Winter Semester 2023/24

Advanced Algorithms

http://www14.in.tum.de/lehre/2023WS/ada/index.html.en

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Organization

Lectures: 3 SWS
Tue, Thu 12–14
Lecture hall: Galileo 8120.EG.001

Exercises: 2 SWS
Teaching assistant: Sebastian Schubert,
Malte Kriegelsteiner

Tue 14–16: Room MI 02.04.011
Thu 14–16: Room MI 01.07.023

Organization

Problem sets: Made available on Tuesday by 10:00 am via Moodle.

Exam: Written exam, date will be announced.

Valuation: 6 ECTS (3 + 2 SWS)

Prerequisites: Grundlagen: Algorithmen und Datenstrukturen (GAD)
Diskrete Strukturen (DS)
Diskrete Wahrscheinlichkeitstheorie (DWT)

Literature

- Th. Cormen, C. Leiserson, R. Rivest, and C. Stein. Introduction to Algorithms, Third Edition, MIT Press, 2009.
- J. Kleinberg and E. Tardos. Algorithm Design. Pearson, Addison Wesley, 2006.
- M. Mitzenmacher and E. Upfal. Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis. Second Edition, Cambridge University Press, 2017.
- Th. Ottmann und P. Widmayer: Algorithmen und Datenstrukturen. 6. Auflage, Springer Verlag, 2017.
- Research papers

Content

Design and analysis techniques for algorithms

- Divide and conquer
- Greedy approaches
- Dynamic programming
- Randomization
- Amortized analysis

Content

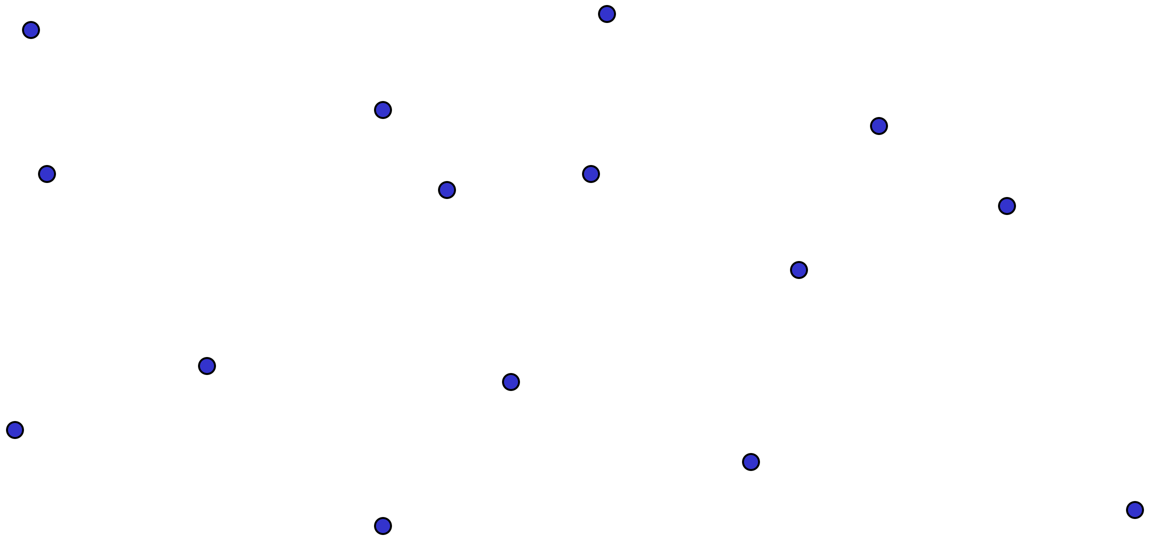
Problems and application areas:

- Geometric algorithms
- Algebraic algorithms
- Graph algorithms
- Data structures
- Algorithms on strings
- Optimization problems
- Complexity

Geometric divide-and-conquer

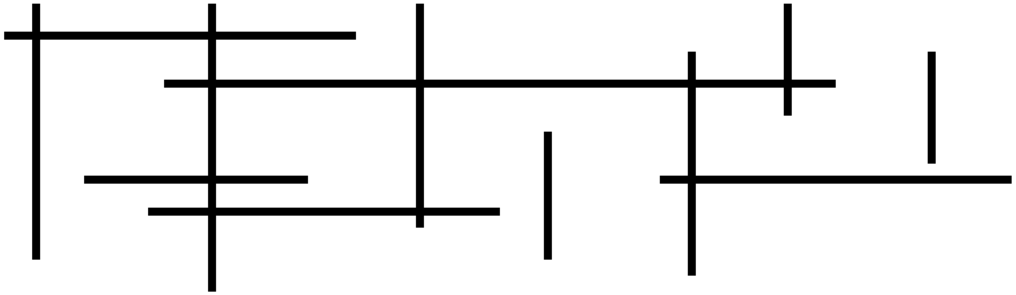
Closest Pair Problem:

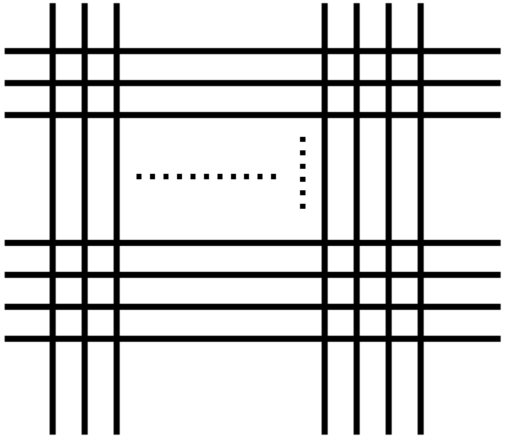
Given a set *S* of *n* points in the plane, find a pair of points with the smallest distance.



Line segment intersection

Find all pairs of intersecting line segments.





Fast Fourier Transform

$p, q \in R[x]$

$$p(x) = a_n x^n + \ldots + a_1 x^1 + a_0$$
$$q(x) = b_n x^n + \ldots + b_1 x^1 + b_0$$

Multiplication

$$p(x)q(x) = (a_n x^n + \ldots + a_0)(b_n x^n + \ldots + b_0)$$
$$= c_{2n} x^{2n} + \ldots + c_1 x^1 + c_0$$

Fast Fourier Transform

- FFT algorithms compute the discrete Fourier transform (DFT).

- Many applications in engineering, science and mathematics.
  - Digital signal processing (e.g. UMTS and LTE)
  - Image processing
  - Data compression
  - Partial differential equations

- Popular algorithm by J. Cooley and J.W. Tukey, 1965, based on earlier ideas of C.F. Gauß in 1805.

- Included in Top 10 Algorithms of the 20th Century by IEEE journal *Computing in Science & Engineering* (2000).

Randomization

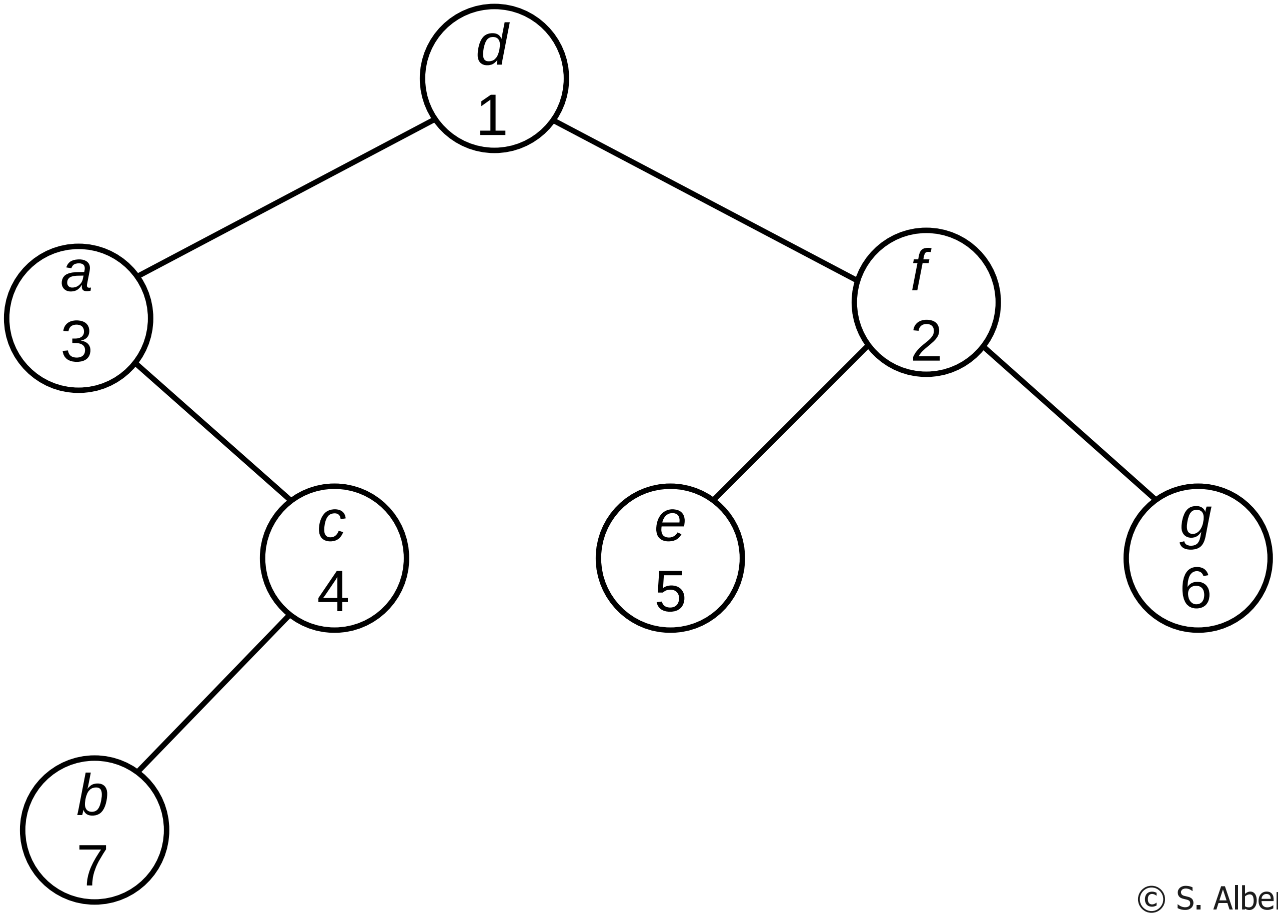
Algorithm may make random choices.

