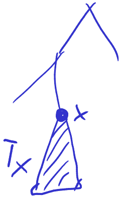
Splay Trees



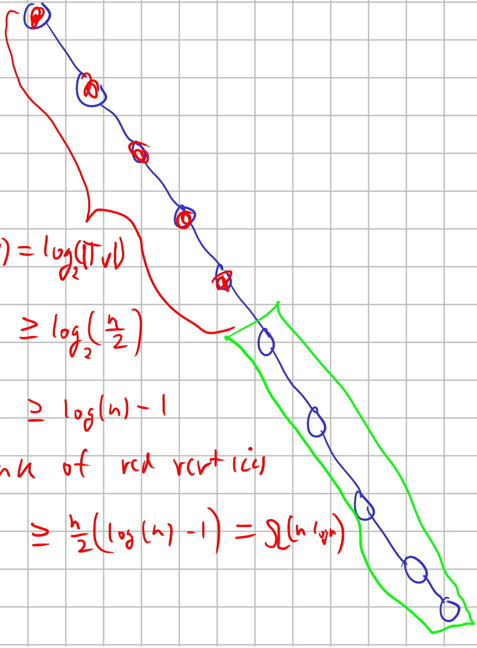
potential function for splay trees:

- size $s(x) = |T_x|$
- rank $r(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

CLAIM: $\Phi(T) = O(n \log n)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.



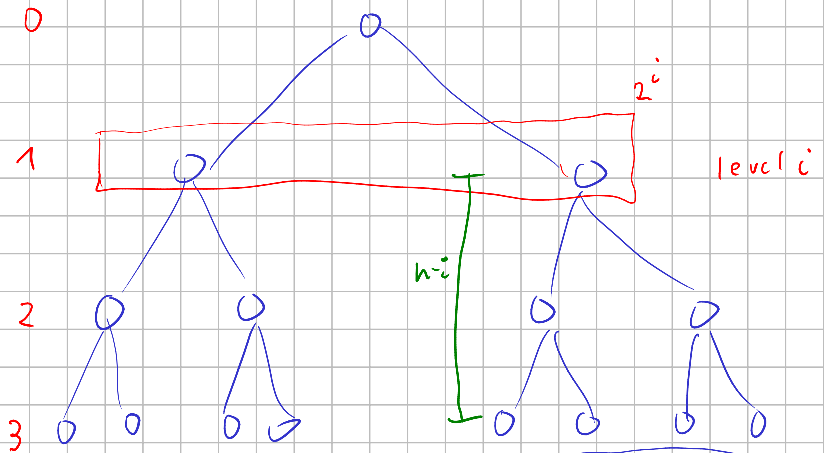
$r(v) = \log_2(|T_v|)$

$\geq \log_2\left(\frac{n}{2}\right)$

$\geq \log(n) - 1$

rank of red vertices

$\geq \frac{n}{2}\left(\log(n) - 1\right) = \Omega(n \log n)$



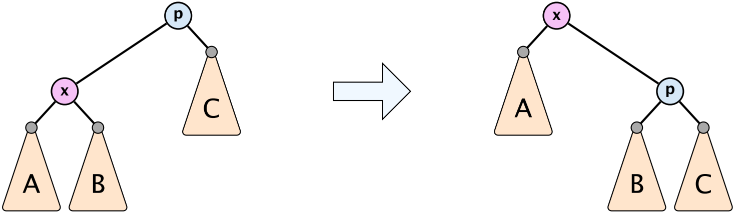
$= h$

$$\sum_{i=0}^{h} \log_2\left(2^{h-i+1}-1\right)\cdot 2^i \le 2^h \boxed{\sum_{i=0}^{h}(h-i+1)\cdot \frac{2^i}{2^h}}$$

$$\begin{array}{llll}
\text{real-cost}_{ZIGZIG} = 2 & \text{cost}_{ZIGZIG} & = & \multirow{2}{*}{$\boxed{\begin{array}{l} 2 + \Delta\Phi(T) \\ \hline 2 + \Delta\Phi(T) \end{array}}$} \\
\text{real-cost}_{ZIGZAG} = 2 & \text{cost}_{ZIGZAG} & = & \\
\text{real-cost}_{ZIG} = 1 & \text{cost}_{ZIG} & = & 1 + \Delta\Phi(T)
\end{array}$$

$$\text{cost}_{SPLAY} = 1 + \sum_{ZIGZIG} \text{cost}_{ZIGZIG} + \sum_{ZIGZAG} \text{cost}_{ZIGZAG} + \text{cost}_{ZIG}$$

