Part II

Foundations

3 Goals

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▶ Learn how to analyze and judge the efficiency of algorithms.

▶ Learn how to design efficient algorithms.

4 Modelling Issues

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  ▶ How do you choose your inputs?
  ▶ May be very time-consuming.
  ▶ Very reliable results if done correctly.
  ▶ Results only hold for a specific machine and for a specific set of inputs.

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    - How do you choose your inputs?
    - May be very time-consuming.
    - Very reliable results if done correctly.
    - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.
    - Gives asymptotic bounds like "this algorithm always runs in time $\mathcal{O}(n^2)$".
    - Typically focuses on the worst case.
    - Can give lower bounds like "any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case".

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Example 1

Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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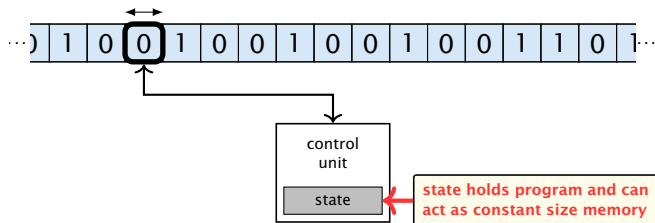
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Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.
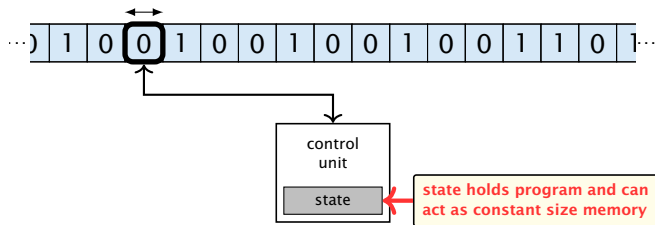
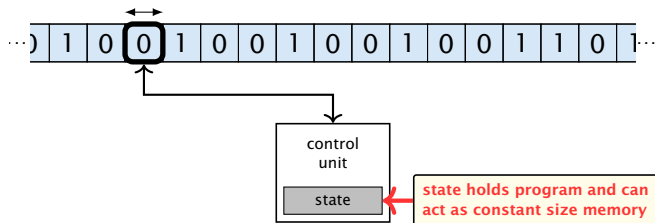# Turing Machine

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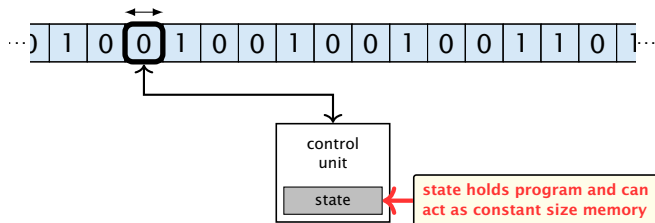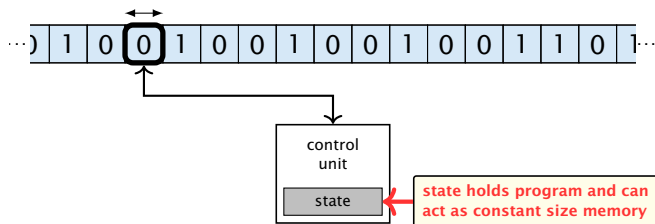
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⇒ **Not a good model for developing efficient algorithms.**

Random Access Machine (RAM)

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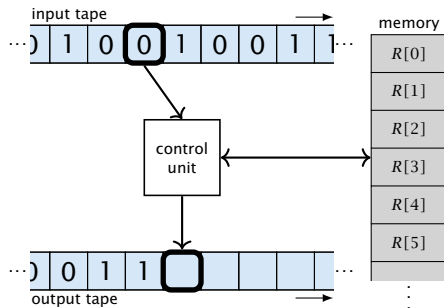
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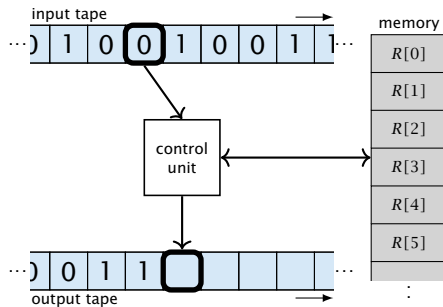▶ Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.

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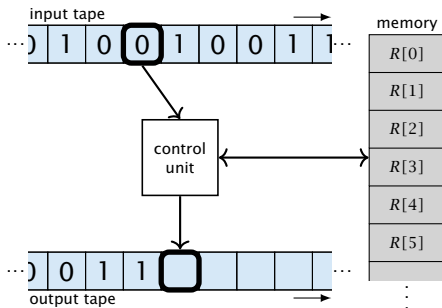
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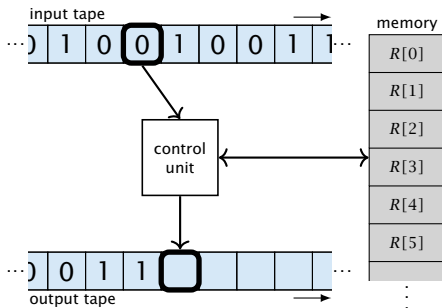
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  - ▶ $R[i] := R[j] + R[k]$;
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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

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Example 2

Algorithm 1 RepeatedSquaring(n)
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2: **for** $i = 1 \rightarrow n$ **do**
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more general: probability measure μ

$$C_{avg}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x)$$

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▶ randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

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▶ Running time should be expressed by simple functions.

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Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0 \ \exists n_0 \in \mathbb{N}_0 \ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

Asymptotic Notation in Equations

How do we interpret an expression like:

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

Asymptotic Notation in Equations

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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

Asymptotic Notation in Equations

How do we interpret an expression like:

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Careful!

"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:
$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)$$
$$\text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \Big\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \, \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

▶ *$c \cdot f(n) \in \Theta(f(n))$ for any constant c*

Harald Räcke

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The expressions also hold for Ω. Note that this means that
$f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Asymptotic Notation

Comments

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  Therefore, we will usually ignore the base of a logarithm within asymptotic notation.

Asymptotic Notation

Comments
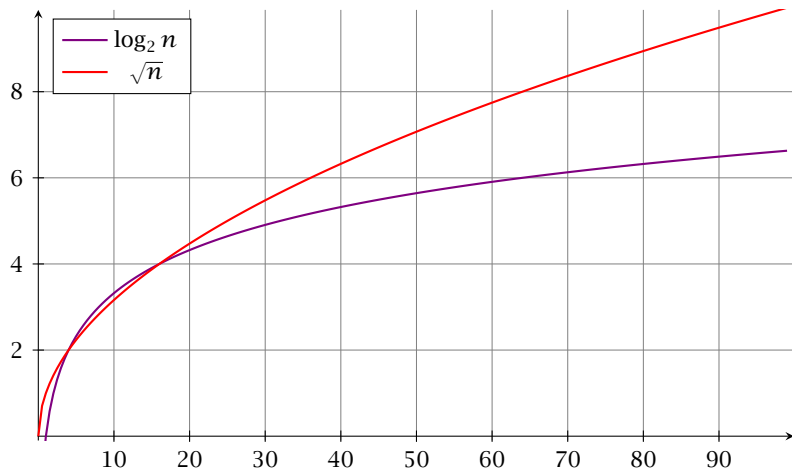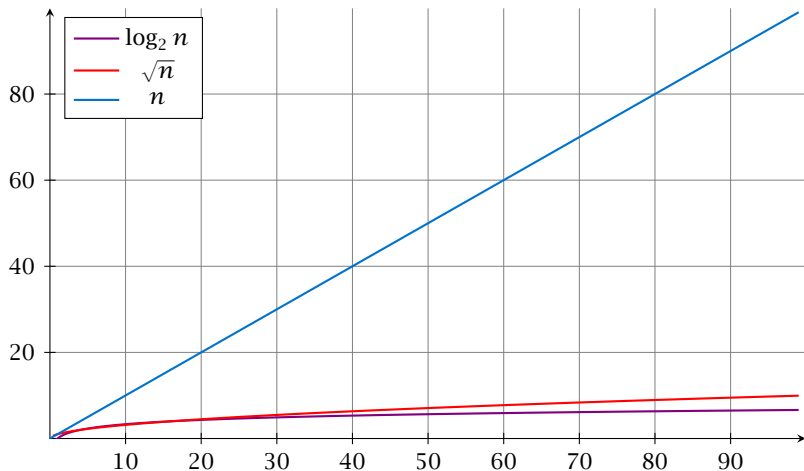
- ▶ Do not use asymptotic notation within induction proofs.

- ▶ For any constants a, b we have $\log_a n = \Theta(\log_b n)$.
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- ▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

Funktionen

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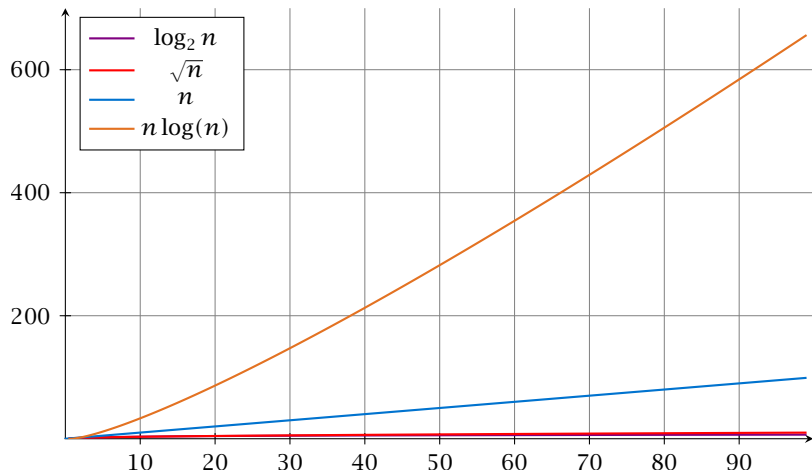
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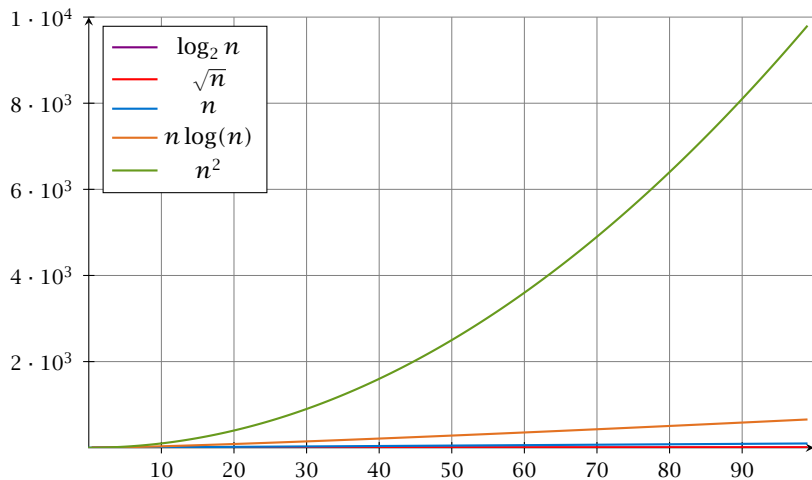
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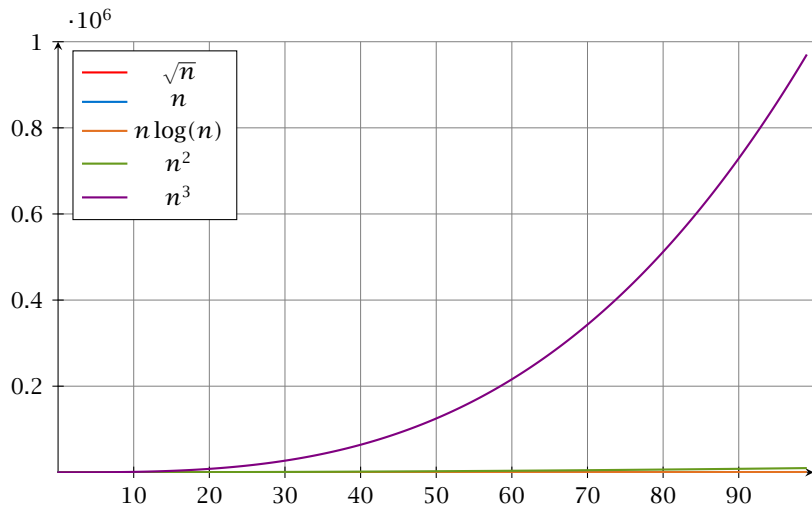
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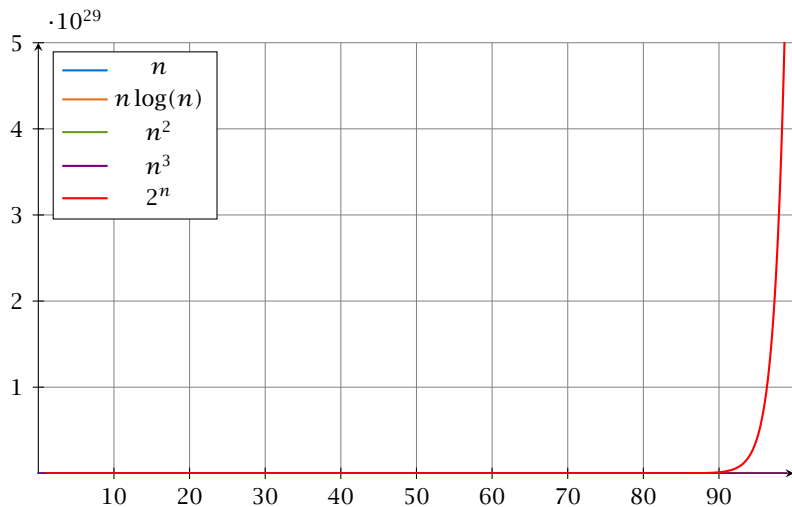
Funktionen

Laufzeiten

Funktion	*Eingabelänge n*							
$f(n)$	**10**	**10^2**	**10^3**	**10^4**	**10^5**	**10^6**	**10^7**	**10^8**
$\log n$	33ns	66ns	0.1µs	0.1µs	0.2µs	0.2µs	0.2µs	0.3µs
$\sqrt{n}$	32ns	0.1µs	0.3µs	1µs	3.1µs	10µs	31µs	0.1ms
n	100ns	1µs	10µs	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3µs	6.6µs	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3µs	10µs	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1µs	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10µs	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$y					
2^n	10µs	$4 \cdot 10^{14}$y						
$n!$	36ms	$3 \cdot 10^{142}$y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$y

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

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Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

Multiple Variables in Asymptotic Notation

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Formal Definition

Let f, g denote functions from $\mathbb{N}^d$ to $\mathbb{R}_0^+$.

▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \; \exists N \in \mathbb{N}_0 \; \forall \vec{n} \text{ with } n_i \geq N \text{ for some } i :$
$$[g(\vec{n}) \leq c \cdot f(\vec{n})]\}$$
(set of functions that asymptotically grow not faster than f)

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Example 4

▶ $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n,m) = n - 1$

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6 Recurrences

Algorithm 2 mergesort(list L)

1: $n \leftarrow \text{size}(L)$
2: **if** $n \leq 1$ **return** L
3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
4: $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
5: $\text{mergesort}(L_1)$
6: $\text{mergesort}(L_2)$
7: $L \leftarrow \text{merge}(L_1, L_2)$
8: **return** L

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5: mergesort(L_1)
6: mergesort(L_2)
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8: **return** L

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

Recurrences

How do we bring the expression for the number of comparisons (≈ running time) into a closed form?

For this we need to solve the recurrence.

Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. Generating Functions
A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence
Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

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$$T(n) \leq \begin{cases} 2T\left(\frac{n}{2}\right) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

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Guess: $T(n) \leq dn \log n$.

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Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

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$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$
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Hence, statement is true if we choose $d \geq c$.

6.1 Guessing+Induction

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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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We also make a guess of $T(n) \leq dn \log n$ and get

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$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1}$$

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$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)}$$

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$$\leq dn \log n - 0.33dn + cn$$

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We also make a guess of $T(n) \leq dn \log n$ and get

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$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 5

Let $a \geq 1, b > 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.
If $f(n) = \mathcal{O}(n^{\log_b(a) - \epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
$k \geq 0$.

Case 3.
If $f(n) = \Omega(n^{\log_b(a) + \epsilon})$ and for sufficiently large n
$af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \,.$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \le c n^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

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$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q-1}} = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\end{aligned}
$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} = cn^{\log_b a - \epsilon}\sum_{i=0}^{\log_b n - 1}(b^\epsilon)^i$$

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Case 1. Now suppose that $f(n) \leq c n^{\log_b a - \epsilon}$.

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$$= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})$$

Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a}$$

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$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\end{aligned}
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
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$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

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Hence,
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

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&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k
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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k}\ell^{k+1}
\end{aligned}
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$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
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$q < 1 : \sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1-q} \leq \frac{1}{1-q}$

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&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
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&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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For this we first need to be able to add two integers A and B:

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$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c}
A \\
B
\end{array}
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & & & & & & 0
\end{array}
$$

A

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
& & & & & & & 0 & 0
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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & & & {}_1 & {}_1 & {}_1 & \\
 & & & & & 0 & 0 & 0 &
\end{array}
$$

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\quad A
$$

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$$
\begin{array}{ccccccccc}
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
 & & & & {}_0 & {}_1 & {}_1 & {}_1 & \\
\hline
 & & & & & 1 & 0 & 0 & 0
\end{array}
$$

A

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
& & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
& & & & 0 & 1 & 0 & 0 & 0 &
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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
& & {}_1 & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & {}_1 & \\
\hline
& & & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

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$$
\begin{array}{cccccccccc}
\color{red}{1} & \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{A} \\
\color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} & \color{blue}{B} \\
 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \\
\hline
 & \mathbf{1} & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
\color{red}1 & \color{red}1 & \color{red}0 & \color{red}1 & \color{red}1 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}1 & & \color{red}A \\
\color{blue}1 & \color{blue}0 & \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}1 & & \color{blue}B \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & & \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & & \\
\end{array}
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$$
\begin{array}{cccccccccc}
\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \quad A \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \quad B \\
\hline
\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

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	1	1	0	1	1	0	1	0	1		A
	1	0	0	0	1	0	0	1	1		B
1	0	0	1	1	0	1	1	1	1		
	0	1	1	0	0	1	0	0	0		

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$$
\begin{array}{ccccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & & B \\
\scriptstyle 1 & \scriptstyle 0 & \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 1 & \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 1 & \scriptstyle 1 & & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & \boxed{1} \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
$$

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$$
\begin{array}{cccccccccc}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \times & \mathbf{1} & \mathbf{0} & \boxed{\mathbf{1}} & \mathbf{1} \\
\hline
 & & & & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\ \hline & & & & & 1 & 0 & 0 & 0 & 1 \\ & & & & & & & & & 0 \end{array}$$

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$$
\begin{array}{ccccccccc}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \times & \mathbf{1} & \mathbf{0} & \boxed{\mathbf{1}} & \mathbf{1} \\
\hline
 & & & & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
 & & & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0}
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{r}
1\ 0\ 0\ 0\ 1 \times 1\ \boxed{0}\ 1\ 1 \\
\hline
1\ 0\ 0\ 0\ 1 \\
1\ 0\ 0\ 0\ 1\ 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

```
  1   0   0   0   1   ×   1   0   1   1
                          1   0   0   0   1
                      1   0   0   0   1   0
                                  0   0
```

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

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\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
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 & & & & & 1 & 0 & 0 & 0 & 1 \\
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1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
  &   &   &   &   & 1 & 0 & 0 & 0 & 1 \\
  &   &   &   & 1 & 0 & 0 & 0 & 1 & 0 \\
  &   &   & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  &   & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
$$

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$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
  &   &   &   & 1 & 0 & 0 & 0 & 1 \\
  &   &   & 1 & 0 & 0 & 0 & 1 & 0 \\
  &   & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

```
1 0 0 0 1 × 1 0 1 1
          1 0 0 0 1
        1 0 0 0 1 0
      0 0 0 0 0 0 0
    1 0 0 0 1 0 0 0
  1 0 1 1 1 0 1 1
```

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Time requirement:

▶ Computing intermediate results: $\mathcal{O}(nm)$.

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$$
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1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Time requirement:

▶ Computing intermediate results: $\mathcal{O}(nm)$.

▶ Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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| b_{n-1} | $\cdots$ | b_0 | $\times$ | a_{n-1} | $\cdots$ | a_0 |

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

| b_{n-1} | $\cdots$ | $b_{\frac{n}{2}}$ $b_{\frac{n}{2}-1}$ | $\cdots$ | b_0 | $\times$ | a_{n-1} | $\cdots$ | $a_{\frac{n}{2}}$ $a_{\frac{n}{2}-1}$ | $\cdots$ | a_0 |

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Suppose that integers A and B are of length $n = 2^k$, for some k.

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Then it holds that

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)
1: **if** $|A| = |B| = 1$ **then**
2:        **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
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5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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$\Longrightarrow$ Not better then the "school method".

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We can use the following identity to compute Z_1:

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$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1 B_1 - A_0 B_0$$

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Hence,

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▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

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Again we are in Case 1. We get a running time of
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

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Consider the recurrence relation:

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Note that we ignore boundary conditions for the moment.

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▶ First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \ldots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space.

The Homogenous Case

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

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This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

The Homogenous Case

Lemma 6

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

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Lemma 6

Assume that the characteristic polynomial has k *distinct roots* $\lambda_1, \ldots, \lambda_k$. *Then all solutions to the recurrence relation are of the form*

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

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Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & & \vdots & \vdots \\
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\end{vmatrix} =
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$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$
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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^{k} (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

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Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n \lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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Hence, the solution is of the form

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \Longrightarrow \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

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There is no general method to find a particular solution.

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$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

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Subtracting the first from the second equation gives,

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[1] = 2$ gives $1 + \beta = 2 \implies \beta = 1$.

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$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
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and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

▶ generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} a_n z^n \; ;$$

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▶ exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n \; .$$

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▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

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There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

▶ **Equality:** f and g are equal if $a_n = b_n$ for all n.

▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.

▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

The arithmetic view:

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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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Then, it is important to think about convergence/convergence radius etc.

What does $\sum_{n \geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n \geq 0}^{\infty} z^n\right) = 1 \ .$$

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This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

6.4 Generating Functions

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Hence, the generating function of the sequence
$a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

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$$= \frac{k!}{(1-z)^{k+1}} \; .$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \; .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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Hence,

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The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	generating function

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$\frac{1}{n!}$	e^z

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$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$

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$n f_n$	$z \dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$

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$n f_n$	$z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

Solving Recursions with Generating Functions

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2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

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6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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2. Plug in:

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1}) z^n$$

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$$A(z) = \frac{1}{1 - 2z}$$

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$$= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} a_n z^n \\
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&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n
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&= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n \\
&= 1 + 3z A(z) + \frac{z}{(1-z)^2}
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$$

4. Solve for $A(z)$:

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5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

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&= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \\
&= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C)
\end{aligned}
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This leads to the following conditions:

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

6.5 Transformation of the Recurrence

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$$f_0 = 1$$
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Define

$$g_n := \log f_n \,.$$

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$$g_n = F_n \ (n\text{-th Fibonacci number})$$
$$f_n = 2^{F_n}$$

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$$f_1 = 1$$
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Define

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Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k, \; k \geq 1$$

6 Recurrences

We get

$$g_k = 3 \left[g_{k-1} \right] + 2^k$$

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$$g_k = 3\left[g_{k-1}\right] + 2^k$$
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\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 3 \cdot 2^{k-1} + 2^k
\end{aligned}
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&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 3 2^{k-1} + 2^k
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&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2}
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&= 2^k \cdot \sum_{i=0}^{k}\left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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$$= 3(2^k)^{\log 3} - 2 \cdot 2^k$$

6.5 Transformation of the Recurrence · 14. Jan. 2024

Harald Räcke · 113/113

6 Recurrences

Let $n = 2^k$:

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$$= 3(2^k)^{\log 3} - 2 \cdot 2^k$$
$$= 3n^{\log 3} - 2n \ .$$