Advantages: Speed and simplicity

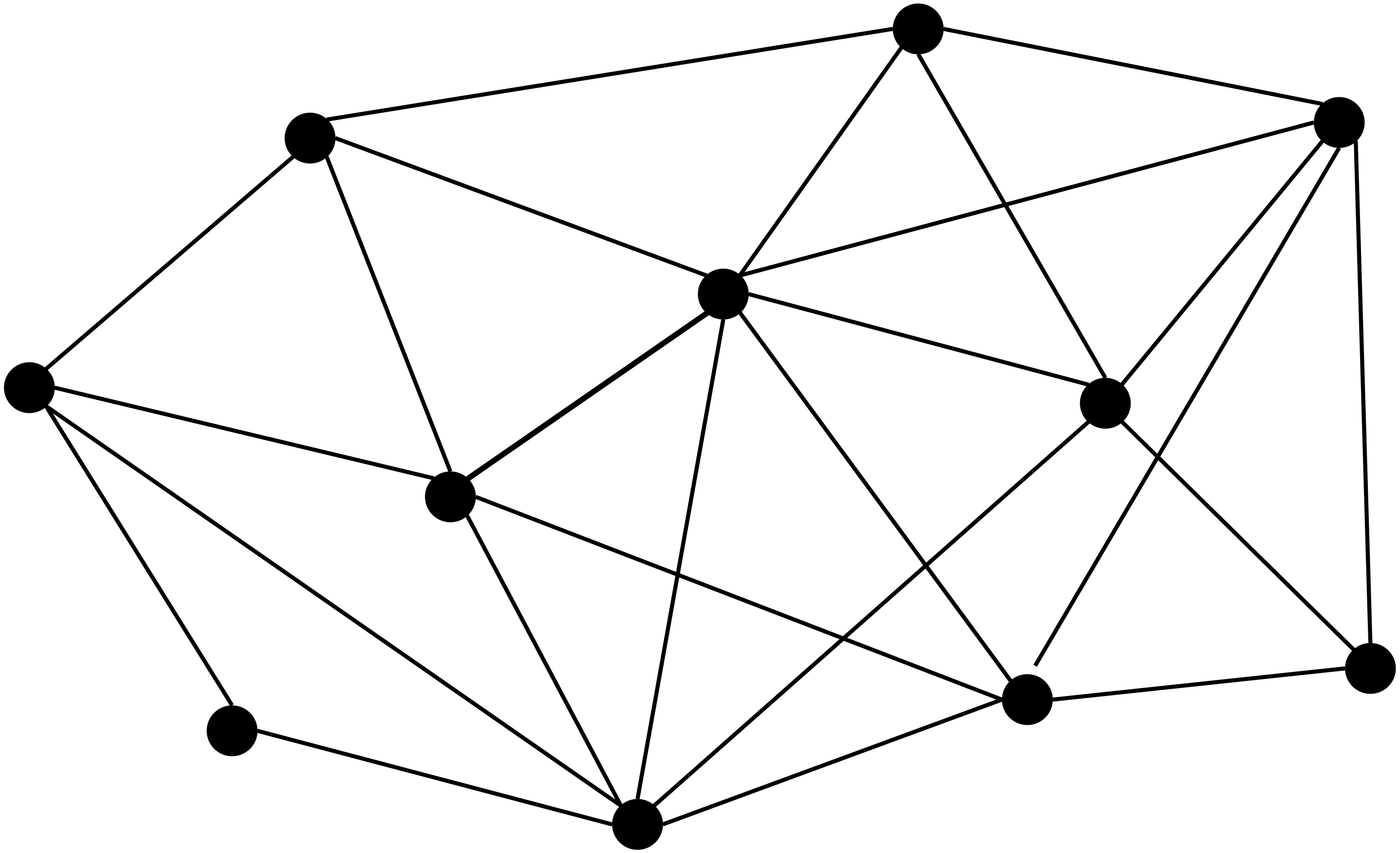
- Types of randomized algorithms
- Randomized primality test
- Cryptography: RSA algorithm

Randomized Search Trees

key	*a*	*b*	*c*	*d*	*e*	*f*	*g*
priority	3	7	4	1	5	2	6



Minimum cuts



Suffix Trees

Static texts

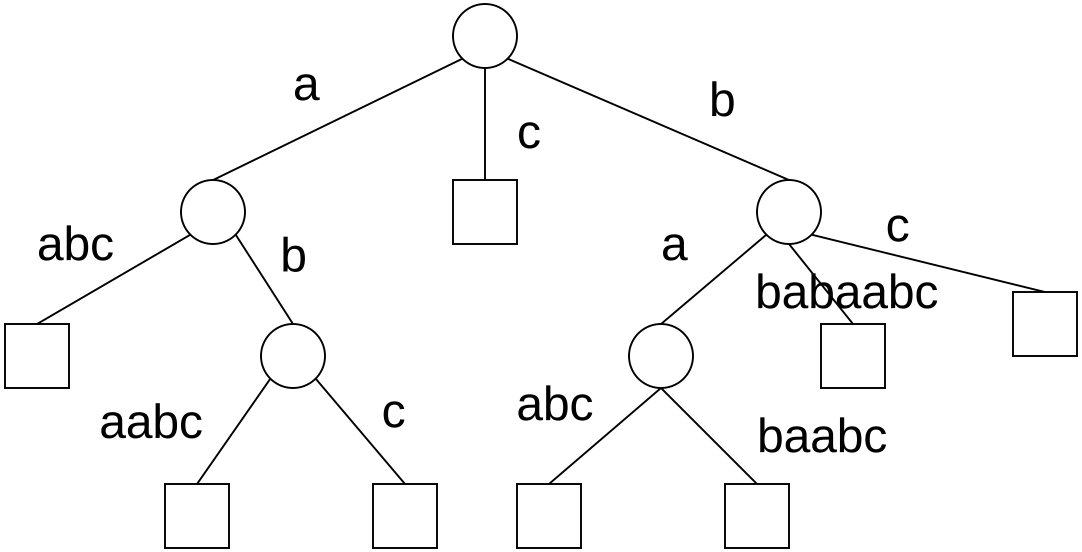
- Literature databases
- Library systems
- Gene databases
- World Wide Web

Search index

for a text σ in order to search for several patterns α.

Substring search in time $O(|\alpha|)$.

Suffix tree



σ = bbabaabc

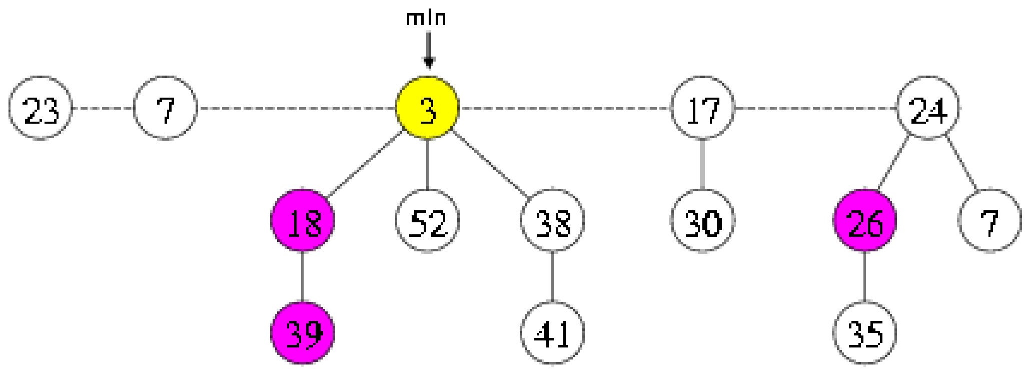
Amortized analysis

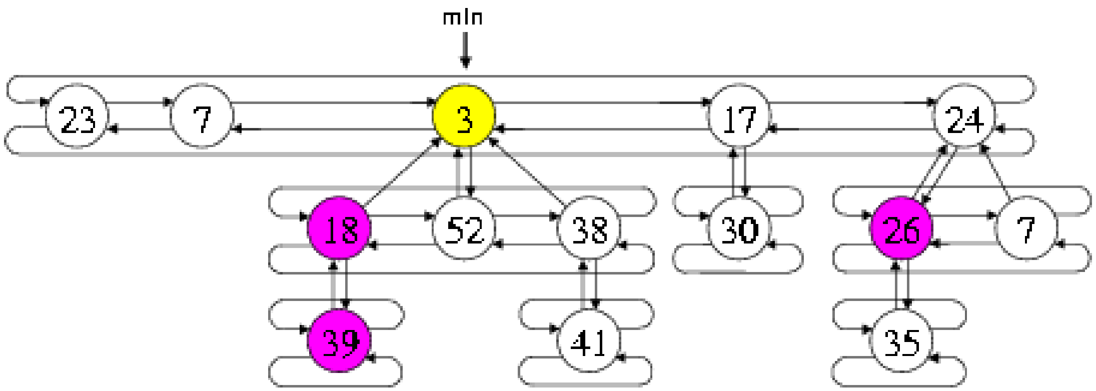
- Best case
- Worst case
- Average case
- Amortized worst case

What is the average cost of an operation in a worst case sequence of operations?

