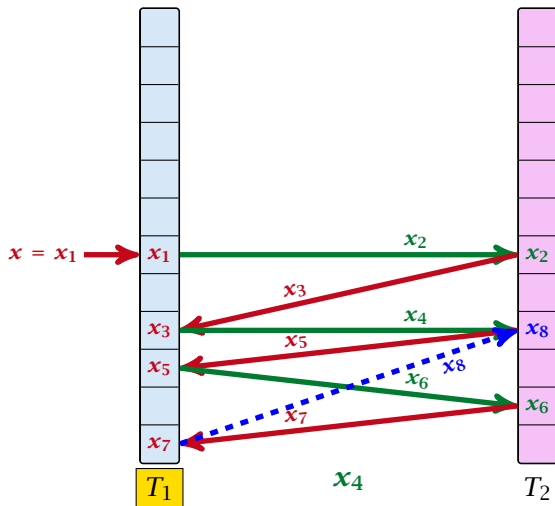
# Cuckoo Hashing: Insert



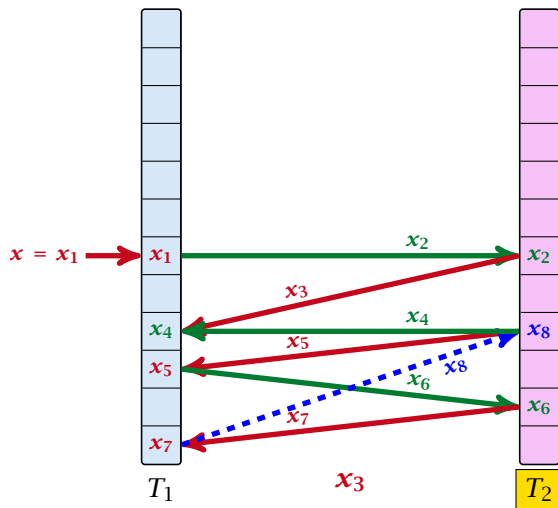
# Cuckoo Hashing: Insert



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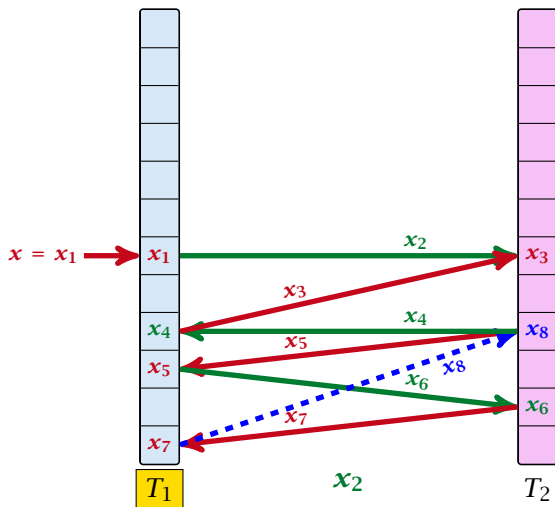


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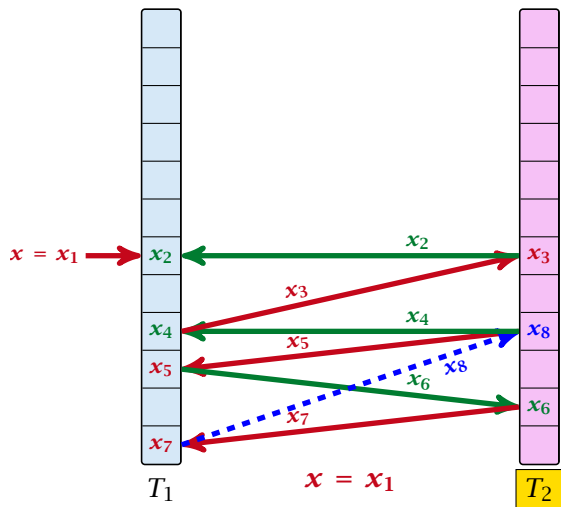