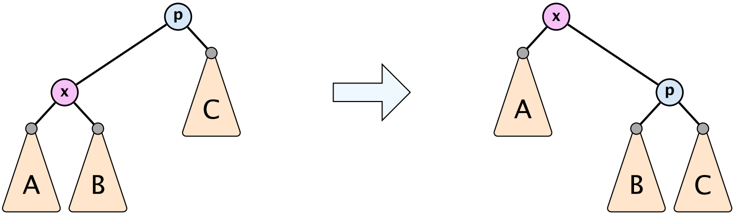
Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\\ &\le r'(x) - r(x)\end{aligned}$$

$$\mathrm{cost}_{\mathrm{zig}} \le 1 + 3(r'(x) - r(x))$$

Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\\ &\le r'(x) - r(x)\end{aligned}$$

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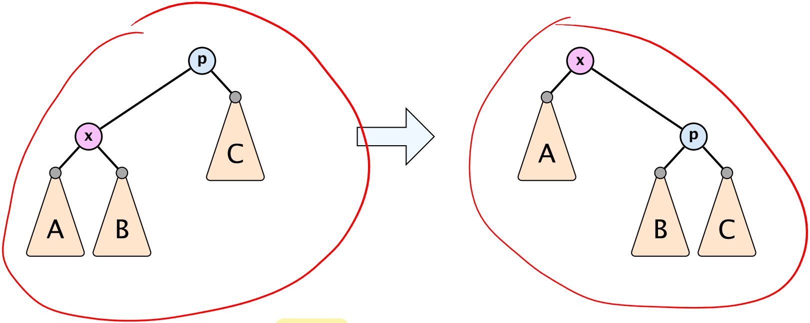
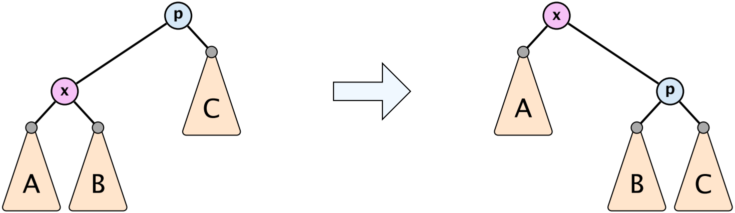


$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\
&= r'(p) - r(x) \\
&\leq r'(x) - r(x)
\end{aligned}
$$

$$\mathrm{cost}_{\mathrm{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zig Case