Average execution time of an operation is small, even though a single operation can have a high execution time.

Fibonacci heaps





Greedy algorithms

In each step make the choice that looks best at the moment

Basic examples

- The coin-changing problem
- The Traveling Salesman Problem

Scheduling problems

- Interval scheduling
- Scheduling to minimize lateness

Discussion: Shortest paths and minimum spanning trees

Dynamic programming

Recursive approach: Solve a problem by solving several smaller analogous subproblems of the same type. Then combine these solutions to generate a solution to the original problem.

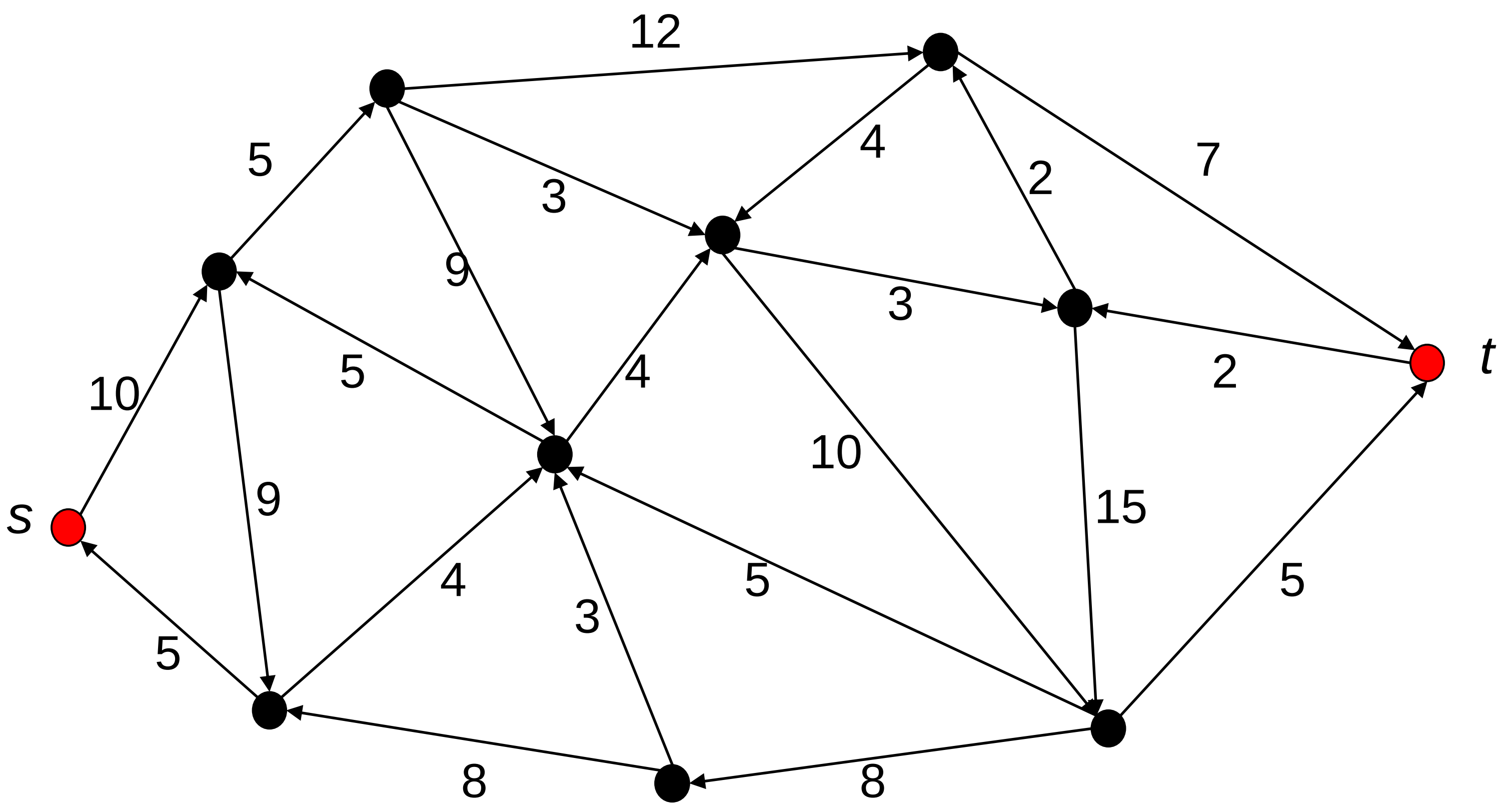
Drawback: Repeated computation of solutions

Dynamic-programming method: Once a subproblem has been solved, store its solution in a table so that it can be retrieved later by simple table lookup.

Dynamic programming

- Matrix chain multiplication
- Segmented least squares
- Optimal binary search trees
- Subset sums & knapsacks

Maximum flow problem

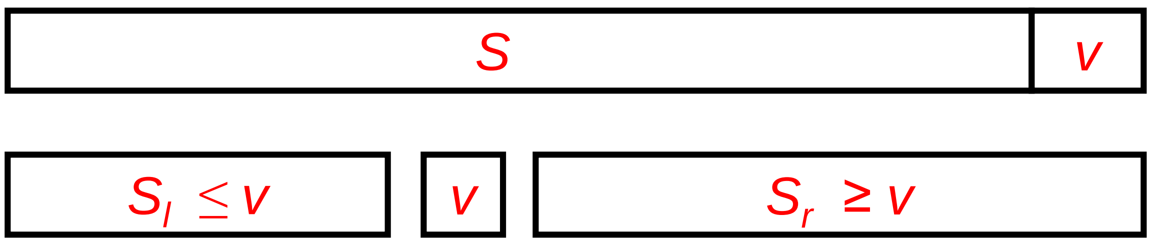


01 - Divide and Conquer

The divide-and-conquer paradigm

- Quicksort
- Formulation and analysis of the paradigm
- Geometric divide-and-conquer
  - Closest pair problem
  - Line segment intersection
  - Voronoi diagrams

Quicksort: Sorting by partitioning



function Quick (S: sequence): sequence;

{returns the sorted sequence S}

begin

if $\#S \leq 1$ then Quick:=S;

else { choose pivot/splitter element v in S;

partition S into S_l with elements $\leq v$,

and S_r with elements $\geq v$;

Quick:= $\boxed{\text{Quick}(S_l) \,|\, v \,|\, \text{Quick}(S_r)}$ }

end;

Formulation of the D&C paradigm

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

n > c: Divide the problem into k subproblems of sizes $n_1,...,n_k$ ($k \geq 2$).

$n \leq c$: Solve the problem directly.

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Merge

Combine the partial solutions to generate a solution for the original instance.

Analysis

$T(n)$: maximum number of steps necessary for solving an instance of size n

$$T(n) = \begin{cases} a & n \le c \\ T(n_1) + \ldots + T(n_k) & n > c \\ + \text{ cost for divide and merge} & \end{cases}$$

Special case: $k = 2$, $n_1 = n_2 = n/2$

cost for divide and merge: DM(n)

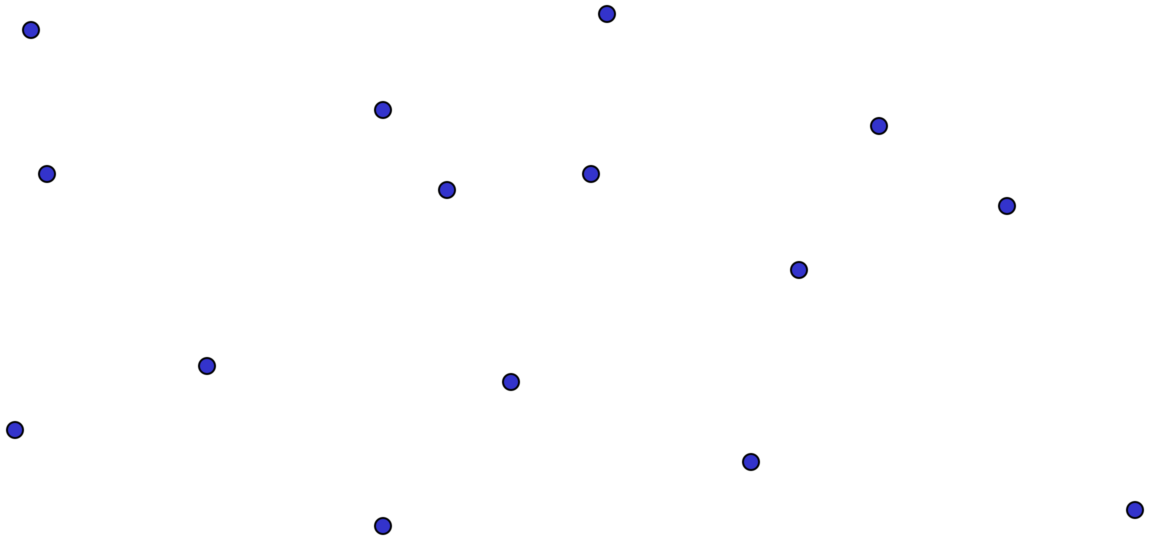
$T(1) = a$

$T(n) = 2T(n/2) + DM(n)$

Geometric divide-and-conquer

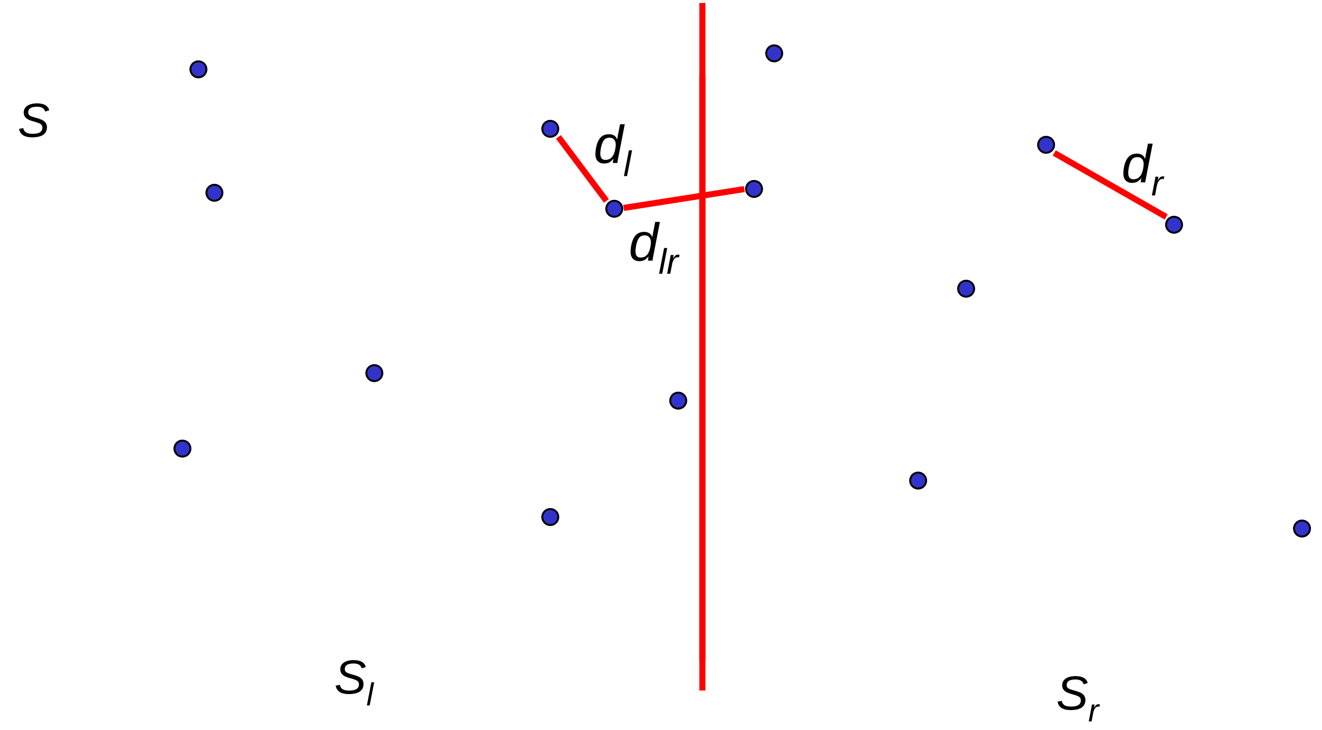
Closest Pair Problem:

Given a set *S* of *n* points in the plane, find a pair of points with the smallest distance.



Divide-and-conquer method

1. **Divide:** Divide S into two equal sized sets S_l und S_r .
2. **Conquer:** d_l = mindist(S_l) d_r = mindist(S_r)
3. **Merge:** $d_{lr} = \min\{ d(p_l, p_r) \mid p_l \in S_l, p_r \in S_r \}$
   return $\min\{d_l, d_r, d_{lr}\}$



Divide-and-conquer method

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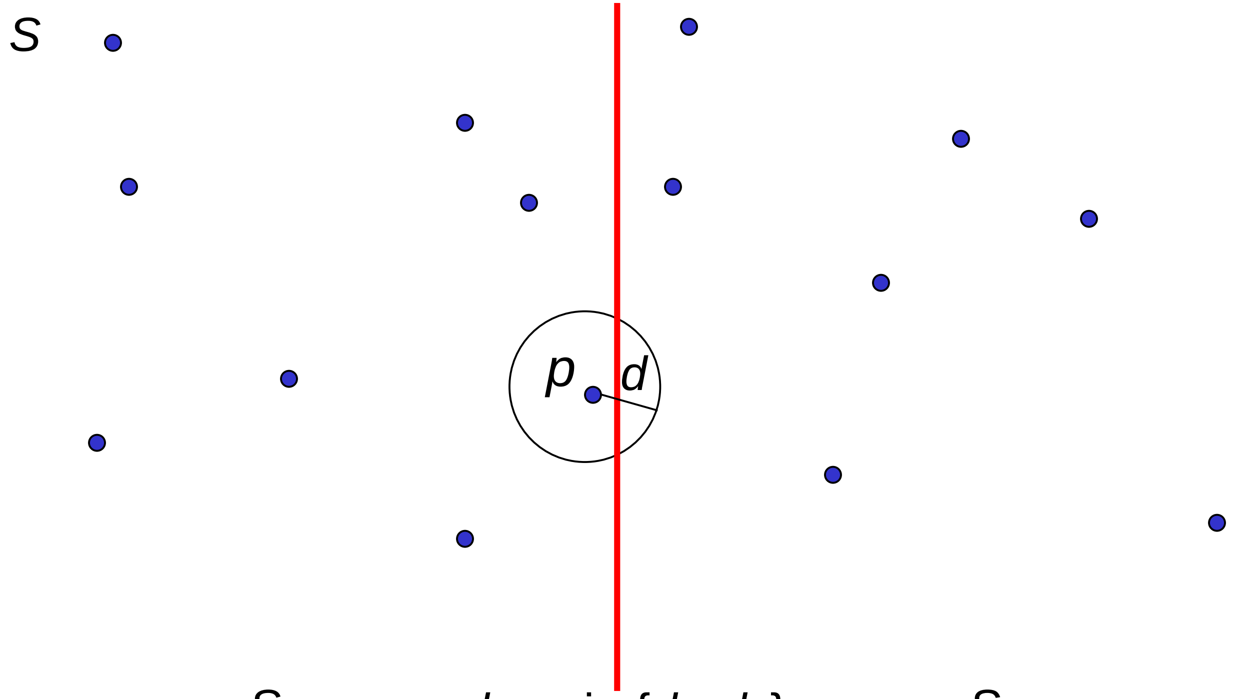
3. **Merge:** $d_{lr} = \min\{ \text{d}(p_l, p_r) \mid p_l \in S_l, \text{p}_\text{r} \in S_r \}$
   return $\min\{d_l, d_r, d_{lr}\}$

Computation of d_{lr} :



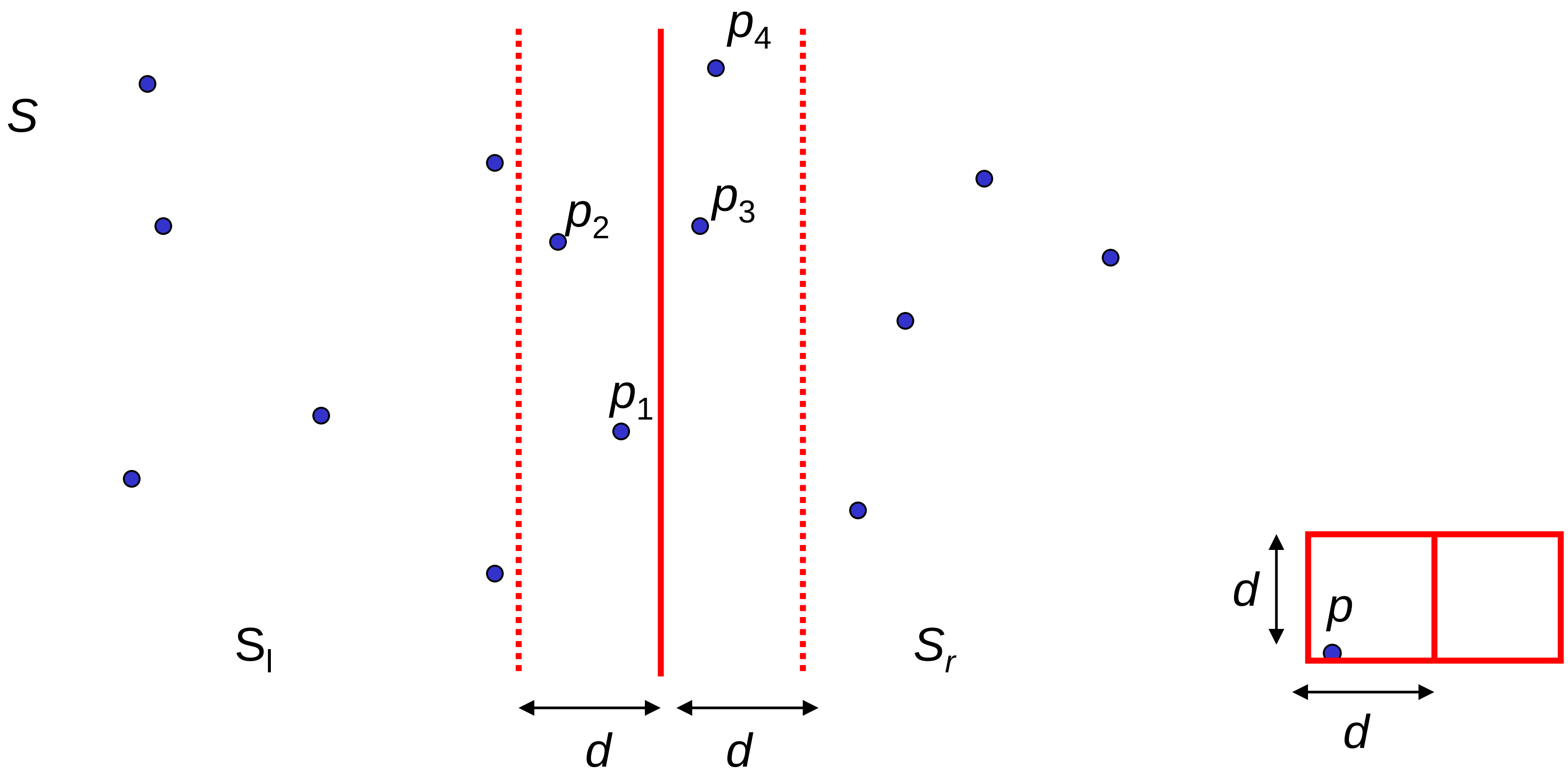
S_l $d = \min \{d_l, d_r\}$ S_r

Merge step

1. Consider only points within distance *d* of the bisection line, in the order of increasing y-coordinates.

2. For each point *p* consider all points *q* within *y*-distance at most *d*; there are at most 7 such points.

Merge step



$d = \min \{ d_l , d_r \}$

Implementation

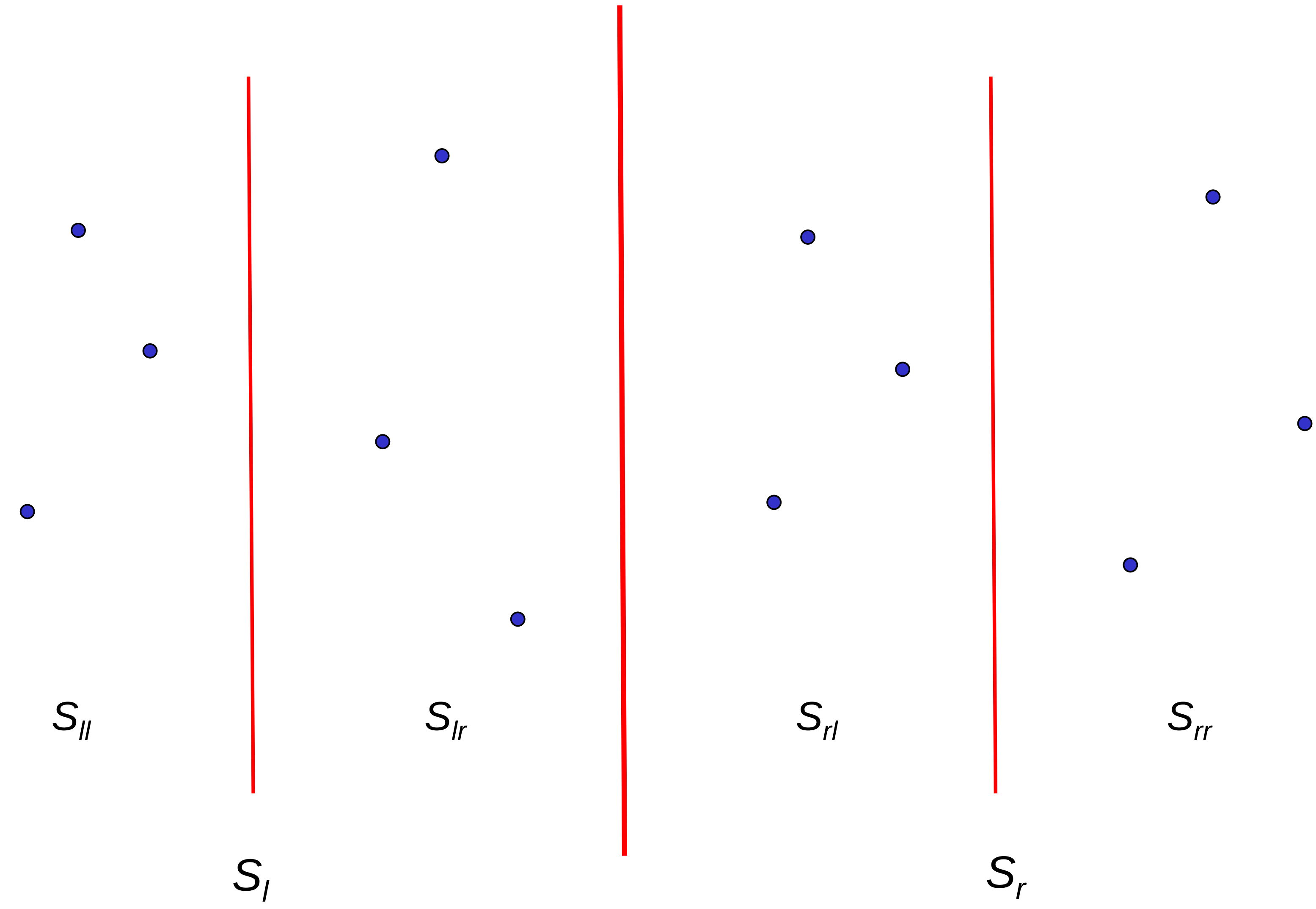
- Initially sort the points in S in order of increasing x-coordinates $O(n \log n)$.

  Each bisection line can be determined in $O(1)$ time.

- Once the subproblems S_l , S_r are solved, generate a list of the points in S in order of increasing y-coordinates.

  This can be done by merging the sorted lists of points of S_l , S_r (merge sort).

Sorted lists



Running time (divide-and-conquer)

$$T(n) = \begin{cases} 2T(n/2) + an & n > 3 \\ a & n \leq 3 \end{cases}$$

- Guess the solution by repeated substitution.
- Verify by induction.

Solution: O(*n* log *n*)

Guess by repeated substitution

$$T(n) = \begin{cases} 2T(n/2) + an & n > 3 \\ a & n \le 3 \end{cases}$$

$$\begin{aligned} T(n) &= 2T(n/2) + an = 2(2T(n/4) + an/2) + an \\ &= 4T(n/4) + 2an = 4(2T(n/8) + an/4) + 2an \\ &= 8T(n/8) + 3an = 8(2T(n/16) + an/8) + 3an \\ &= 16T(n/16) + 4an \end{aligned}$$

Verify by induction

$$T(n) \leq an \log n \qquad T(n) = \begin{cases} 2T(n/2) + an & n > 3 \\ a & n \leq 3 \end{cases}$$

$n = 2^i$

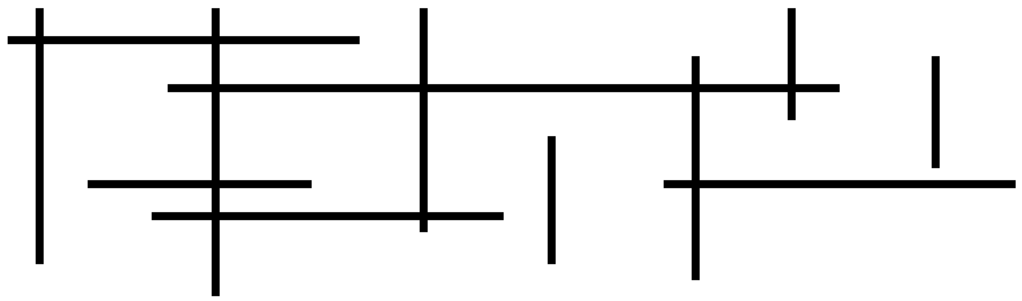
$i = 1$: ok

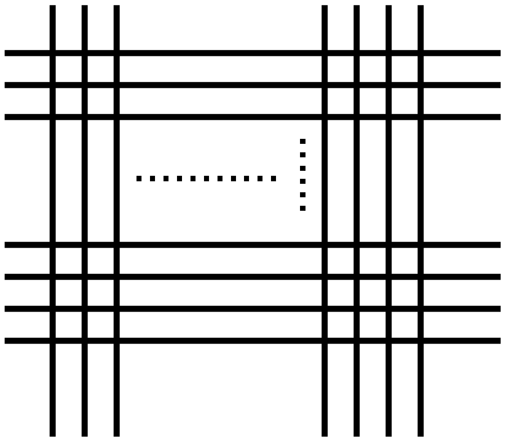
$i > 1$

$$\begin{aligned} T(2^i) &= 2T(2^{i-1}) + a2^i \\ &\leq 2a2^{i-1}(i-1) + a2^i \\ &= a2^i(i-1) + a2^i \\ &= a2^i i \\ &= an \log n \end{aligned}$$

Line segment intersection

Find all pairs of intersecting line segments.





Line segment intersection

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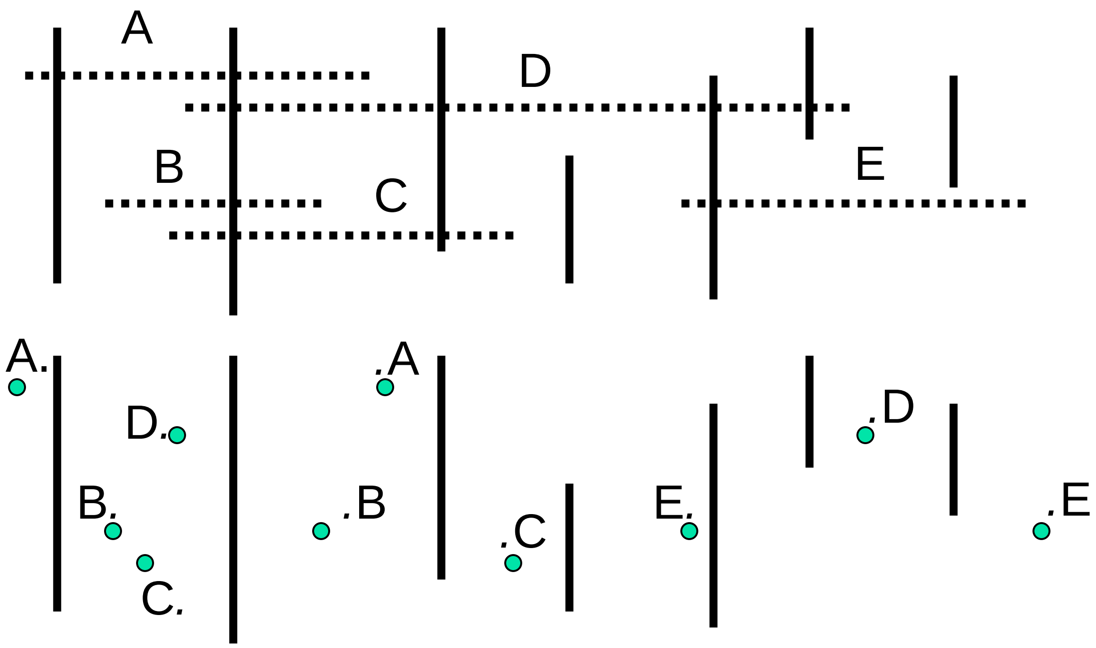


The representation of the horizontal line segments by their endpoints allows for a vertical partitioning of all objects.

ReportCuts

Input: Set S of vertical line segments and endpoints of horizontal line segments.

Output: All intersections of vertical line segments with horizontal line segments, for which at least one endpoint is in S.

1. Divide

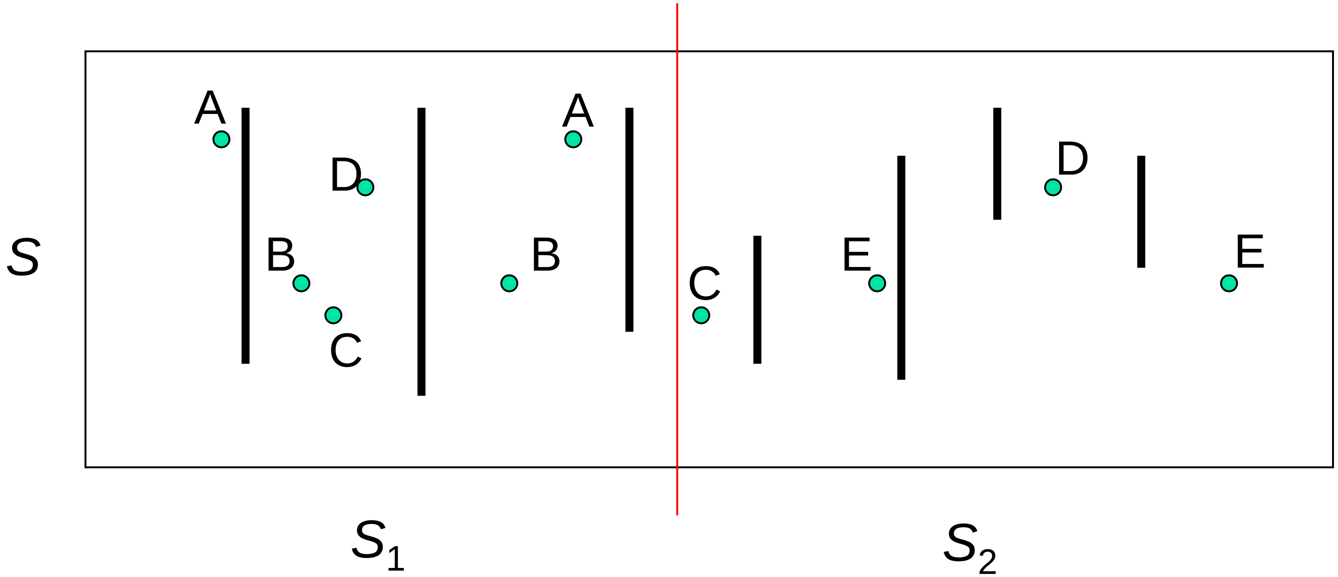
if $|S| > 1$

then using vertical bisection line L, divide S into equal size sets S_1 (to the left of L) and S_2 (to the right of L)

else S contains no intersections

ReportCuts

1. Divide:



2. Conquer:

ReportCuts(S_1); ReportCuts(S_2)

ReportCuts

3. Merge: ???

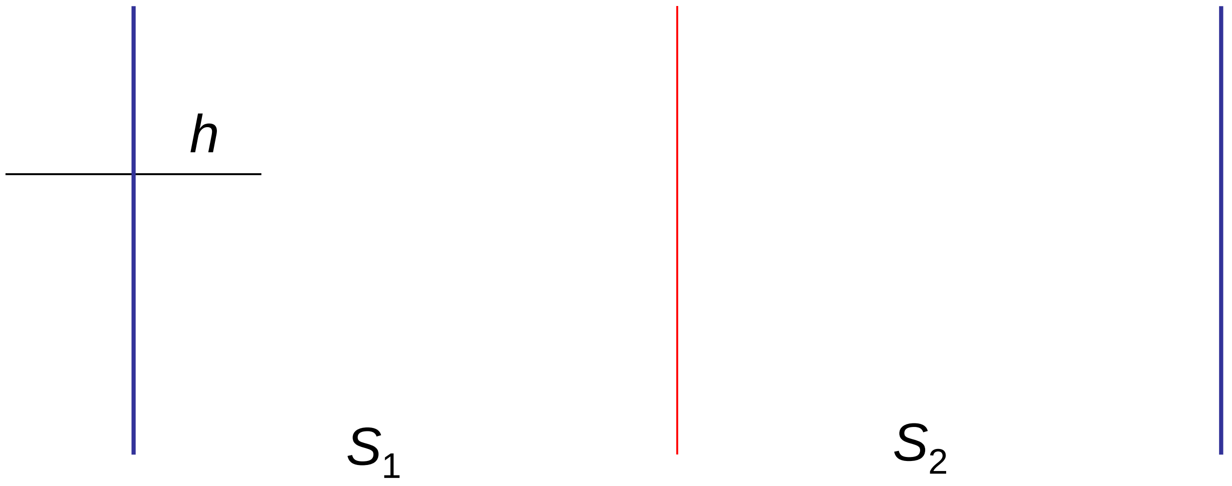
Possible intersections of a horizontal line segment h in S_1

Case 1: both endpoints in S_1

ReportCuts

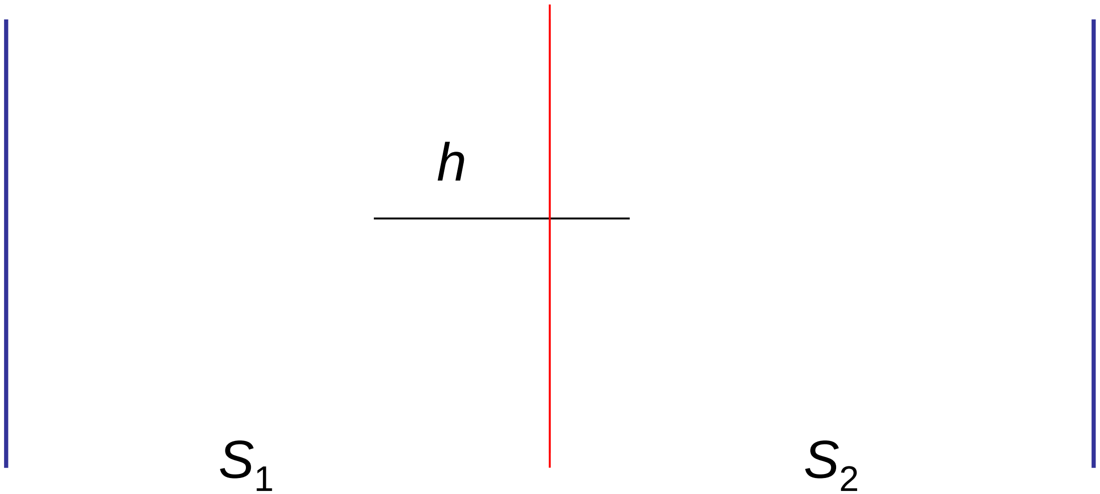
Case 2: only one endpoint of h in S_1

2 a) right endpoint in S_1

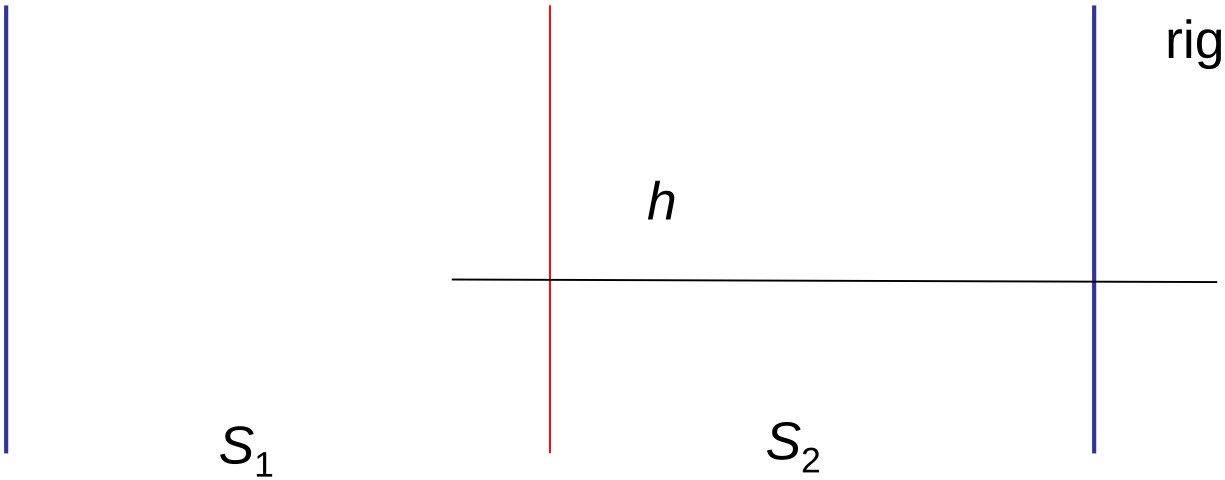


ReportCuts

2 b) left endpoint of h in S_1



right endpoint in S_2

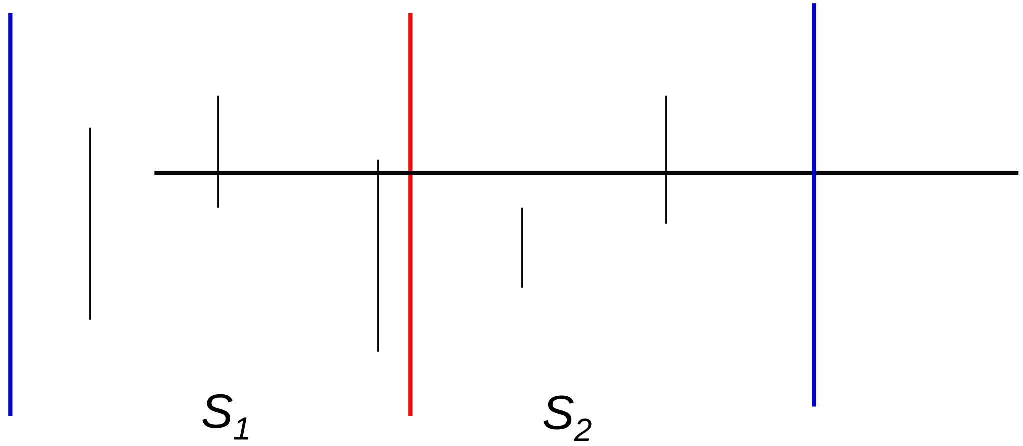


right endpoint not in S_2

Procedure: ReportCuts(S)

3. Merge:

Return the intersections of vertical line segments in S_2 with horizontal line segments in S_1, for which the left endpoint is in S_1 and the right endpoint is neither in S_1 nor in S_2 .
Proceed analogously for S_1 .



Implementation

Set S

$L(S)$: y-coordinates of all segments whose left endpoint in S, but right endpoint is not in S.

$R(S)$: y-coordinates of all segments whose right endpoint is in S, but left endpoint is not in S.

$V(S)$: y-intervals of all vertical line segments in S.

Base cases

S contains only one element e.

Case 1: $e = (x,y)$ is a left endpoint of horizontal line segment s

$L(S) = \{(y,s)\}$ $R(S) = \emptyset$ $V(S) = \emptyset$

Case 2: $e = (x,y)$ is a right endpoint of horizontal line segment s

$L(S) = \emptyset$ $R(S) = \{(y,s)\}$ $V(S) = \emptyset$

Case 3: $e = (x, y_1, y_2)$ is a vertical line segment s

$L(S) = \emptyset$ $R(S) = \emptyset$ $V(S) = \{([y_1, y_2],s)\}$

Merge step

Assume that $L(S_i)$, $R(S_i)$, $V(S_i)$ are known for i = 1,2.
$S = S_1 \cup S_2$

$$L(S) = L(S_1)\backslash R(S_2) \cup L(S_2)$$

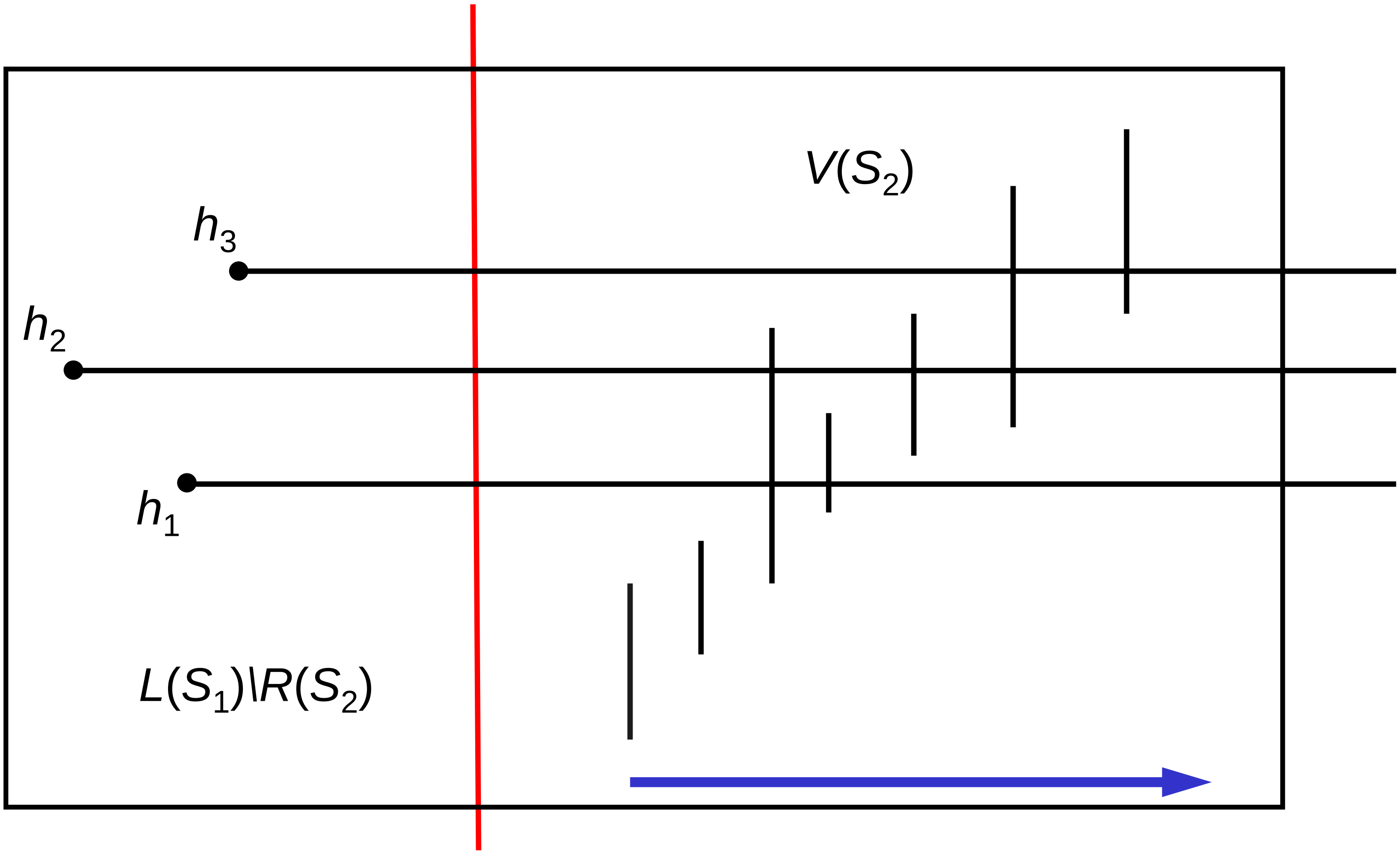
$$R(S) = R(S_2)\backslash L(S_1) \cup R(S_1)$$

$$V(S) = V(S_1) \cup V(S_2)$$

L, R: ordered by increasing y-coordinates (and segment number)
linked lists

V: ordered by increasing lower endpoints
linked list

Output of the intersections



Running time

Initially, the input (vertical line segments, left/right endpoints of horizontal line segments) has to be sorted and stored in an array.

Divide-and-conquer:

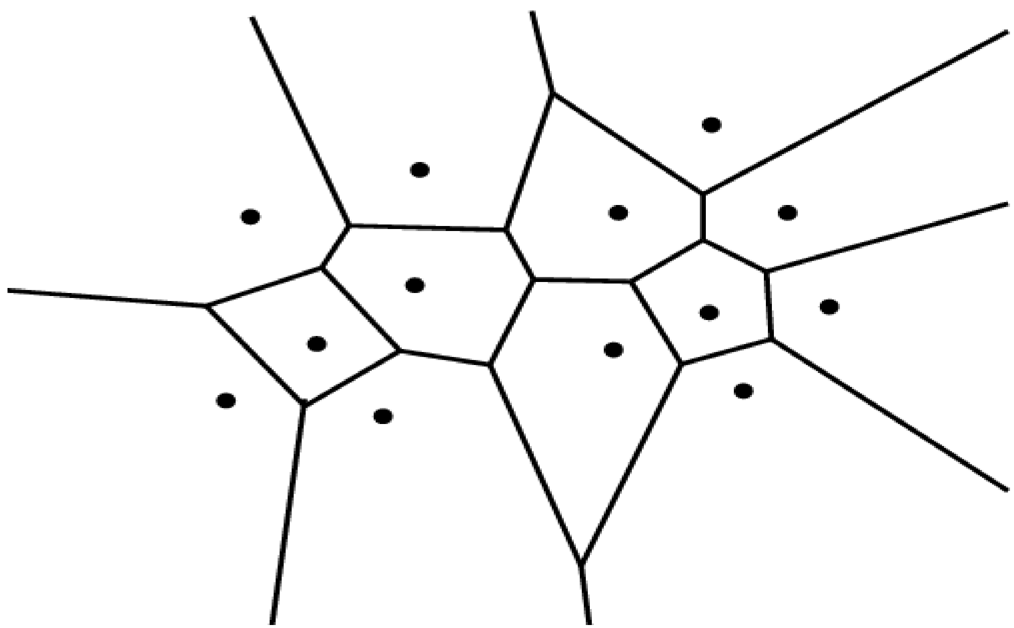
$$T(n) = 2T(n/2) + a \cdot n + \text{size of output}$$
$$T(1) = O(1)$$

$O(n \log n + k)$ k = # intersections

Computation of a Voronoi diagram

Input: Set of sites

Output: Partition of the plane into regions, each consisting of the points closer to one particular site than to any other site.



Definition of Voronoi diagrams

P : Set of sites

$H(p \mid p') = \{x \mid x$ is closer to p than to $p'\}$

Voronoi region of *p:*

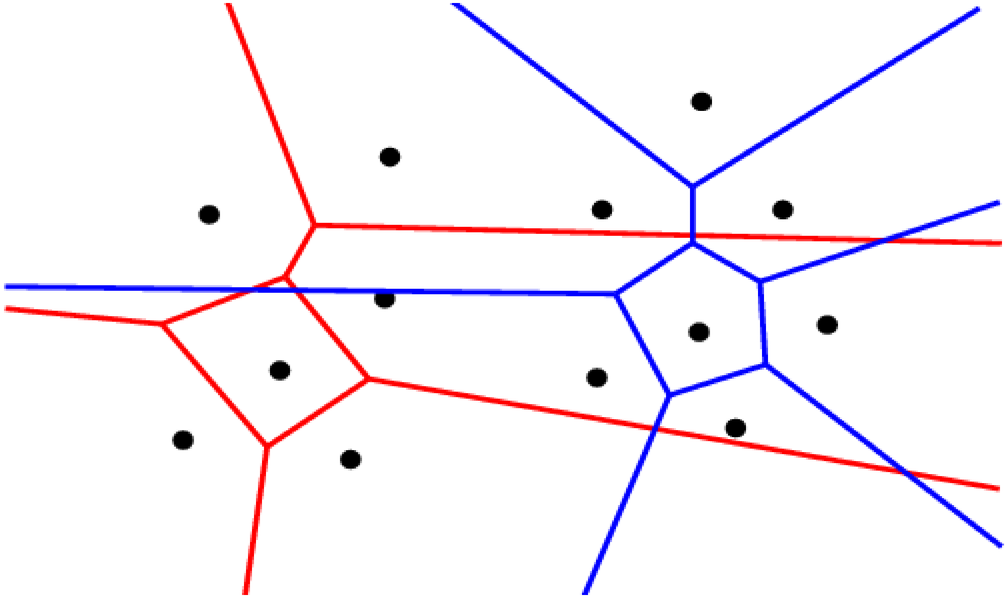
$$VR(p) = \bigcap_{p' \in P \setminus \{p\}} H(p \mid p')$$

Computation of a Voronoi Diagram

Divide: Partition the set of sites into two equal sized sets.

Conquer: Recursive computation of the two smaller Voronoi diagrams.

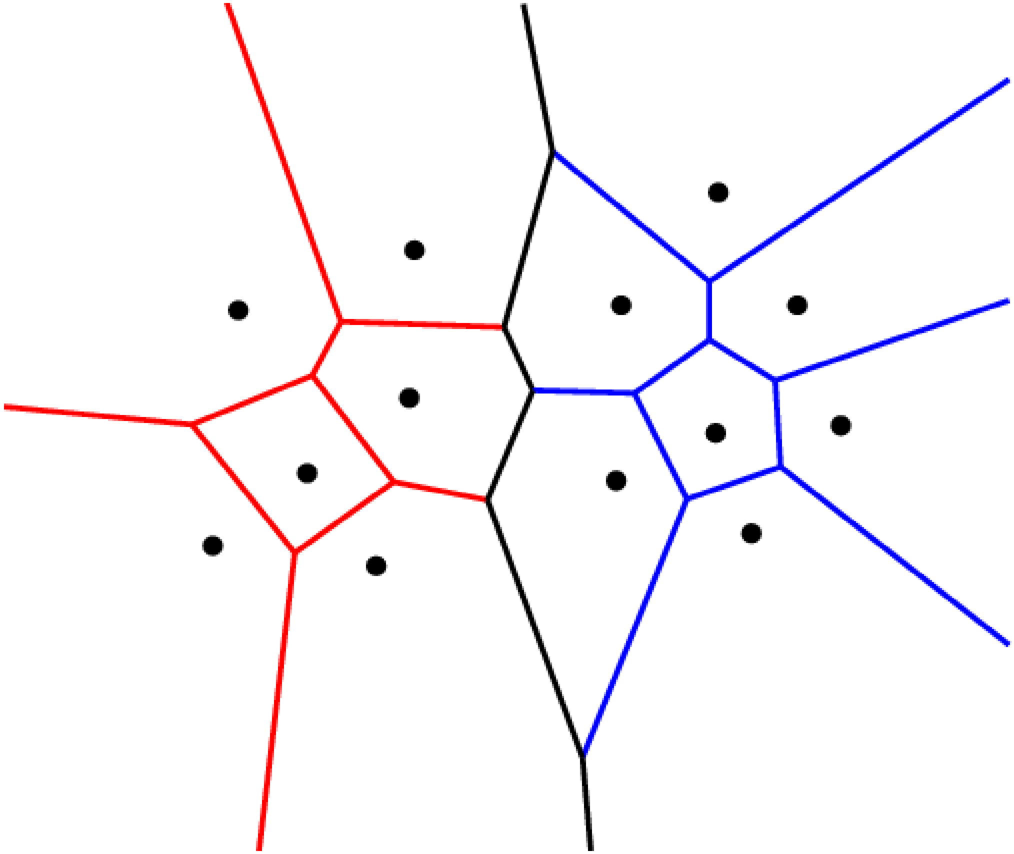
Stopping condition: The Voronoi diagram of a single site is the whole plane.



Merge: Connect the diagrams by adding new edges.

Computation of a Voronoi diagram

Output: The complete Voronoi diagram.



Running time: O(n log n), where n is the number of sites.