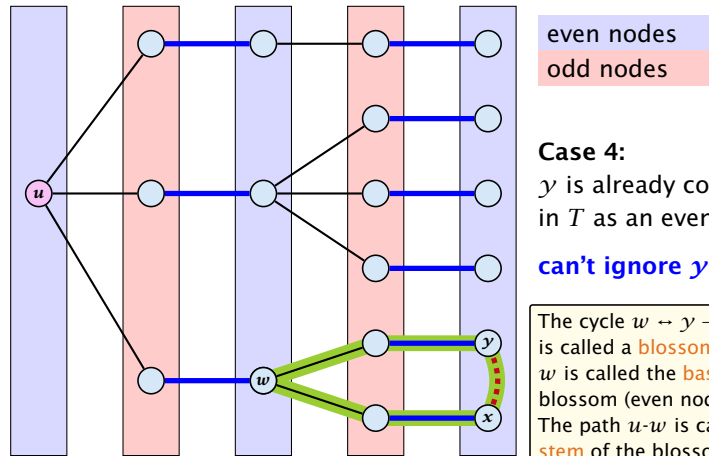


How to find an augmenting path?

Construct an alternating tree.



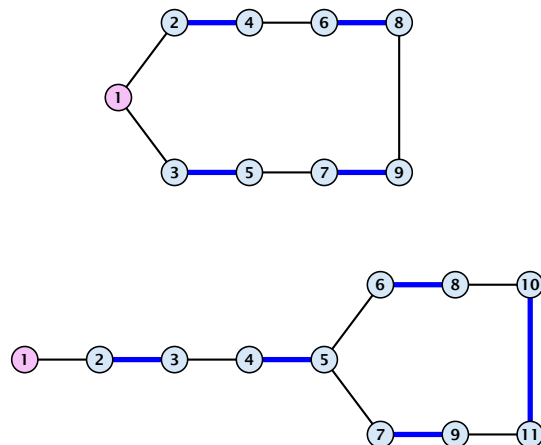
Flowers and Blossoms

Definition 87

A **flower** in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r , is a subgraph with two components:

- ▶ A **stem** is an even length alternating path that starts at the root node r and terminates at some node w . We permit the possibility that $r = w$ (empty stem).
- ▶ A **blossom** is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the **base** of the blossom.

Flowers and Blossoms



Flowers and Blossoms

Properties:

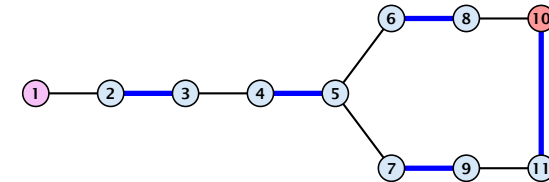
1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.
2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

- Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
- The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms



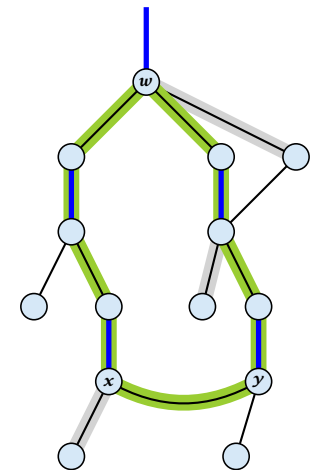
Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B .

- Delete all vertices in B (and its incident edges) from G .
- Add a new (pseudo-)vertex b . The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B .

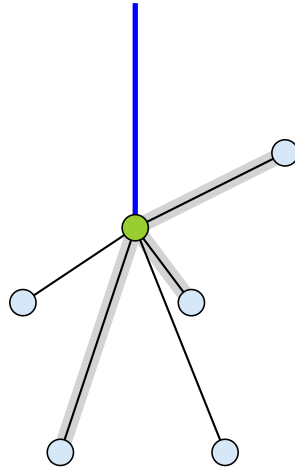
Shrinking Blossoms

- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b .
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M' .
- Nodes that are connected in G to at least one node in B become connected to b in G' .



Shrinking Blossoms

- ▶ Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b .
- ▶ Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M' .
- ▶ Nodes that are connected in G to at least one node in B become connected to b in G' .



Example: Blossom Algorithm

Animation of Blossom Shrinking algorithm is only available in the lecture version of the slides.

Correctness

Assume that in G we have a flower w.r.t. matching M . Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b . Let M' be the matching in the contracted graph.

Lemma 88

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M .

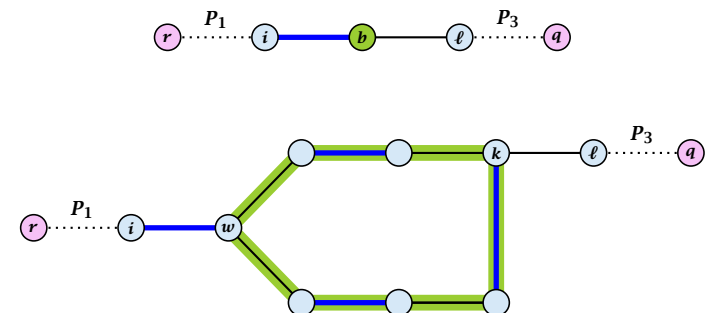
Correctness

Proof.

If P' does not contain b it is also an augmenting path in G .

Case 1: non-empty stem

- ▶ Next suppose that the stem is non-empty.



Correctness

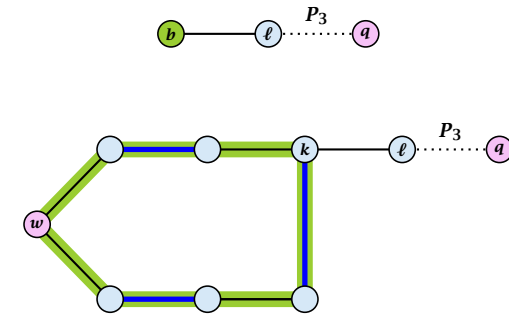
- ▶ After the expansion ℓ must be incident to some node in the blossom. Let this node be k .
- ▶ If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.
- ▶ $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- ▶ If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

Correctness

Proof.

Case 2: empty stem

- ▶ If the stem is empty then after expanding the blossom, $w = r$.



- ▶ The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 89

If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M' .

Correctness

Proof.

- ▶ If P does not contain a node from B there is nothing to prove.
- ▶ We can assume that r and q are the only free nodes in G .

Case 1: empty stem

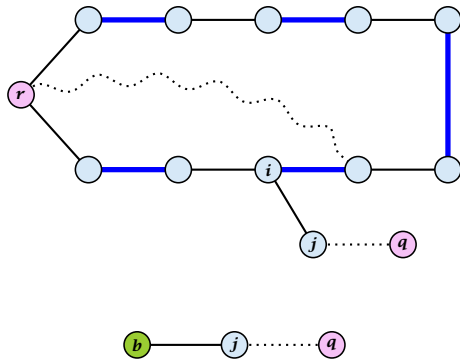
Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

Correctness

Illustration for Case 1:



Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w ; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+ , r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+ , since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+ .

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+ . It must also have an augmenting path w.r.t. M' , as both matchings have the same cardinality.

This path must go between r and q .

The lecture slides
contain a step by
step explanation.

Algorithm 49 search(r , found)

- 1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
- 2: $found \leftarrow false$
- 3: unlabel all nodes;
- 4: give an even label to r and initialize $list \leftarrow \{r\}$
- 5: **while** $list \neq \emptyset$ **do**
- 6: delete a node i from $list$
- 7: examine(i , found)
- 8: **if** found = true **then return**

Search for an augmenting path
starting at r .

The lecture slides
contain a step by
step explanation.

Algorithm 50 examine(i , found)

- 1: **for all** $j \in \bar{A}(i)$ **do**
- 2: **if** j is even **then** contract(i , j) and **return**
- 3: **if** j is unmatched **then**
- 4: $q \leftarrow j$;
- 5: pred(q) $\leftarrow i$;
- 6: found $\leftarrow true$;
- 7: **return**
- 8: **if** j is matched and unlabeled **then**
- 9: pred(j) $\leftarrow i$;
- 10: pred(mate(j)) $\leftarrow j$;
- 11: add mate(j) to $list$

Examine the neighbours of a node i

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label b even and add to $list$
- 4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

Contract blossom identified by nodes i and j

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label b even and add to $list$
- 4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

Get all nodes of the blossom.
Time: $\mathcal{O}(m)$

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label b even and add to $list$
- 4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

Identify all neighbours of b .
Time: $\mathcal{O}(m)$ (how?)

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label b even and add to $list$
- 4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

b will be an even node, and it has unexamined neighbours.

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label b even and add to *list*
- 4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

Every node that was adjacent to a node in B is now adjacent to b

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
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- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

Only for making a blossom expansion easier.

Algorithm 51 contract(i, j)

- 1: trace pred-indices of i and j to identify a blossom B
- 2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label b even and add to *list*
- 4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in B
- 6: delete nodes in B from the graph

Only delete links from nodes not in B to B .
When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

- ▶ A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.
- ▶ The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.
- ▶ There are at most n contractions as each contraction reduces the number of vertices.
- ▶ The expansion can trivially be done in the same time as needed for all contractions.
- ▶ An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.
- ▶ In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) .$$

Example: Blossom Algorithm

Animation of Blossom Shrinking algorithm is only available in the lecture version of the slides.