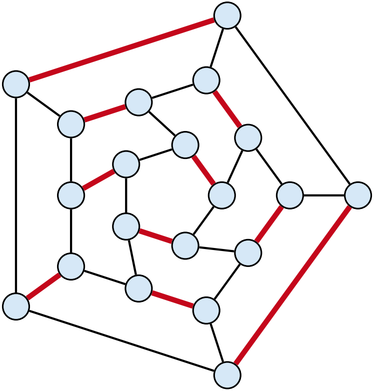
Part V

Matchings

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



16 Bipartite Matching via Flows

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

17 Augmenting Paths for Matchings

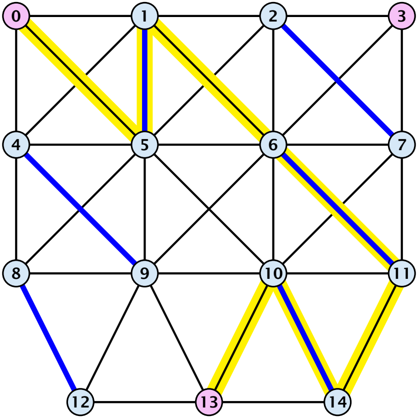
Definitions.

- Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.
- For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.
- An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

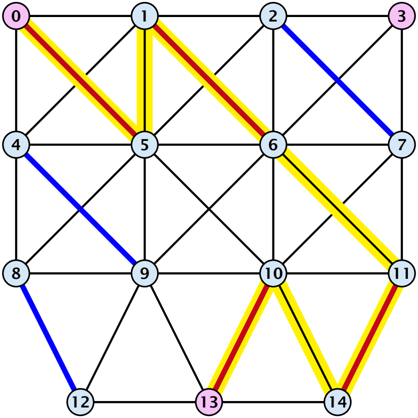
Theorem 89

A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

Augmenting Paths in Action



Augmenting Paths in Action



17 Augmenting Paths for Matchings

Proof.

- ⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.
- ⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

  Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

  As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

17 Augmenting Paths for Matchings

Algorithmic idea:
As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

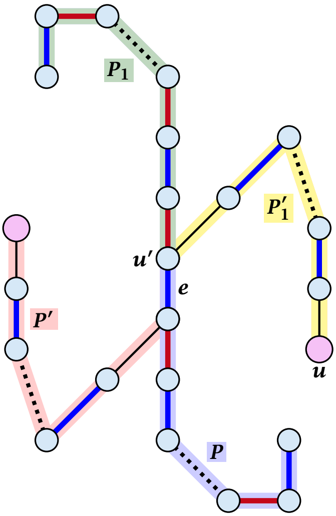
Theorem 90
Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

The above theorem allows for an easier implementation of an augmenting path algorithm. Once we checked for augmenting paths starting from u we don't have to check for such paths in future rounds.

17 Augmenting Paths for Matchings

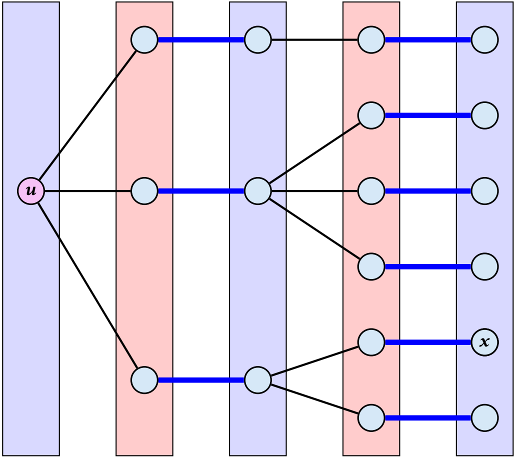
Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
- u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.
- $P_1 \circ P_1'$ is augmenting path in M (↯).



How to find an augmenting path?

Construct an alternating tree.



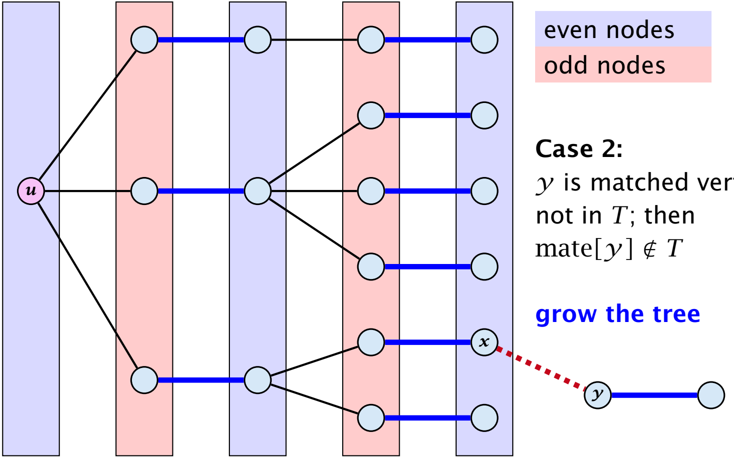
even nodes
odd nodes

Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.