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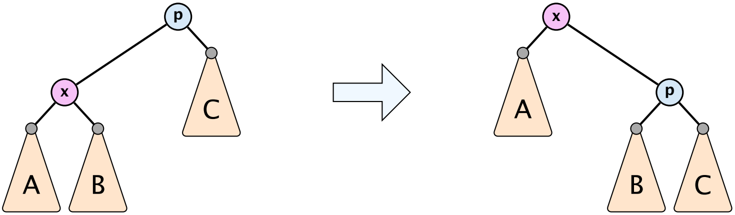
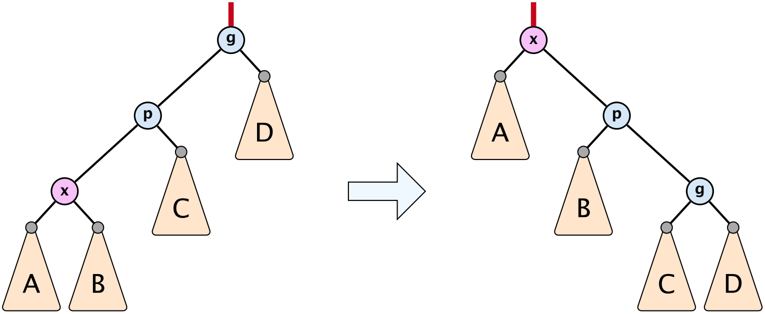
Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= \boxed{r'(x)} + r'(p) - r(x) - \boxed{r(p)}\\ &= r'(p) - r(x)\\ &\leq r'(x) - r(x)\end{aligned}$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\
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&= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\\
&\le -2 + 3(r'(x) - r(x)) \quad \Rightarrow \mathrm{cost}_{\mathrm{zigzig}} \le 3(r'(x) - r(x))
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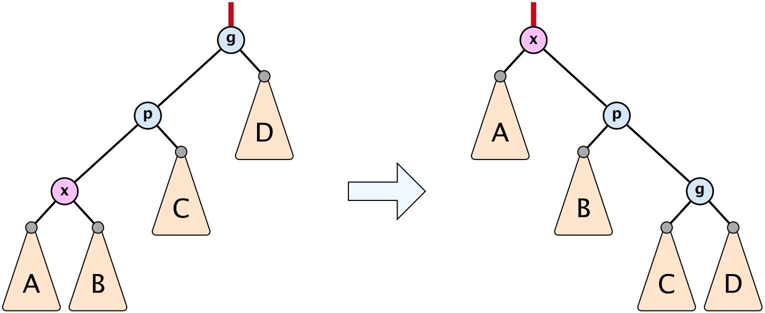
Splay: Zigzig Case



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\begin{aligned}
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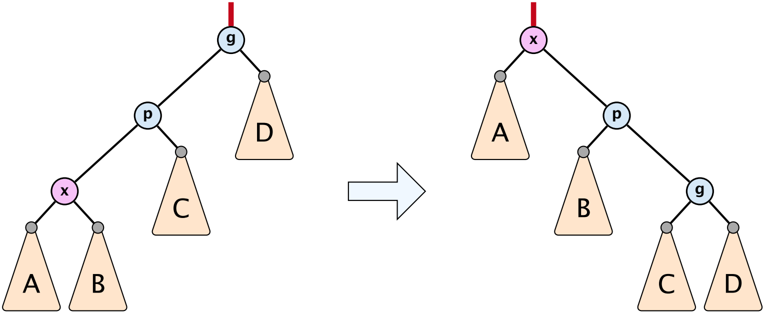
Splay: Zigzig Case



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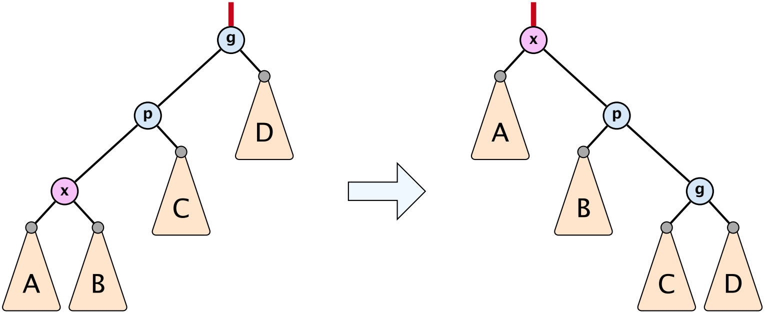

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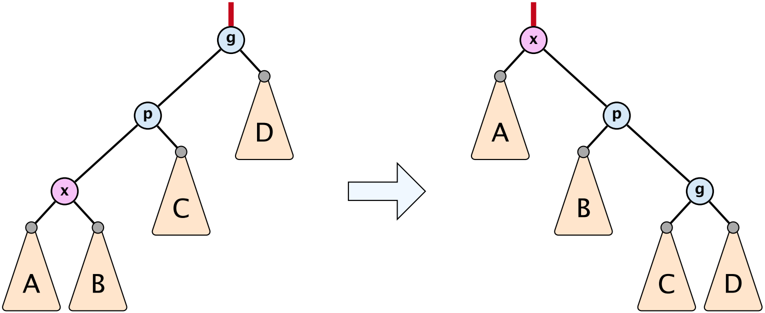
Splay: Zigzig Case



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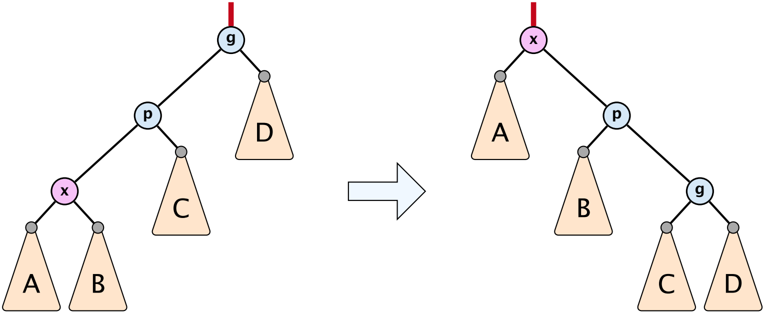
Splay: Zigzig Case



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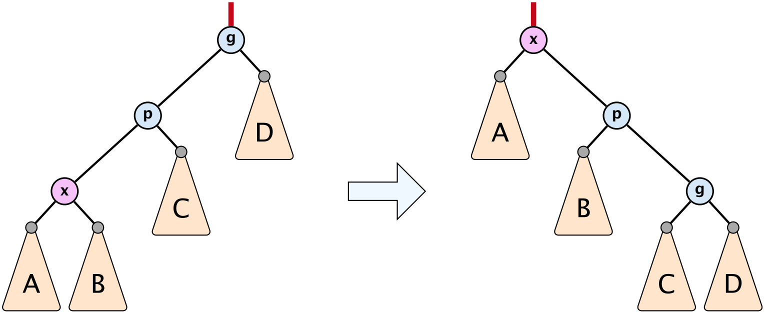
Splay: Zigzig Case



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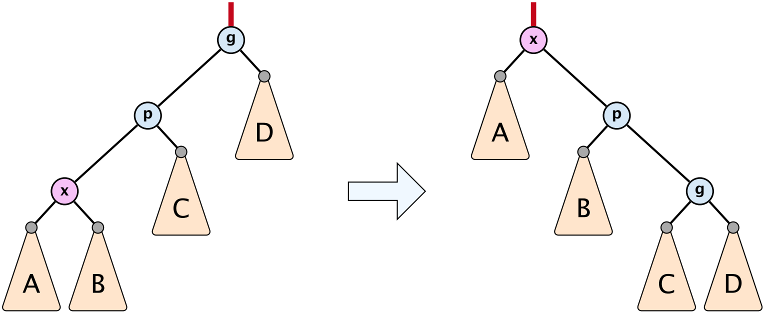
Splay: Zigzig Case



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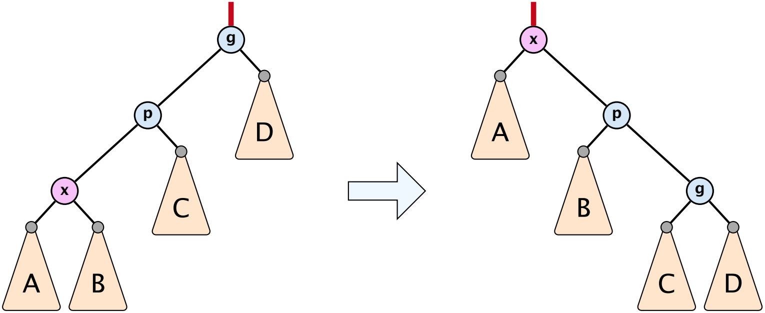
Splay: Zigzig Case



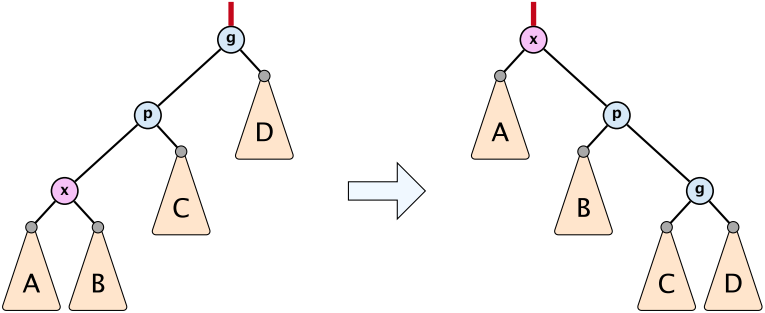
$$\frac{1}{2}\Big(r(x)+r'(g)-2r'(x)\Big)$$

$$=\frac{1}{2}\Big(\log(s(x))+\log(s'(g))-2\log(s'(x))\Big)$$

$$=\frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big)+\frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\le\log\Big(\frac{1}{2}\frac{s(x)}{s'(x)}+\frac{1}{2}\frac{s'(g)}{s'(x)}\Big)\le\log\Big(\frac{1}{2}\Big)=-1$$

Splay: Zigzig Case



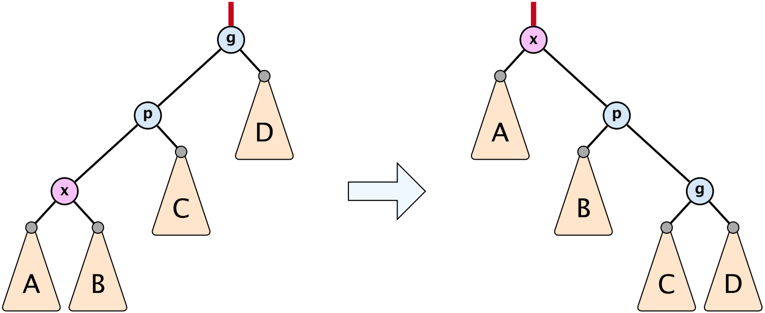
$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

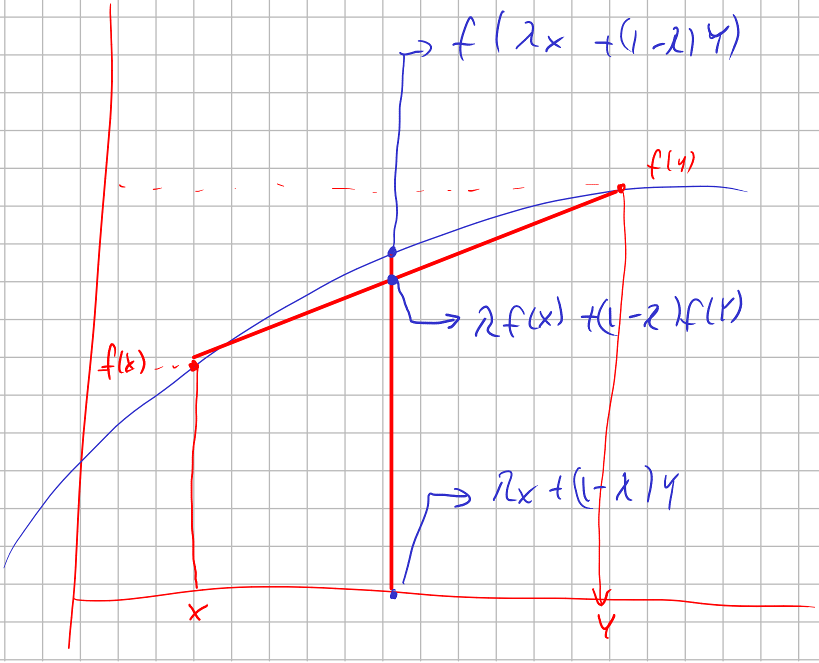
$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big) = -1$$

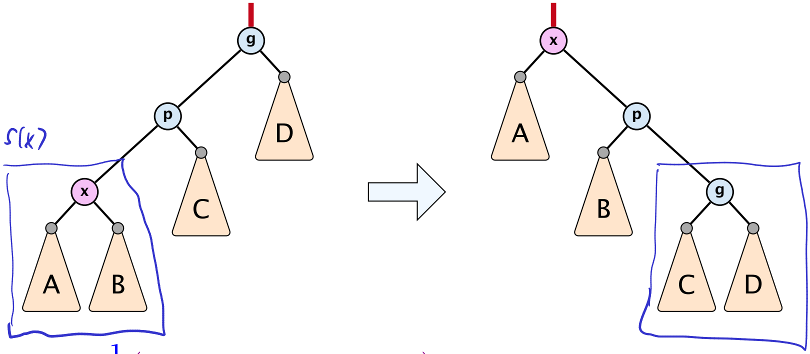
Splay: Zigzig Case



$$
\begin{aligned}
\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big) \\
&= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big) \\
&= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big) \\
&\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big) = -1
\end{aligned}
$$


$f(\lambda x + (1-\lambda) y)$
$f(y)$
$\lambda f(x) + (1-\lambda) f(y)$
$f(x)$
$\lambda x + (1-\lambda) y$
x
y

Splay: Zigzig Case



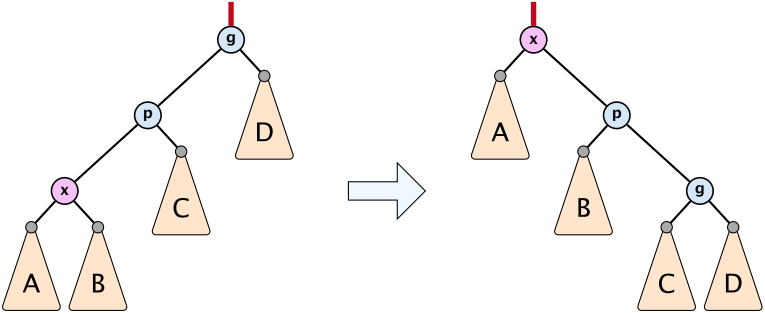
$$\frac{1}{2}\Big(r(x)+r'(g)-2r'(x)\Big)$$

$$=\frac{1}{2}\Big(\log(s(x))+\log(s'(g))-2\log(s'(x))\Big)$$

$$=\frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big)+\frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq\log\Big(\frac{1}{2}\frac{s(x)}{s'(x)}+\frac{1}{2}\frac{s'(g)}{s'(x)}\Big)\leq\log\Big(\frac{1}{2}\Big)=-1$$

Splay: Zigzig Case



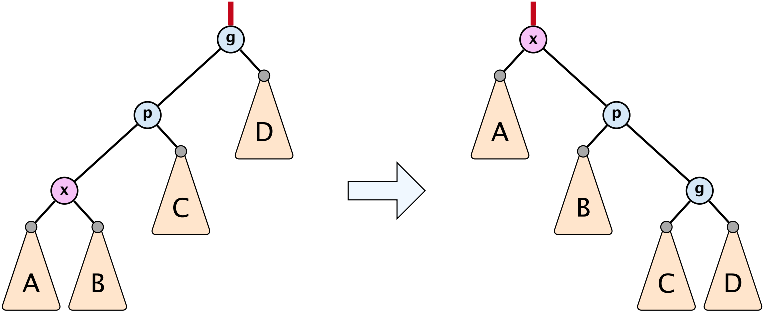
$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2}\log\left(\frac{s'(g)}{s'(x)}\right)$$

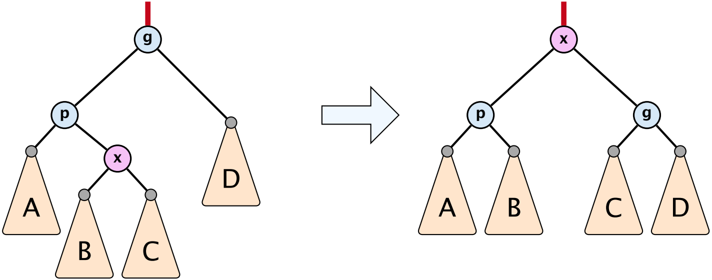
$$\le \log\left(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \le \log\left(\frac{1}{2}\right) = -1$$