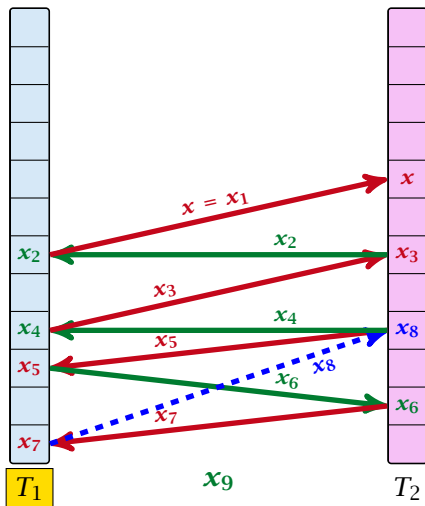
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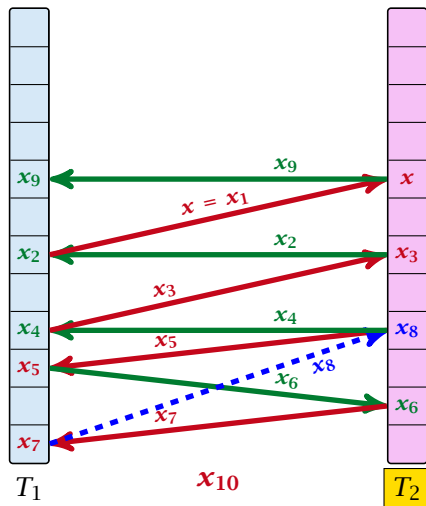
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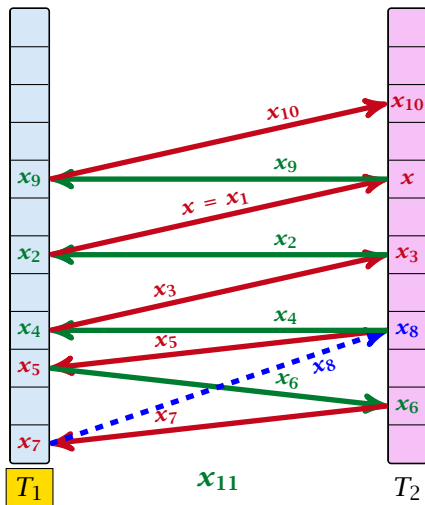
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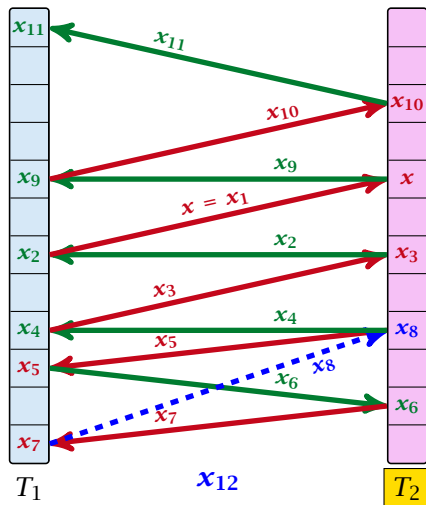
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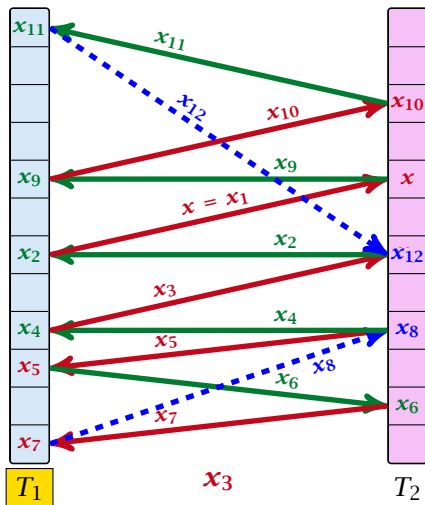
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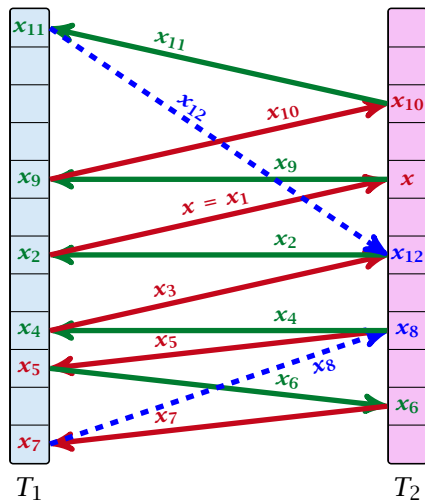
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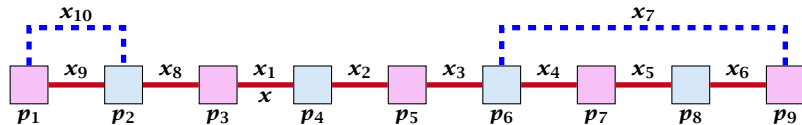


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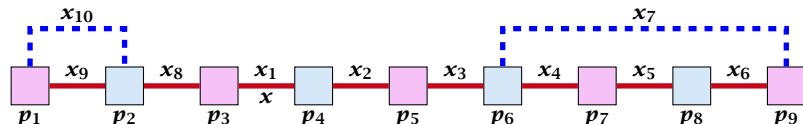


# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

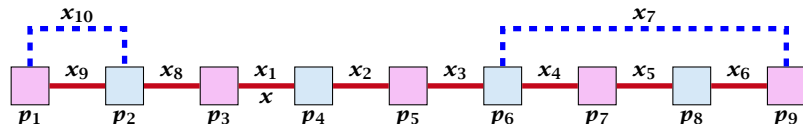
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A cycle-structure of size  $s$  is defined by

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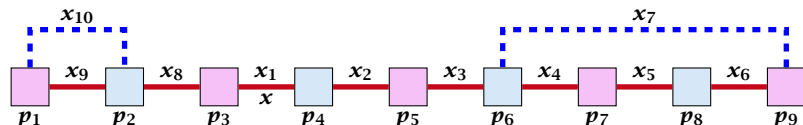
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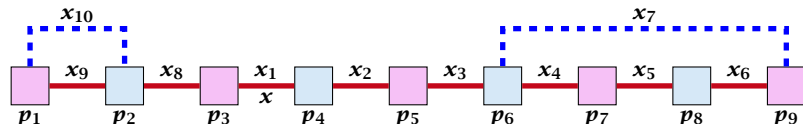
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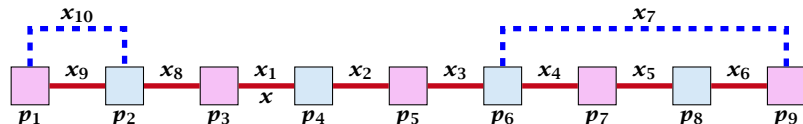
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- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .



# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

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This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

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This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

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These events are independent.

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The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

What is the probability that **there exists** an active cycle structure of size  $s$ ?

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$



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- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

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The probability that there exists an active cycle-structure is therefore at most

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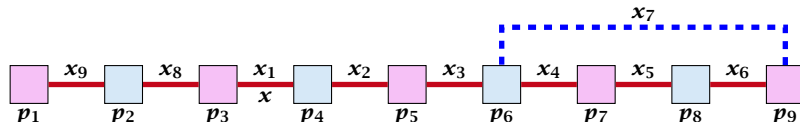
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

# Cuckoo Hashing

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## Lemma 30

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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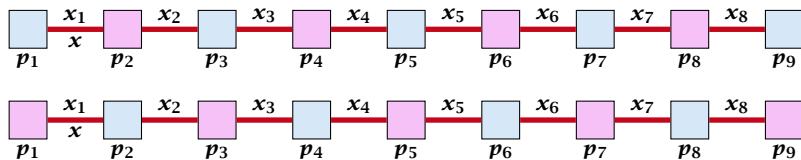
As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

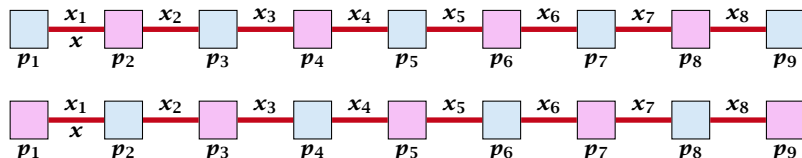


# Cuckoo Hashing



A path-structure of size  $s$  is defined by

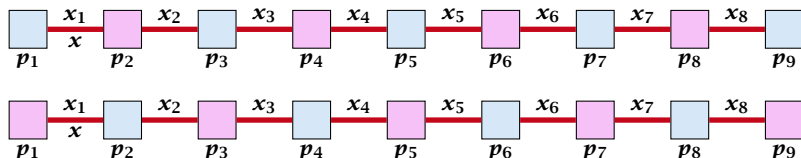
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A path-structure of size  $s$  is defined by

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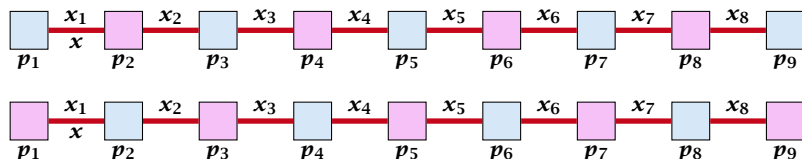
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A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

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This gives  $\text{maxsteps} = \Theta(\log m)$ .

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).



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Therefore the expected cost for re-hashes is  $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$ .

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Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (**table-expand**).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (**table-shrink**).



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- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (table-expand).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (table-shrink).
- ▶ Note that right after a change in table-size we have  $m = \alpha n/2$ . In order for a table-expand to occur at least  $\alpha n/2$  insertions are required. Similar, for a table-shrink at least  $\alpha n/4$  deletions must occur.

# Cuckoo Hashing

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- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

# Cuckoo Hashing

## Lemma 31

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .