10 van Emde Boas Trees

Dynamic Set Data Structure S:

- $S.\mathrm{insert}(x)$
- $S.\mathrm{delete}(x)$
- $S.\mathrm{search}(x)$
- $S.\mathrm{min}()$
- $S.\mathrm{max}()$
- $S.\mathrm{succ}(x)$
- $S.\mathrm{pred}(x)$

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For this chapter we ignore the problem of storing satellite data:

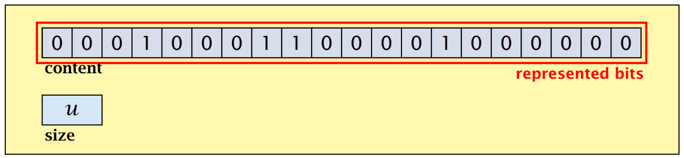
- $\boldsymbol{S}.\mathbf{insert}(\boldsymbol{x})$**:** Inserts x into S.
- $\boldsymbol{S}.\mathbf{delete}(\boldsymbol{x})$**:** Deletes x from S. Usually assumes that $x \in S$.
- $\boldsymbol{S}.\mathbf{member}(\boldsymbol{x})$**:** Returns 1 if $x \in S$ and 0 otw.
- $\boldsymbol{S}.\mathbf{min}()$**:** Returns the value of the minimum element in S.
- $\boldsymbol{S}.\mathbf{max}()$**:** Returns the value of the maximum element in S.
- $\boldsymbol{S}.\mathbf{succ}(\boldsymbol{x})$**:** Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.
- $\boldsymbol{S}.\mathbf{pred}(\boldsymbol{x})$**:** Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

10 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u-1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 array.insert(x)
content[x] $\leftarrow 1$;

Algorithm 2 array.delete(x)
content[x] $\leftarrow 0$;

Algorithm 3 array.member(x)
return content[x];

- Note that we assume that x is valid, i.e., it falls within the array boundaries.
- Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 array.max()
for (i = size -1; $i \geq 0$; i--) **do**
  if content[i] $= 1$ **then return** i;
return null;

Algorithm 5 array.min()
for ($i = 0$; $i <$ size; i++) **do**
  if content[i] $= 1$ **then return** i;
return null;

- Running time is $\mathcal{O}(u)$ in the worst case.

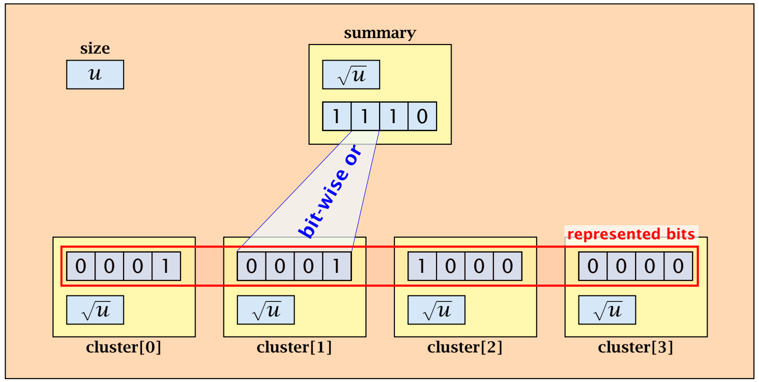
Implementation 1: Array

Algorithm 6 array.succ(x)
for ($i = x + 1$; $i <$ size; i++) **do**
  if content[i] $= 1$ **then return** i;
return null;

Algorithm 7 array.pred(x)
for ($i = x - 1$; $i \geq 0$; i--) **do**
  if content[i] $= 1$ **then return** i;
return null;

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\mathrm{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\mathrm{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 8 member(x)

return cluster[high(x)].member(low(x));

Algorithm 9 insert(x)

cluster[high(x)].insert(low(x));
summary.insert(high(x));

- The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 10 delete(x)

cluster[high(x)].delete(low(x));
if cluster[high(x)].min() = null **then**
    summary.delete(high(x));

- The running time is dominated by the cost of a minimum computation on an array of size $\sqrt{u}$. Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 max()

maxcluster ← summary.max();
if *maxcluster* = null **return** null;
offs ← cluster[*maxcluster*].max()
return *maxcluster* ∘ *offs*;

The operator ∘ stands for the concatenation of two bitstrings. This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

Algorithm 12 min()

mincluster ← summary.min();
if *mincluster* = null **return** null;
offs ← cluster[*mincluster*].min();
return *mincluster* ∘ *offs*;

- Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 succ(x)

$m \leftarrow$ cluster[high(x)].succ(low(x))
if $m \neq$ null **then return** high(x) $\circ$ m;
succcluster $\leftarrow$ summary.succ(high(x));
if *succcluster* $\neq$ null **then**
    offs $\leftarrow$ cluster[*succcluster*].min();
    return *succcluster* $\circ$ *offs*;
return null;

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 pred(x)

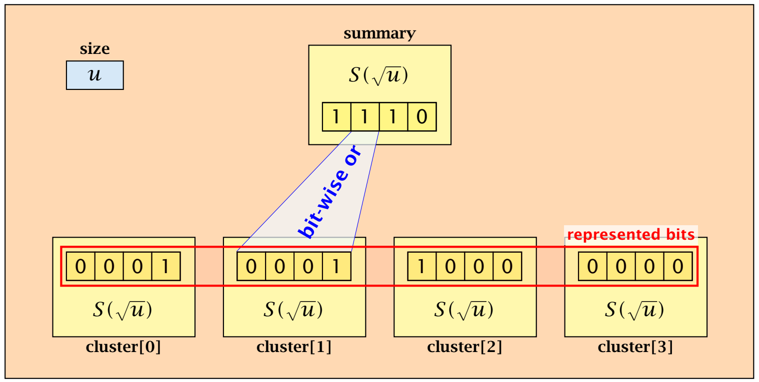
$m \leftarrow$ cluster[high(x)].pred(low(x))
if $m \neq$ null **then return** high(x) $\circ$ m;
predcluster $\leftarrow$ summary.pred(high(x));
if *predcluster* $\neq$ null **then**
    offs $\leftarrow$ cluster[*predcluster*].max();
    return *predcluster* $\circ$ *offs*;
return null;

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\text{min}()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\text{min}()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 member(x)

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x))$;

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 16 insert(x)

1: $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x))$;
2: $\text{summary}.\text{insert}(\text{high}(x))$;

- $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 17 delete(x)

1: $\text{cluster}[\text{high}(x)].\text{delete}(\text{low}(x))$;
2: **if** $\text{cluster}[\text{high}(x)].\text{min}() = \text{null}$ **then**
3: $\quad$ $\text{summary}.\text{delete}(\text{high}(x))$;

- $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 18 min()

1: *mincluster* ← summary.min();
2: **if** *mincluster* = null **return** null;
3: *offs* ← cluster[*mincluster*].min();
4: **return** *mincluster* ∘ *offs*;

- $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 19 succ(x)

1: m ← cluster[high(x)].succ(low(x))
2: **if** $m \neq$ null **then return** high(x) ∘ m;
3: *succcluster* ← summary.succ(high(x));
4: **if** *succcluster* ≠ null **then**
5: *offs* ← cluster[*succcluster*].min();
6: **return** *succcluster* ∘ *offs*;
7: **return** null;

- $T_{\mathrm{succ}}(u) = 2T_{\mathrm{succ}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1}$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$
$$= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\mathrm{mem}}(u) = \mathcal{O}(\log\log u)$.

Implementation 3: Recursion

$\boldsymbol{T_{\mathrm{ins}}(u) = 2T_{\mathrm{ins}}(\sqrt{u}) + 1}$.

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\mathrm{ins}}(2^\ell) = T_{\mathrm{ins}}(u) = 2T_{\mathrm{ins}}(\sqrt{u}) + 1$$
$$= 2T_{\mathrm{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\mathrm{ins}}(u) = \mathcal{O}(\log u)$.

The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

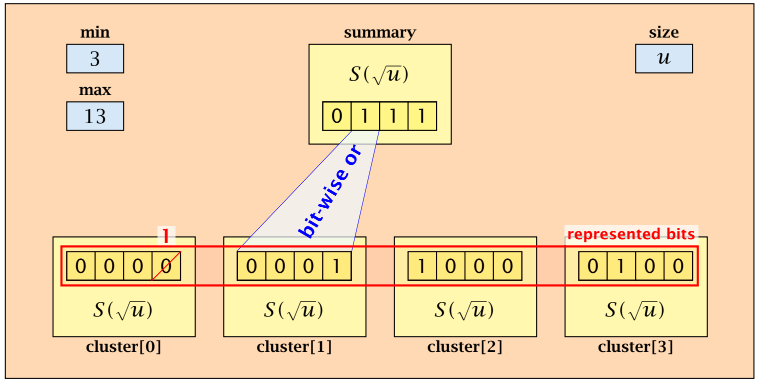
Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{del}}(2^\ell) &= T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c\log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X(\tfrac{\ell}{2}) + c\ell \ . \end{aligned}$$

Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log\log u)$.

The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- The bit referenced by min is not set within sub-datastructures.
- The bit referenced by max is set within sub-datastructures (if $\max \neq \min$).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- Recursive calls for min and max are constant time.
- $\min = \text{null}$ means that the data-structure is empty.
- $\min = \max \neq \text{null}$ means that the data-structure contains exactly one element.
- We can insert into an empty datastructure in constant time by only setting $\min = \max = x$.
- We can delete from a data-structure that just contains one element in constant time by setting $\min = \max = \text{null}$.

Implementation 4: van Emde Boas Trees

Algorithm 20 max()

1: **return** max;

Algorithm 21 min()

1: **return** min;

- Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 member(x)

if $x = \min$ **then return** 1; // TRUE
return $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x))$;

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 succ(x)

if $\min \neq \text{null} \wedge x < \min$ **then return** min;
$\textit{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\max()$;
if $\textit{maxincluster} \neq \text{null} \wedge \text{low}(x) < \textit{maxincluster}$ **then**
  $\textit{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$;
  return $\text{high}(x) \circ \textit{offs}$;
else
  $\textit{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$;
  if $\textit{succcluster} = \text{null}$ **then return** null;
  $\textit{offs} \leftarrow \text{cluster}[\textit{succcluster}].\min()$;
  return $\textit{succcluster} \circ \textit{offs}$;

- $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{succ}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 65 insert(x)

1: **if** $\min = \text{null}$ **then**
2:   $\min = x$; $\max = x$;
3: **else**
4:   **if** $x < \min$ **then** exchange x and min;
5:   **if** $\text{cluster}[\text{high}(x)].\min = \text{null}$; **then**
6:     $\text{summary}.\text{insert}(\text{high}(x))$;
7:     $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x))$;
8:   **else**
9:     $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x))$;
10:   **if** $x > \max$ **then** $\max = x$;

- $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{ins}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 7 takes constant time as the if-condition in Line 5 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 6 and in Line 9. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 65 delete(x)

1: **if** min = max **then**
2: min = null; max = null;
3: **else**
4: **if** x = min **then** — find new minimum
5: *firstcluster* ← summary.min();
6: *offs* ← cluster[*firstcluster*].min();
7: x ← *firstcluster* ∘ *offs*;
8: min ← x;
9: cluster[high(x)].delete(low(x)); — delete
10: **if** cluster[high(x)].min() = null **then**
11: summary.delete(high(x));

continued...

Implementation 4: van Emde Boas Trees

Algorithm 65 delete(x)

...continued — fix maximum

10: **if** x = max **then**
11: *summax* ← summary.max();
12: **if** *summax* = null **then** max ← min;
13: **else**
14: *offs* ← cluster[*summax*].max();
15: max ← *summax* ∘ *offs*

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\text{cluster}[\text{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\text{del}}(u) = \mathcal{O}(\log\log u)$.

10 van Emde Boas Trees

Space requirements:

- The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \ .$$

- Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- Let the "real" recurrence relation be

$$S(k^2) = (k+1)S(k) + c_1 \cdot k;\ S(4) = c_2$$

- Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k+1)R(k) + ck;\ R(4) = 1$$

where $c = c_1/c_2 < 1$.

- Now, we show $R(k) \leq k - 2$ for squares $k \geq 4$.
  - Obviously, this holds for $k = 4$.
  - For $k = \ell^2 > 4$ with ℓ integral we have

$$\begin{aligned} R(k) &= (1+\ell)R(\ell) + c\ell \\ &\leq (1+\ell)(\ell-2) + \ell \leq k - 2 \end{aligned}$$

  - This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

van Emde Boas Trees

Bibliography

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
Introduction to Algorithms (3rd ed.),
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See Chapter 20 of [CLRS90].