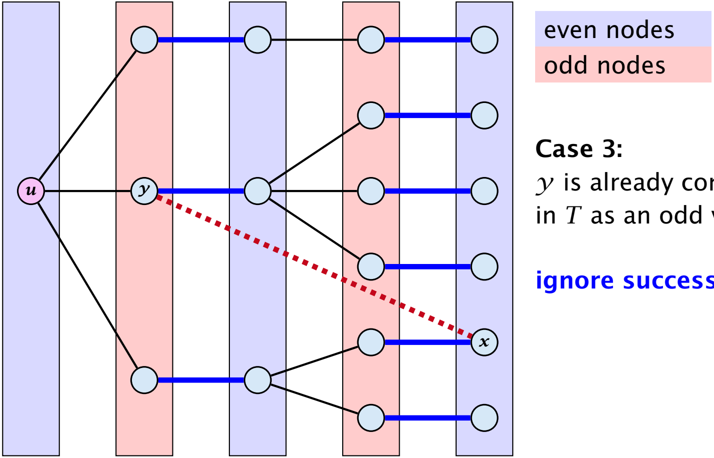


Case 2:
y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.

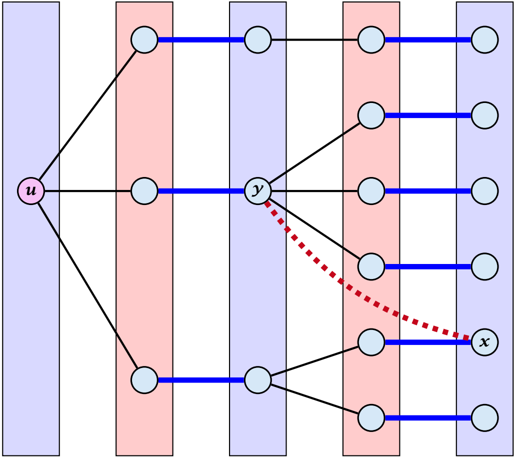


Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

Case 4:

y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

The lecture slides contain a step by step explanation.

Algorithm 49 BiMatch(G, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

graph $G = (S \cup S', E)$

$$S = \{1, \ldots, n\}$$

$$S' = \{1', \ldots, n'\}$$

18 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

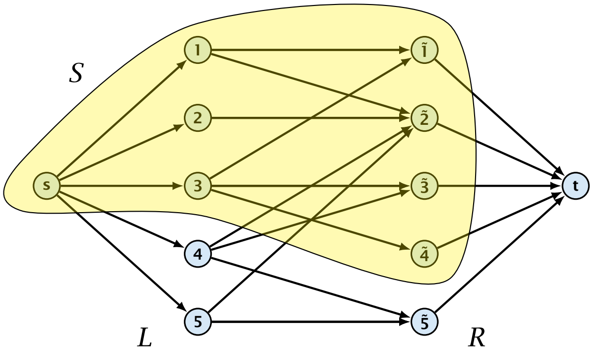
Simplifying Assumptions (wlog [why?]):

- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- can assume goal is to construct maximum weight perfect matching

Weighted Bipartite Matching

Theorem 91 (Halls Theorem)
A bipartite graph $G = (L \cup R, E)$ *has a perfect matching if and only if for all sets* $S \subseteq L$, $|\Gamma(S)| \geq |S|$, *where* $\Gamma(S)$ *denotes the set of nodes in* R *that have a neighbour in* S.

18 Weighted Bipartite Matching



Halls Theorem

Proof:

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neighbours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.
  - Let S denote a minimum cut and let $L_S \overset{\text{def}}{=} L \cap S$ and $R_S \overset{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.
  - Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.
  - This gives $R_S \geq |\Gamma(L_S)|$.
  - The size of the cut is $|L| - |L_S| + |R_S|$.
  - Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node v.

- Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

- Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.
- Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

Algorithm Outline

Reason:

- The weight of your matching M^* is

$$\sum_{(u,v)\in M^*} w_{(u,v)} = \sum_{(u,v)\in M^*} (x_u + x_v) = \sum_v x_v \enspace .$$

- Any other perfect matching M (in G, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v)\in M} w_{(u,v)} \le \sum_{(u,v)\in M} (x_u + x_v) = \sum_v x_v \enspace .$$

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

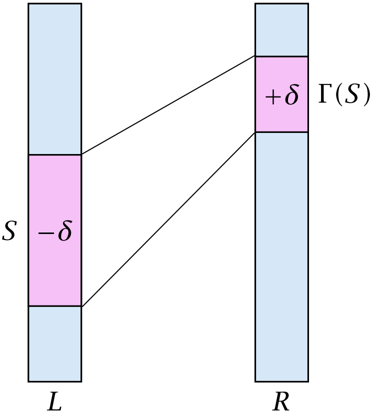
- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

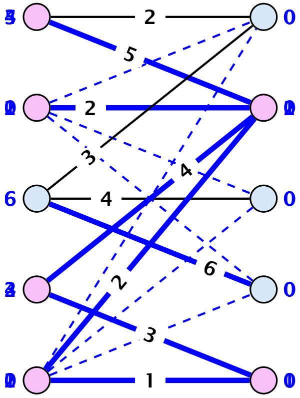
- Total node-weight decreases.
- Only edges from S to $R - \Gamma(S)$ decrease in their weight.
- Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between S and $\Gamma(S)$) we can do this decrement for small enough $\delta > 0$ until a new edge gets tight.



Weighted Bipartite Matching

Edges not drawn have weight 0.

$\delta = 1$ $\delta = 1$



Analysis

How many iterations do we need?

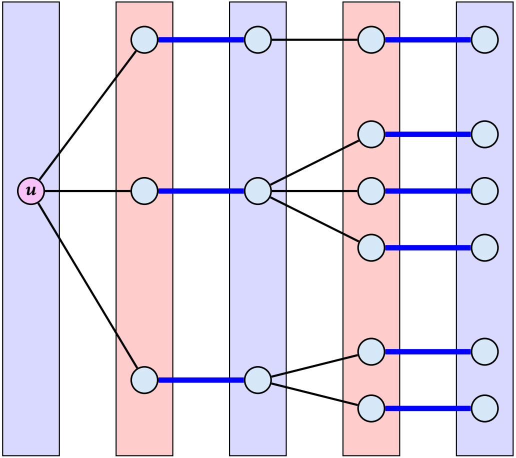
- One reweighting step increases the number of edges out of S by at least one.
- Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).
- This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.
- Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

Analysis

- We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- This gives a polynomial running time.

How to find an augmenting path?

Construct an alternating tree.



Analysis

How do we find S?

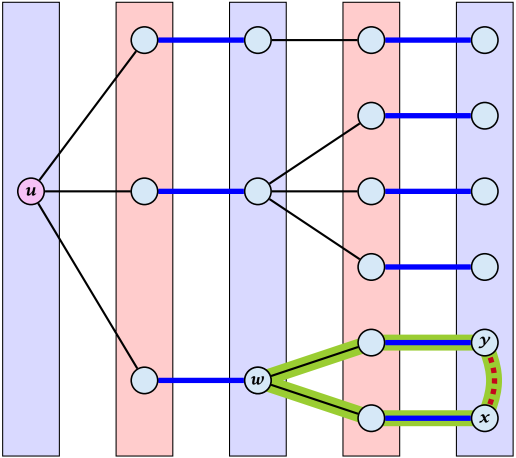
- Start on the left and compute an alternating tree, starting at any free node u.
- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).
- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
- All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

Analysis

- The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- An augmentation takes at most $\mathcal{O}(n)$ time.
- In total we obtain a running time of $\mathcal{O}(n^4)$.
- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.

How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w is called the stem of the blossom.

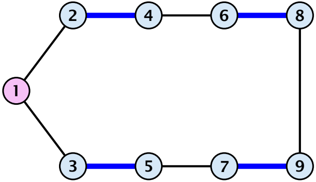
Flowers and Blossoms

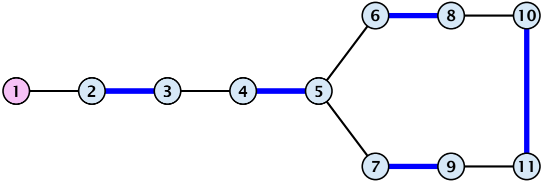
Definition 92

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

- A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

Flowers and Blossoms





Flowers and Blossoms

Properties:

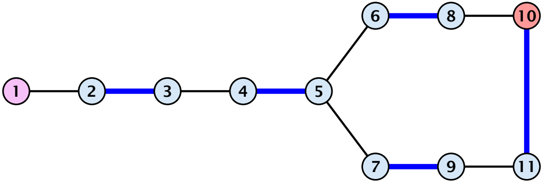
1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.
2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
5. The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms



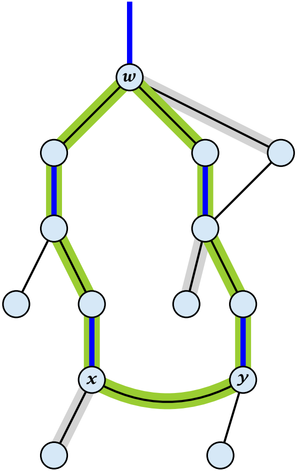
Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- Delete all vertices in B (and its incident edges) from G.
- Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

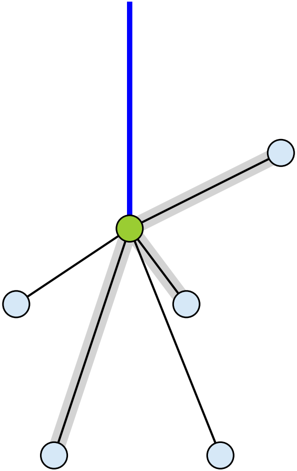
Shrinking Blossoms

- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.
- Nodes that are connected in G to at least one node in B become connected to b in G'.



Shrinking Blossoms

- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.
- Nodes that are connected in G to at least one node in B become connected to b in G'.



Example: Blossom Algorithm

Animation of Blossom Shrinking algorithm is only available in the lecture version of the slides.

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 93

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

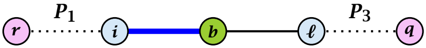
Correctness

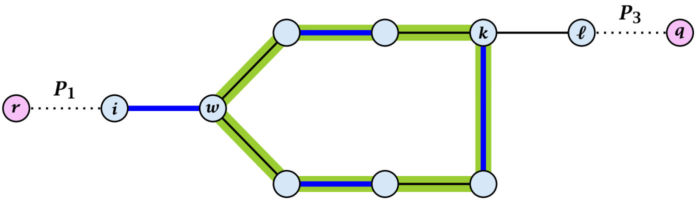
Proof.

If P' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

- Next suppose that the stem is non-empty.





Correctness

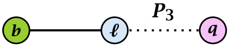
- After the expansion ℓ must be incident to some node in the blossom. Let this node be k.
- If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.
- $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

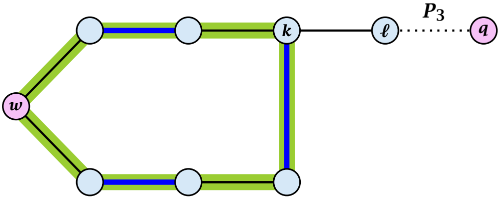
Correctness

Proof.

Case 2: empty stem

- If the stem is empty then after expanding the blossom, $w = r$.





- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 94
If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

- If P does not contain a node from B there is nothing to prove.
- We can assume that r and q are the only free nodes in G.

Case 1: empty stem

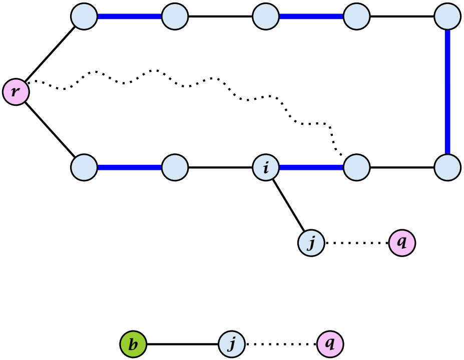
Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

Correctness

Illustration for Case 1:



Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

This path must go between r and q.

The lecture slides contain a step by step explanation.

Algorithm 50 search(r, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
$found \leftarrow$ false
unlabel all nodes;
give an even label to r and initialize $list \leftarrow \{r\}$
while $list \neq \emptyset$ **do**
    delete a node i from *list*
    examine(i, *found*)
    if $found =$ true **then return**

Search for an augmenting path starting at r.

The lecture slides contain a step by step explanation.

Algorithm 51 examine($i, found$)

for all $j \in \bar{A}(i)$ **do**
    if j is even **then** contract(i, j) and **return**
    if j is unmatched **then**
        $q \leftarrow j$;
        $\text{pred}(q) \leftarrow i$;
        $found \leftarrow \text{true}$;
        return
    if j is matched and unlabeled **then**
        $\text{pred}(j) \leftarrow i$;
        $\text{pred}(\text{mate}(j)) \leftarrow j$;
        add $\text{mate}(j)$ to $list$

Examine the neighbours of a node i

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Contract blossom identified by nodes i and j

Algorithm 52 contract(i, j)

trace pred-indices of i and j to identify a blossom B
create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
label b even and add to *list*
update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in B
delete nodes in B from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

b will be an even node, and it has unexamined neighbours.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Every node that was adjacent to a node in B is now adjacent to b

Algorithm 52 contract(i, j)

trace pred-indices of i and j to identify a blossom B
create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
label b even and add to *list*
update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in B
delete nodes in B from the graph

Only for making a blossom expansion easier.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only delete links from nodes not in B to B.

When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.
- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.
- There are at most n contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.
- In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

Example: Blossom Algorithm

Animation of Blossom Shrinking algorithm is only available in the lecture version of the slides.