

5.2 Red Black Trees

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4. If a node is red then both its children are black.

5.2 Red Black Trees

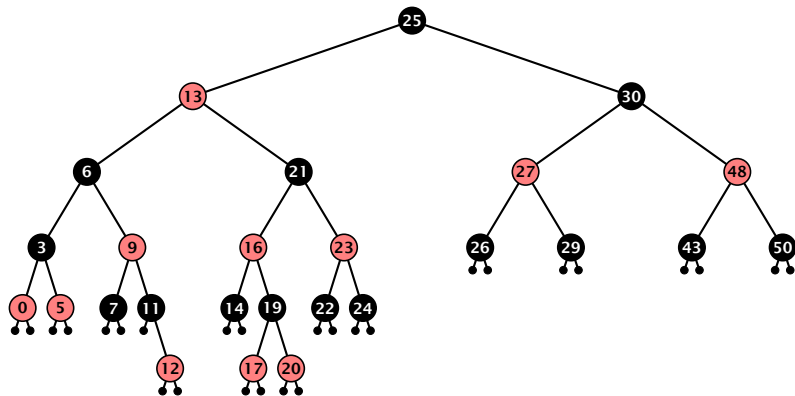
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The **null**-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

Red Black Trees: Example



5.2 Red Black Trees

Lemma 4

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The **black height** $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

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We first show:

Lemma 6

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

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Proof of Lemma 6.

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- ▶ If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

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- ▶ The black height of v is 0.
- ▶ The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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induction step

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- ▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.

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- ▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- ▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.



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Hence, the black height of the root is at least $h/2$.

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence,
 $2^{h/2} - 1 \leq n$.

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence,
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Hence, $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$. □

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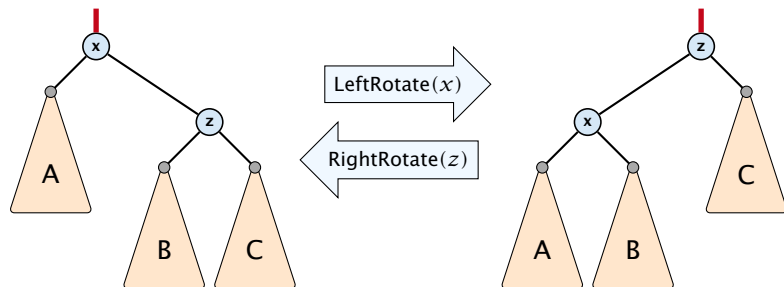
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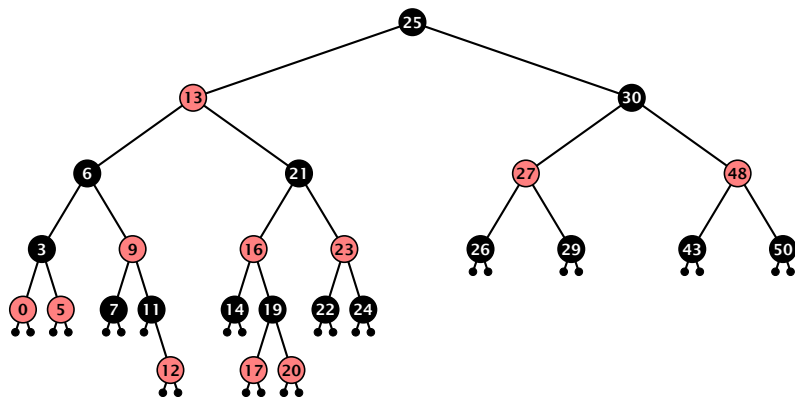
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

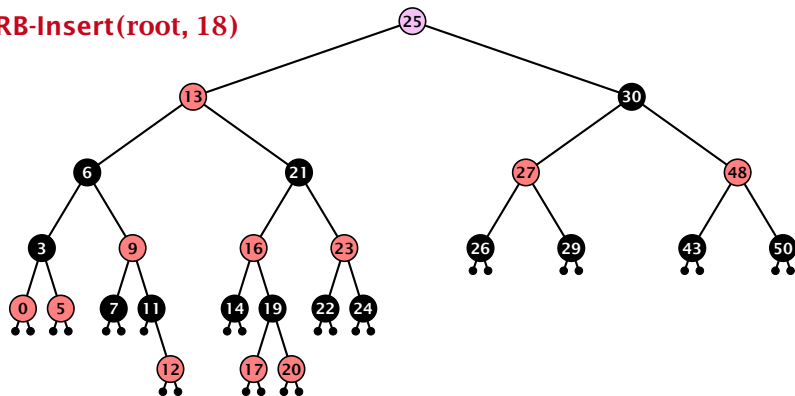


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

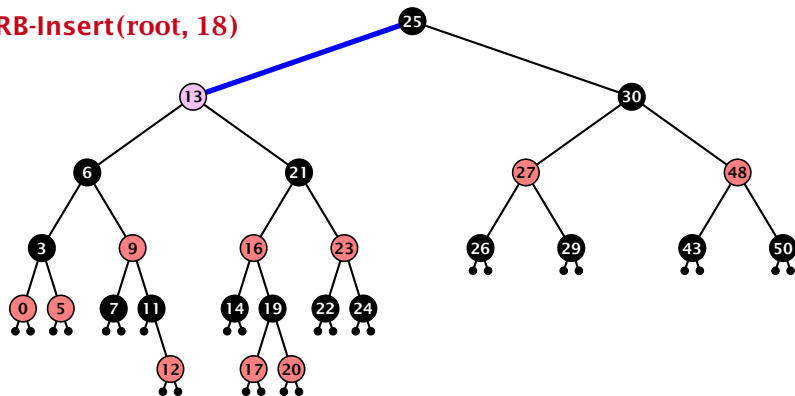


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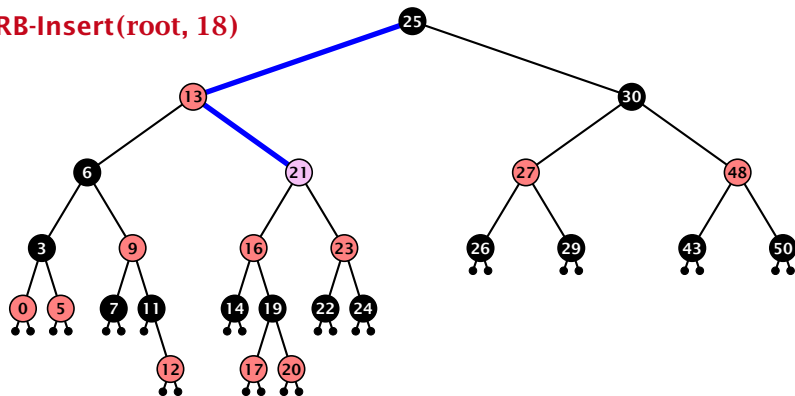


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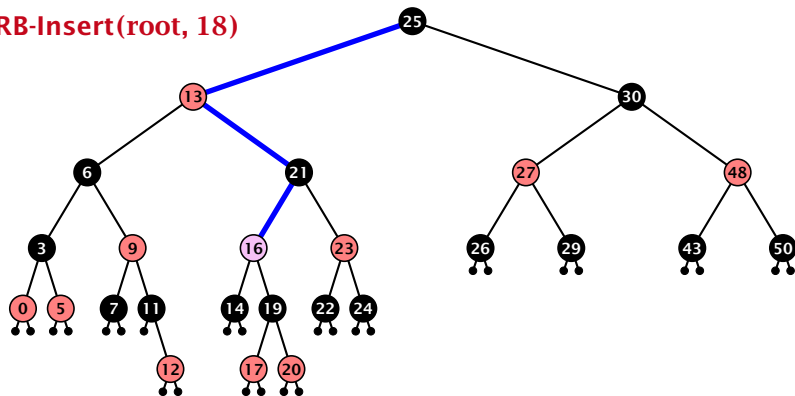


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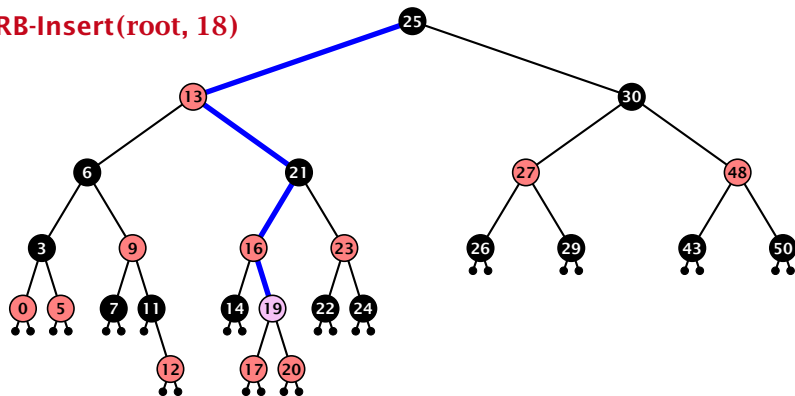


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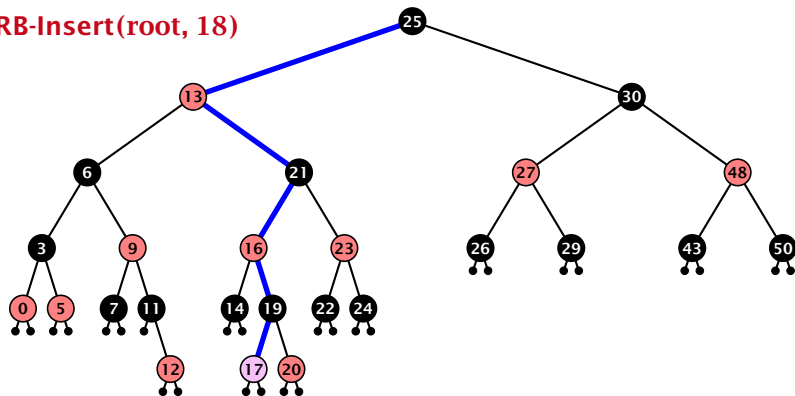


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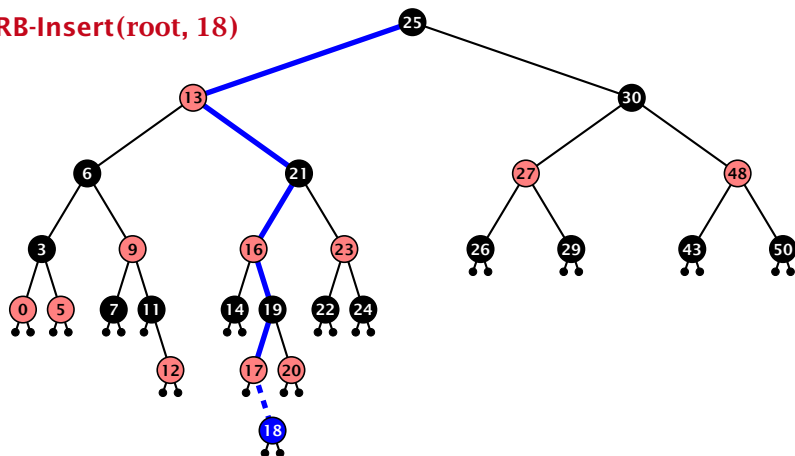


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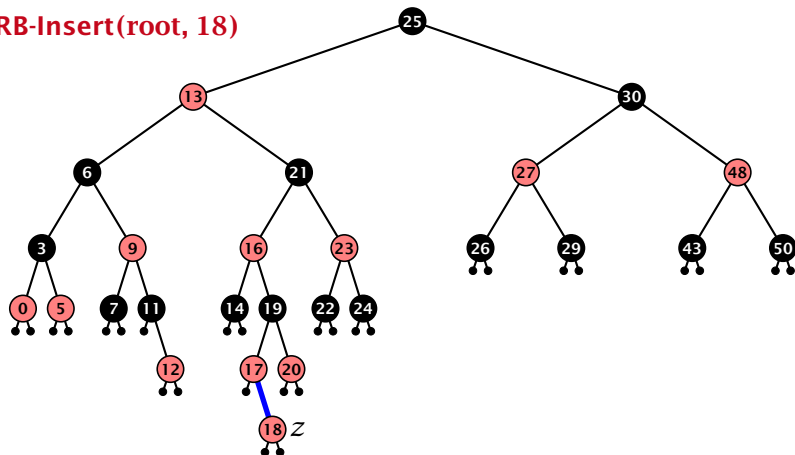


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 - ▶ or the parent does not exist
(violation since root must be black)

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- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$
 - ▶ either both of them are red
(most important case)
 - ▶ or the parent does not exist
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If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 6 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
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4:     if col[ $uncle$ ] = red then Case 1: uncle red
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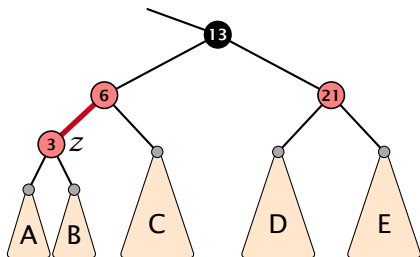
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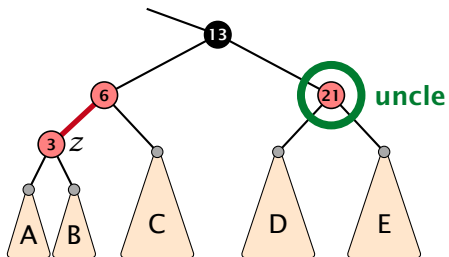
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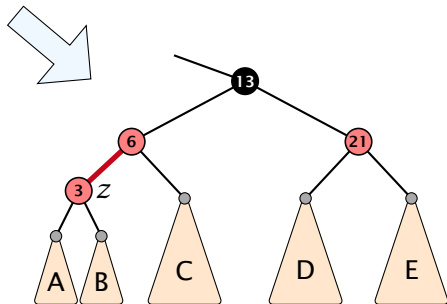
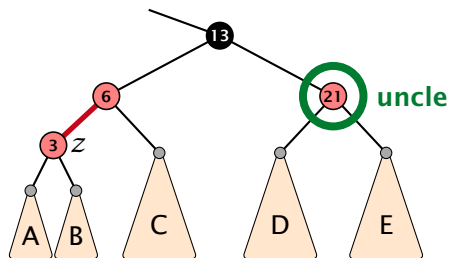
Case 1: Red Uncle



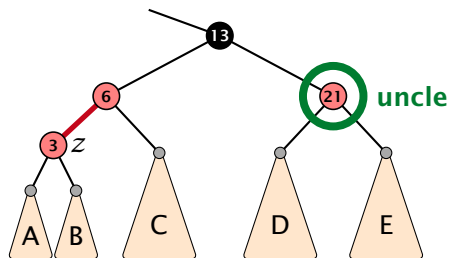
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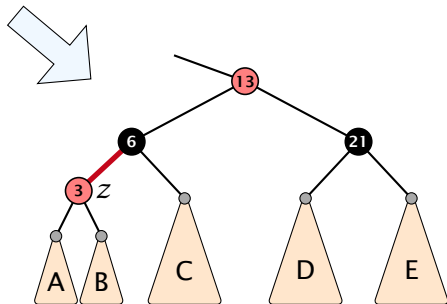
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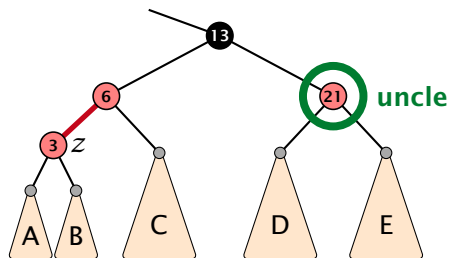
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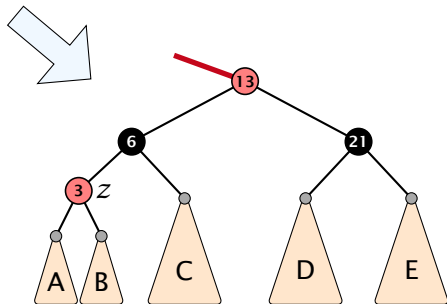
1. recolour



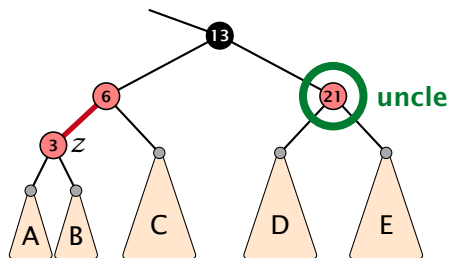
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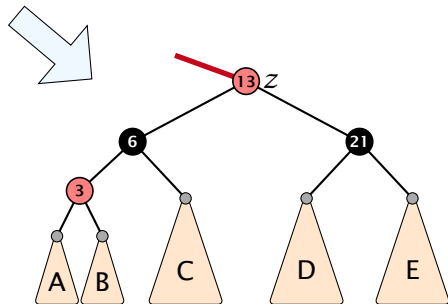
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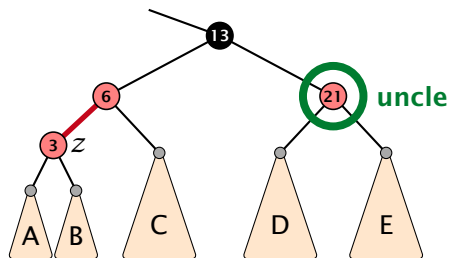
Case 1: Red Uncle



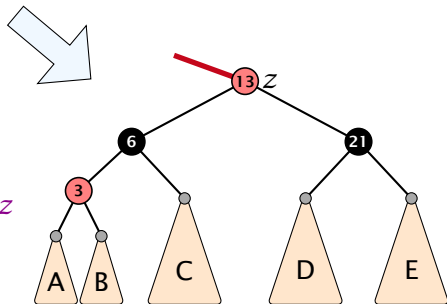
1. recolour
2. move z to grand-parent



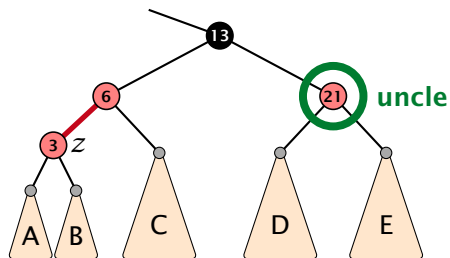
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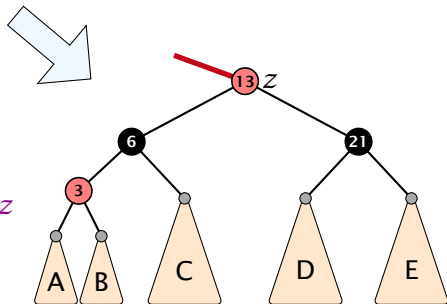
1. recolor
2. move z to grand-parent
3. invariant is fulfilled for new z



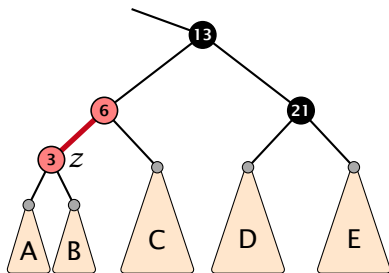
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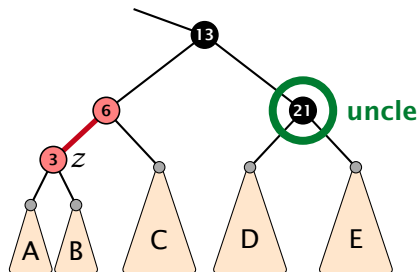
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress



Case 2b: Black uncle and z is left child

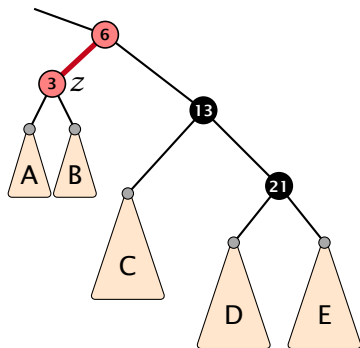
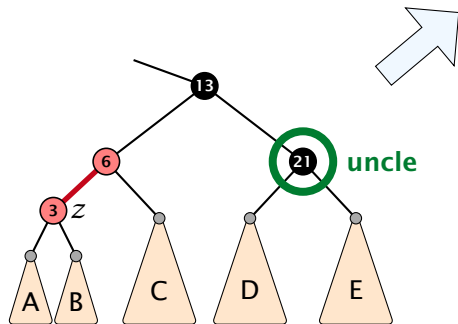


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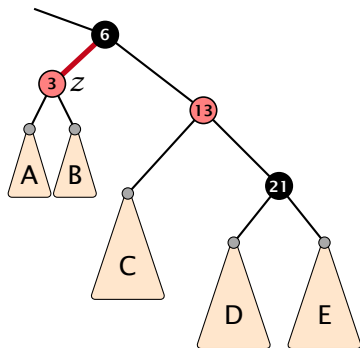
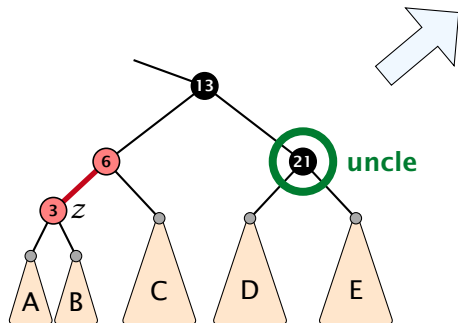
Case 2b: Black uncle and z is left child

1. rotate around grandparent



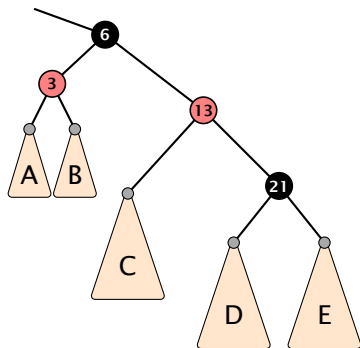
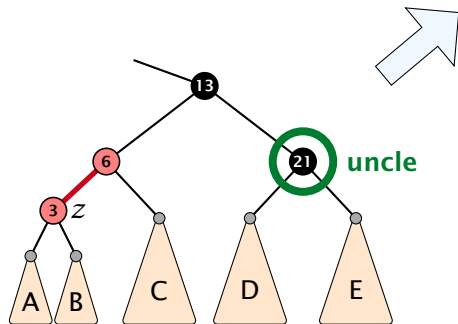
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

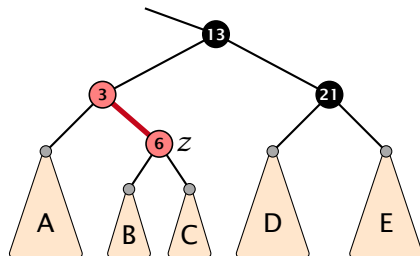


Case 2b: Black uncle and z is left child

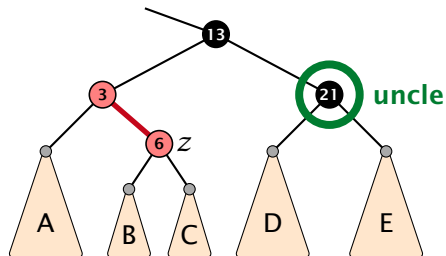
1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



Case 2a: Black uncle and z is right child

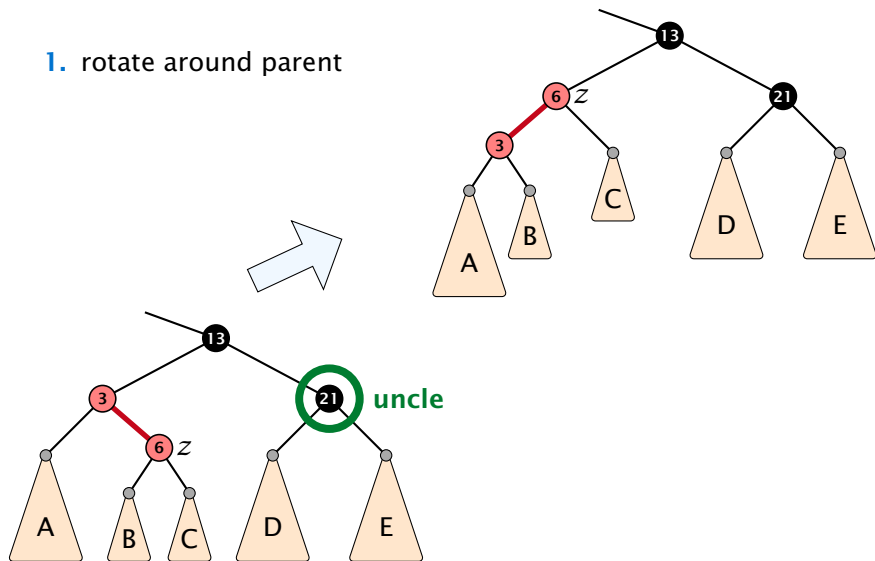


Case 2a: Black uncle and z is right child



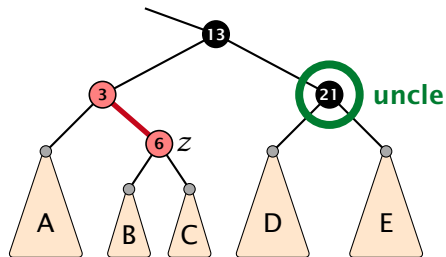
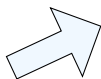
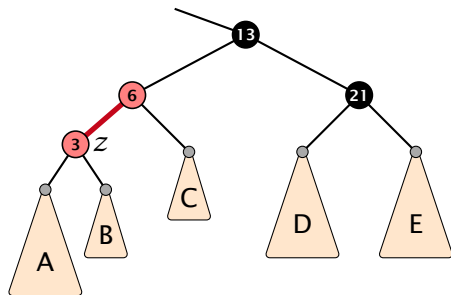
Case 2a: Black uncle and z is right child

1. rotate around parent



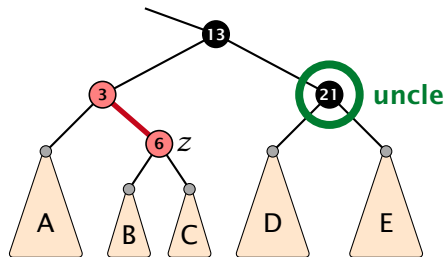
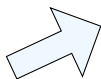
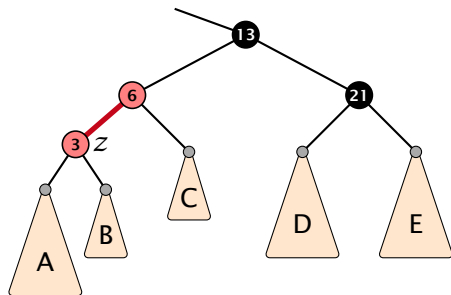
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards



Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree
- ▶ Case 2b \rightarrow red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree
- ▶ Case 2b \rightarrow red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

Red Black Trees: Delete

First do a standard delete.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

Red Black Trees: Delete

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If it was black there may be the following problems.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

Red Black Trees: Delete

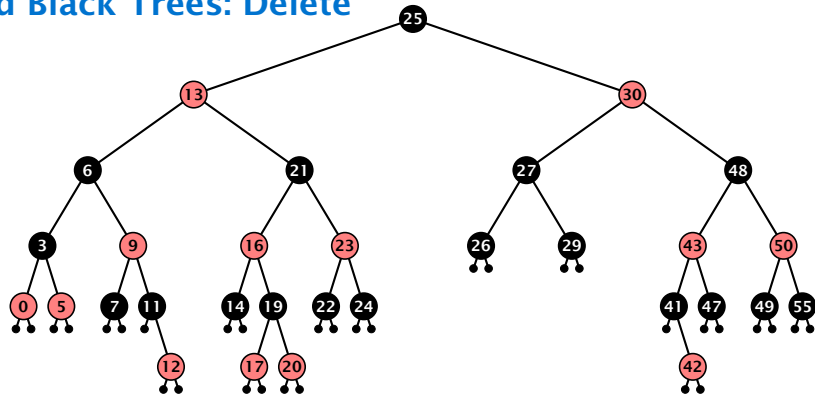
First do a standard delete.

If the spliced out node x was red everything is fine.

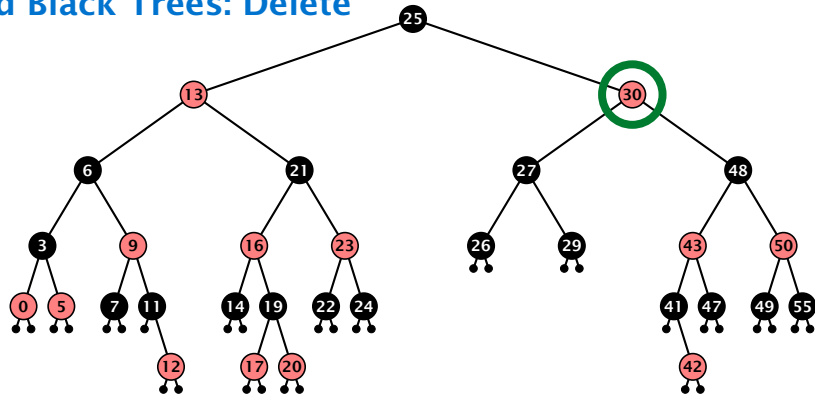
If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

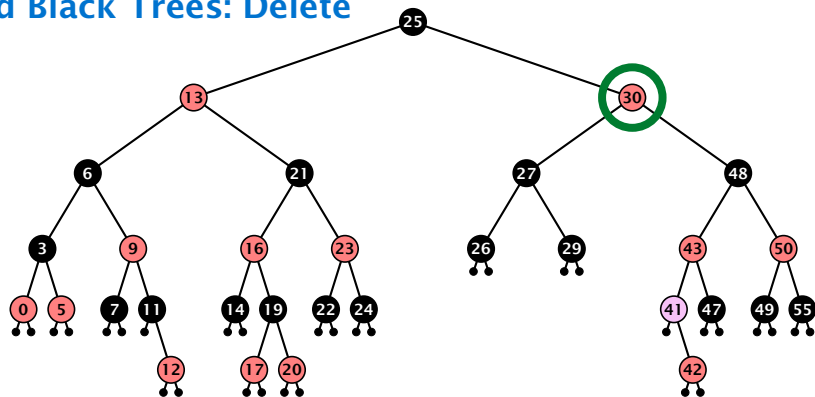


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

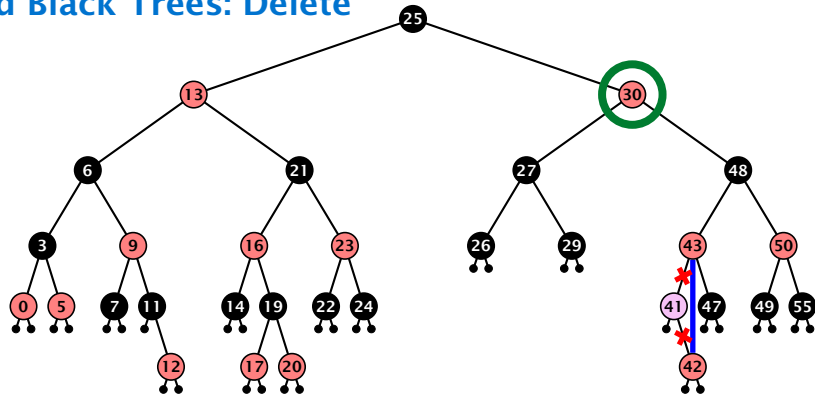


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Red Black Trees: Delete

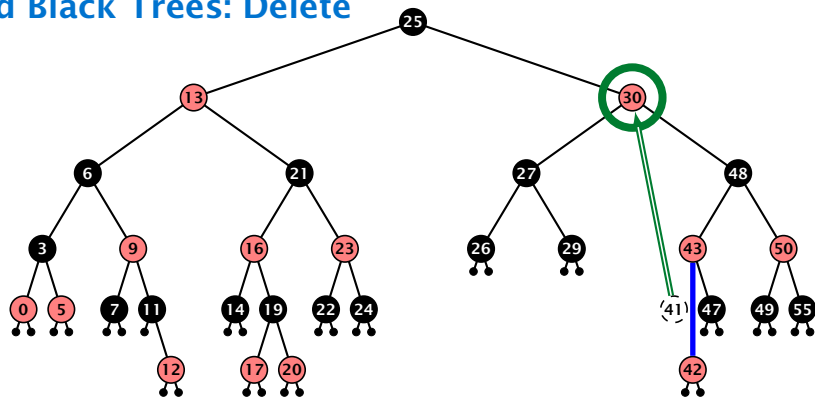


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Element has two children

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Red Black Trees: Delete

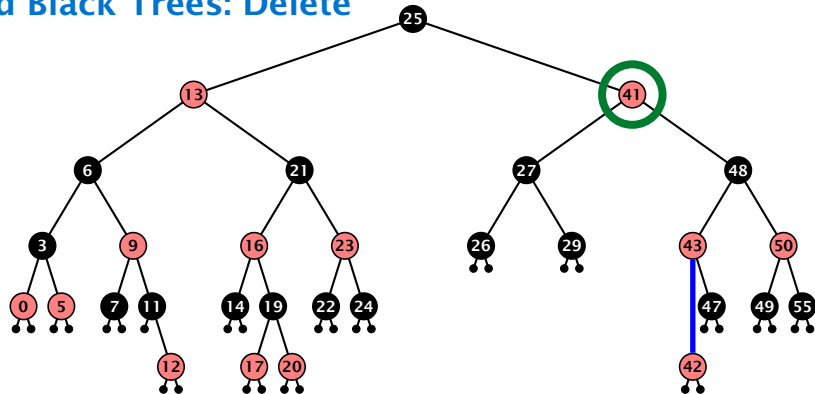


Case 3:

Element has two children

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Red Black Trees: Delete

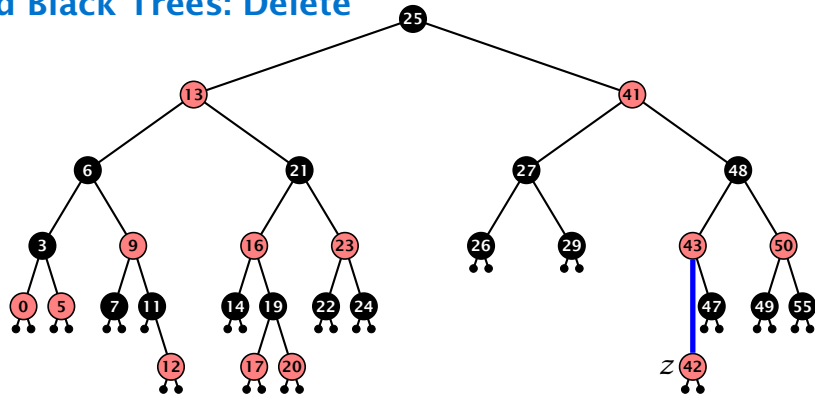


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

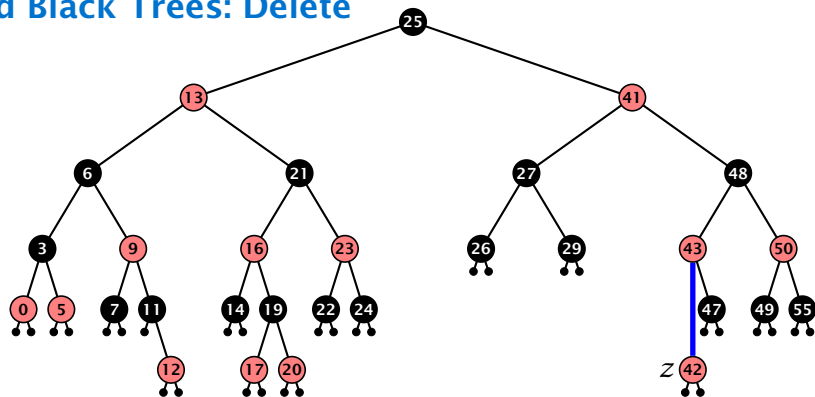
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property

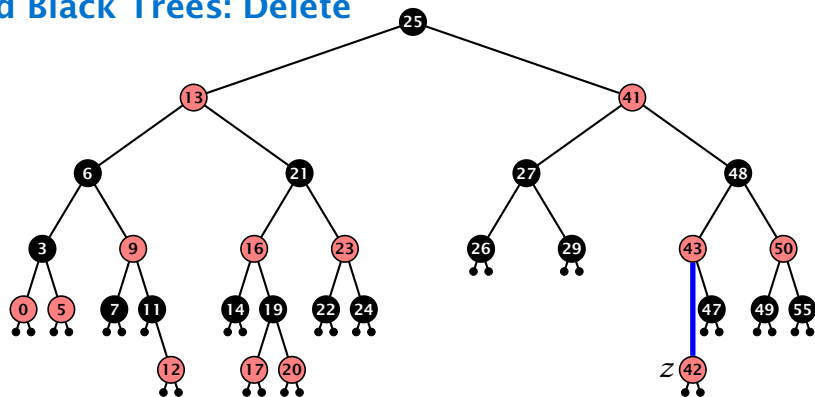
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

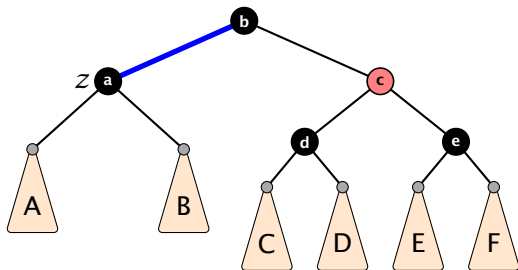
Red Black Trees: Delete

Invariant of the fix-up algorithm

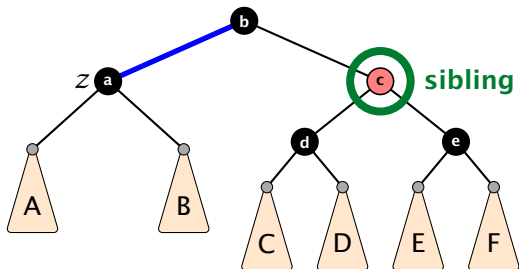
- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

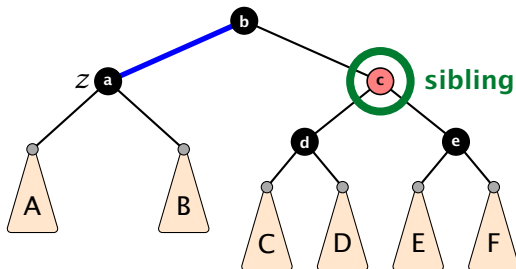
Case 1: Sibling of z is red



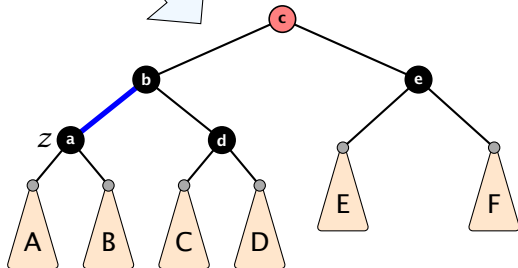
Case 1: Sibling of z is red



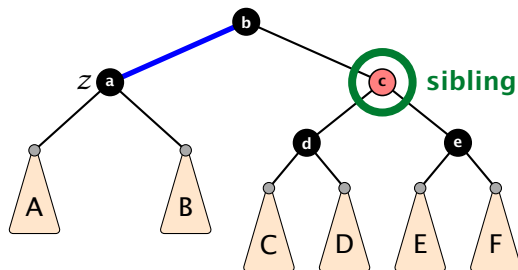
Case 1: Sibling of z is red



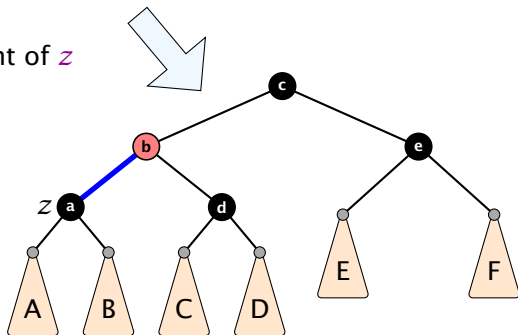
1. left-rotate around parent of z



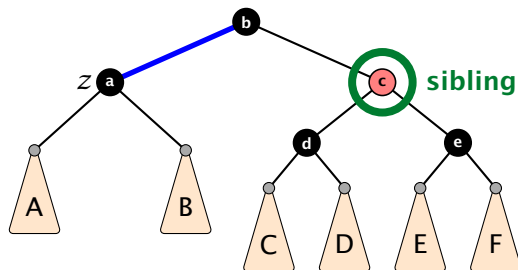
Case 1: Sibling of z is red



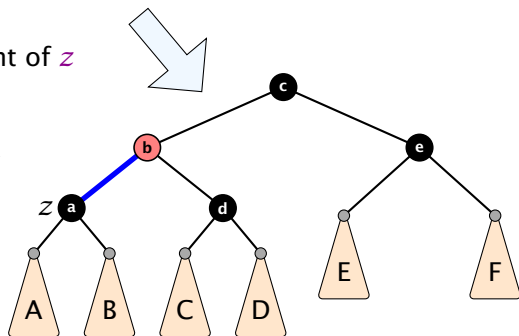
1. left-rotate around parent of z
2. recolor nodes b and c



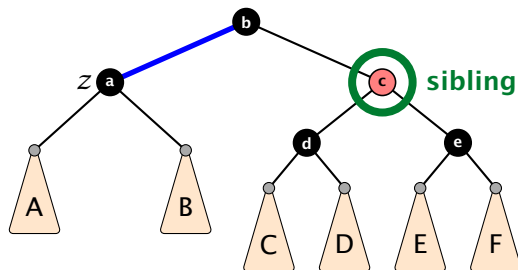
Case 1: Sibling of z is red



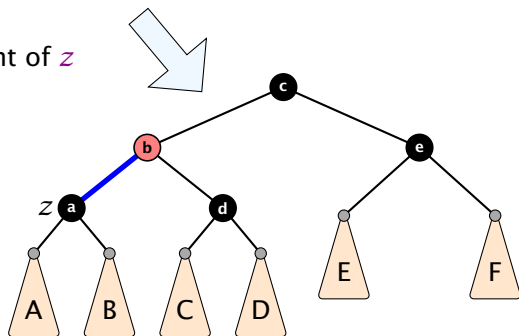
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)



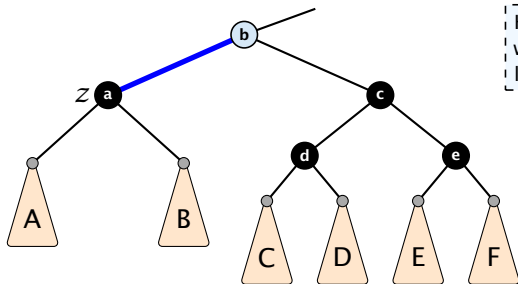
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4

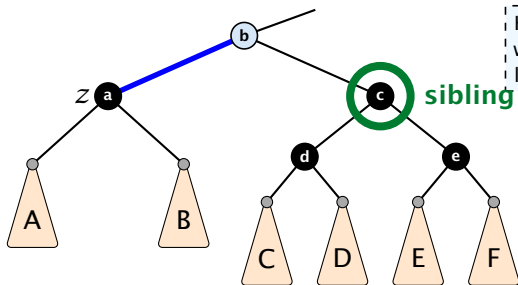


Case 2: Sibling is black with two black children



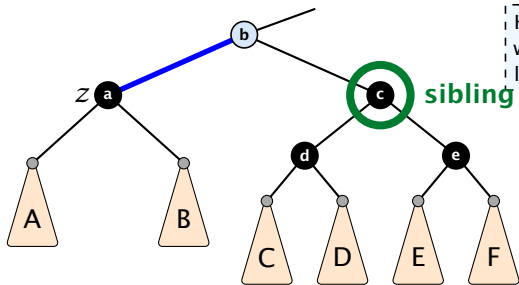
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

Case 2: Sibling is black with two black children

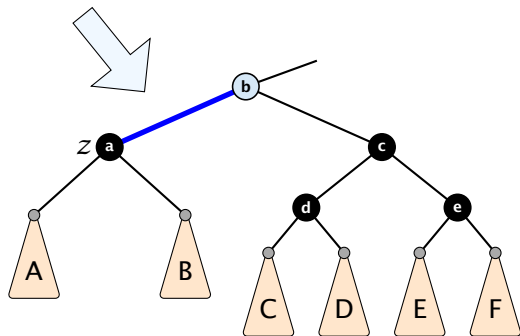


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

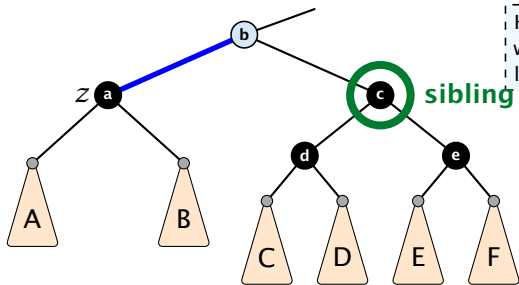
Case 2: Sibling is black with two black children



Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

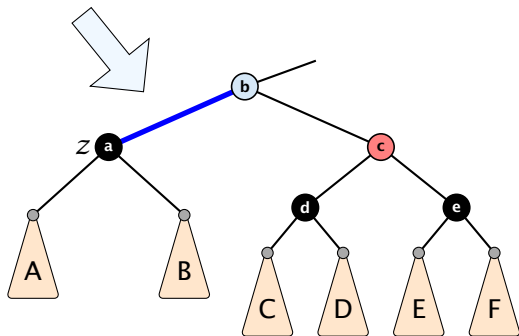


Case 2: Sibling is black with two black children

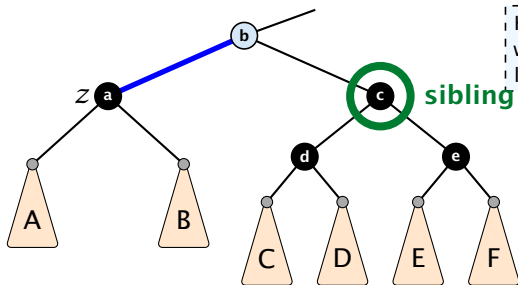


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**

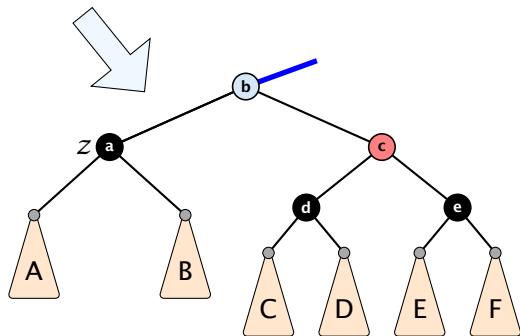


Case 2: Sibling is black with two black children

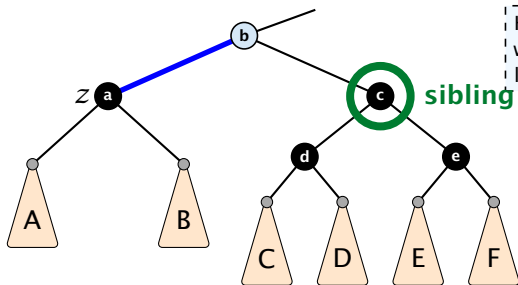


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards

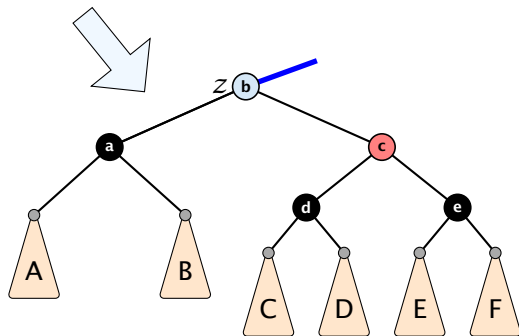


Case 2: Sibling is black with two black children

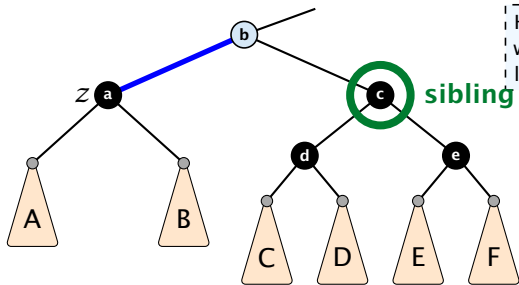


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards

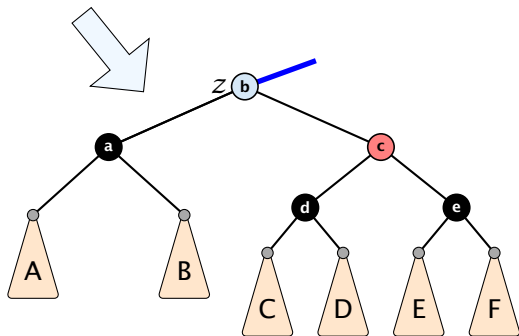


Case 2: Sibling is black with two black children

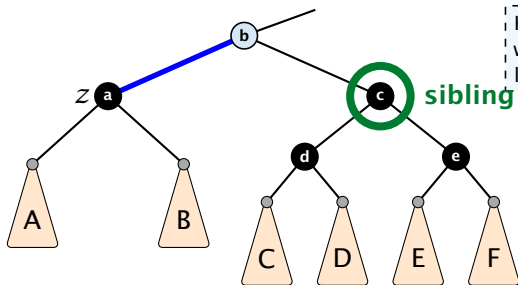


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress

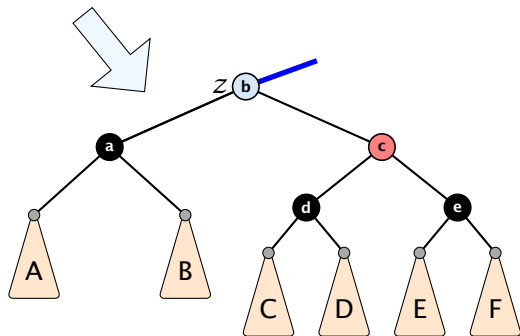


Case 2: Sibling is black with two black children

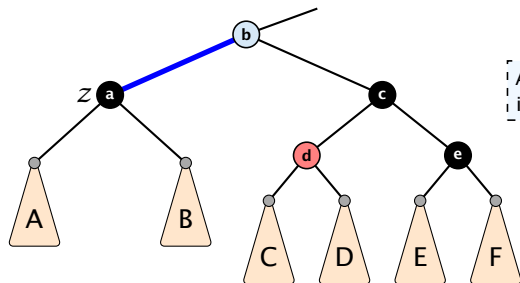


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done

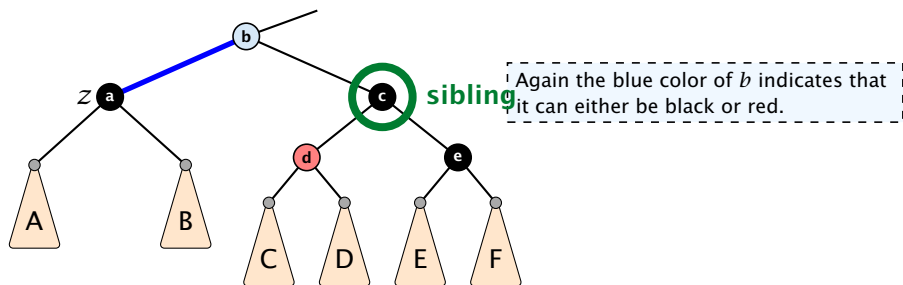


Case 3: Sibling black with one black child to the right



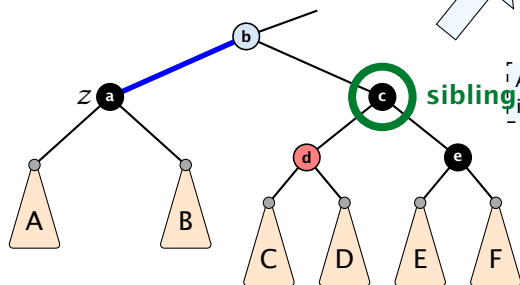
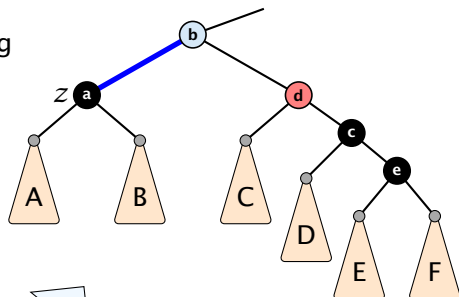
Again the blue color of b indicates that it can either be black or red.

Case 3: Sibling black with one black child to the right



Case 3: Sibling black with one black child to the right

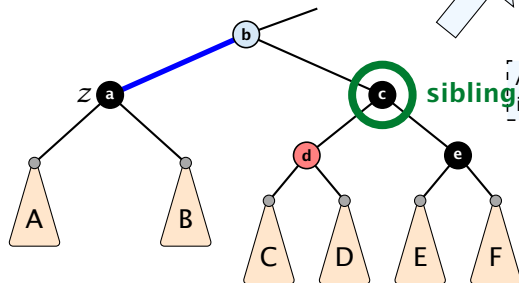
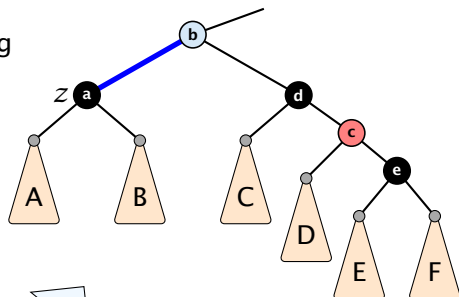
1. do a right-rotation at sibling



Again the blue color of b indicates that it can either be black or red.

Case 3: Sibling black with one black child to the right

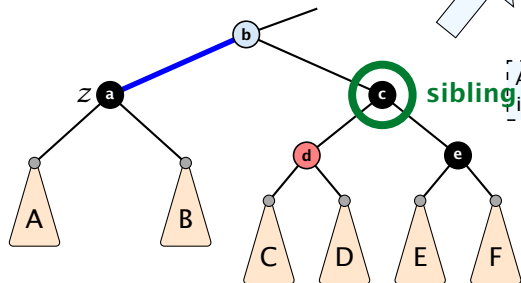
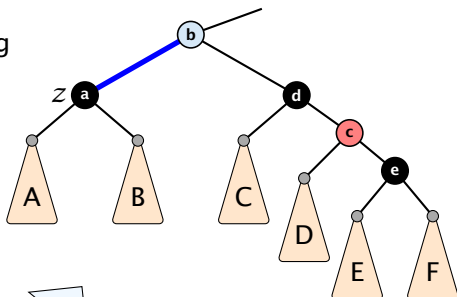
1. do a right-rotation at sibling
2. recolor c and d



Again the blue color of b indicates that it can either be black or red.

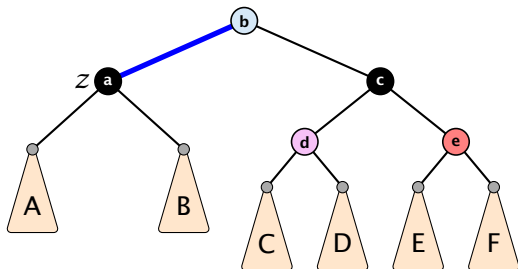
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



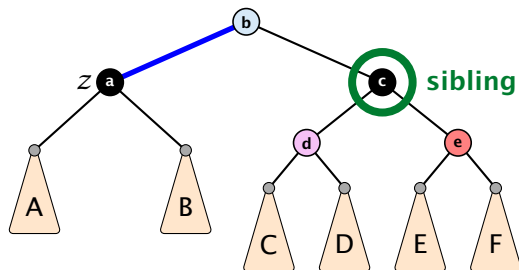
Again the blue color of b indicates that it can either be black or red.

Case 4: Sibling is black with red right child



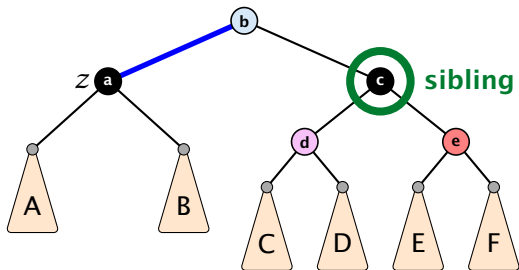
- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

Case 4: Sibling is black with red right child



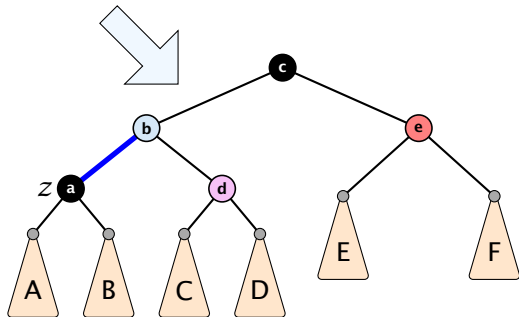
- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

Case 4: Sibling is black with red right child

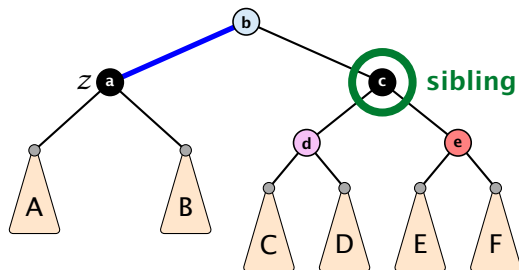


- Here **b** and **d** are either red or black but have possibly different colors.
- We recolor **c** by giving it the color of **b**.

1. left-rotate around **b**

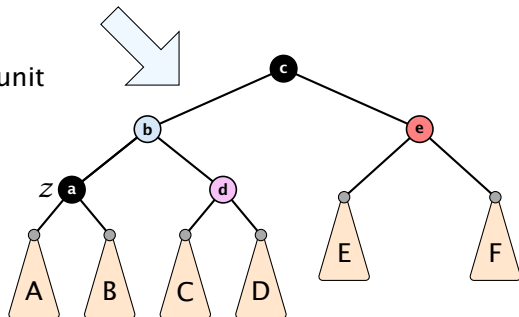


Case 4: Sibling is black with red right child

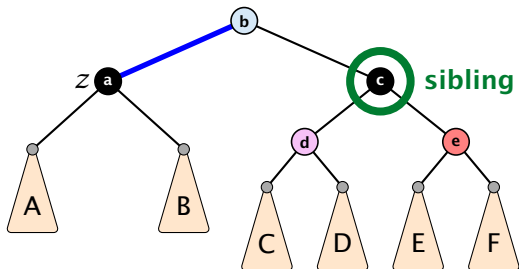


- Here **b** and **d** are either red or black but have possibly different colors.
- We recolor **c** by giving it the color of **b**.

1. left-rotate around **b**
2. remove the fake black unit

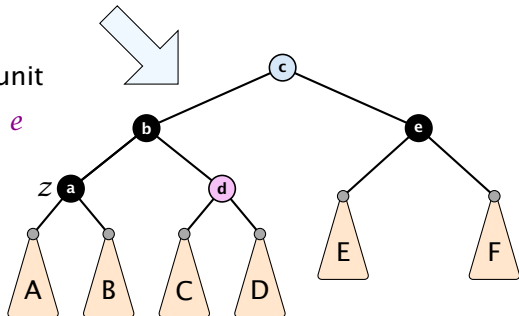


Case 4: Sibling is black with red right child

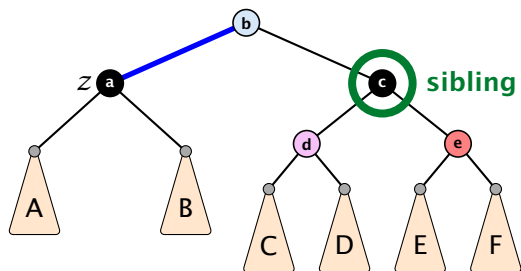


- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b .

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b , c , and e

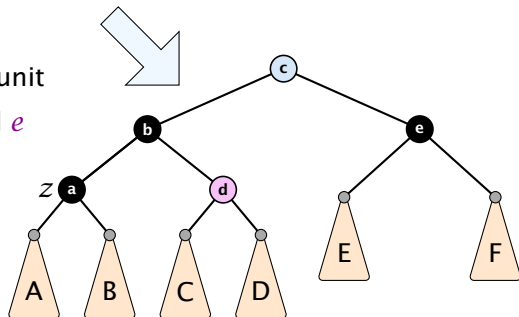


Case 4: Sibling is black with red right child



- Here **b** and **d** are either red or black but have possibly different colors.
- We recolor **c** by giving it the color of **b**.

1. left-rotate around **b**
2. remove the fake black unit
3. recolor nodes **b**, **c**, and **e**
4. you have a valid red black tree



Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
Case 1 → Case 3 → Case 4 → red black tree
Case 1 → Case 4 → red black tree

Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
Case 1 → Case 3 → Case 4 → red black tree
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree

Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
Case 1 → Case 3 → Case 4 → red black tree
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
Case 1 → Case 3 → Case 4 → red black tree
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.