Splay: Zigzig Case



$$
\begin{aligned}
\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)
&= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)\\
&= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)\\
&\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big) = -1
\end{aligned}
$$

Splay: Zigzag Case



$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
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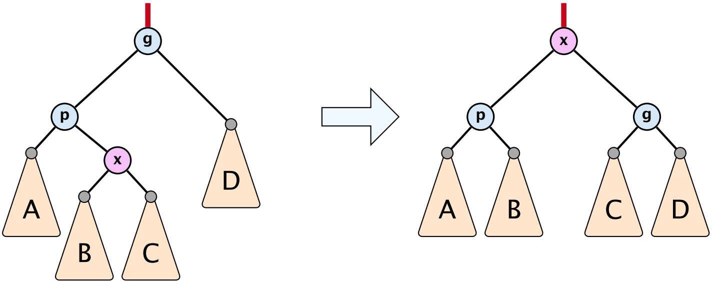
&\le r'(p) + r'(g) - r(x) - r(x) \\
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\
&\le -2 + 2(r'(x) - r(x)) \qquad \Rightarrow \mathrm{cost}_{\mathrm{zigzag}} \le 3(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzag Case



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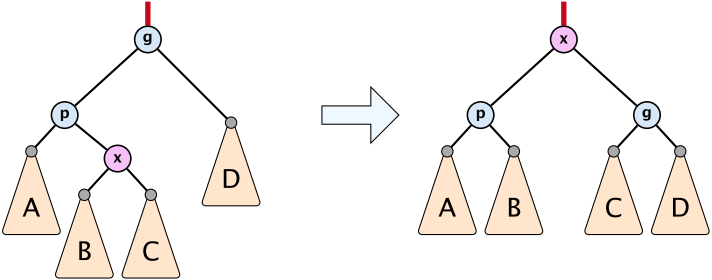
Splay: Zigzag Case



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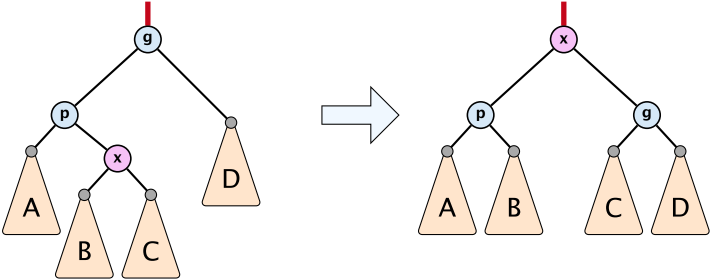
Splay: Zigzag Case



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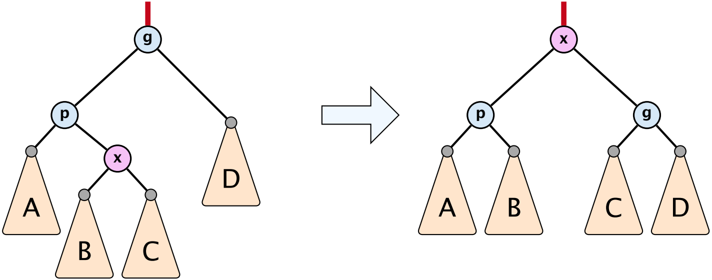
$$

Splay: Zigzag Case



$$
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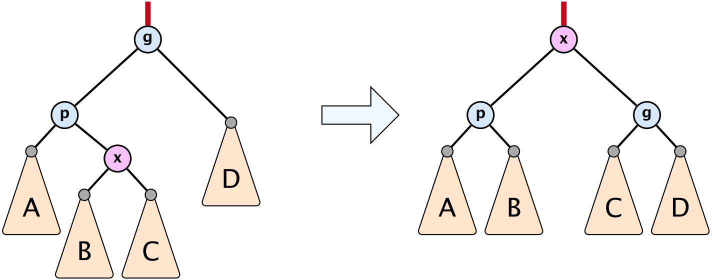
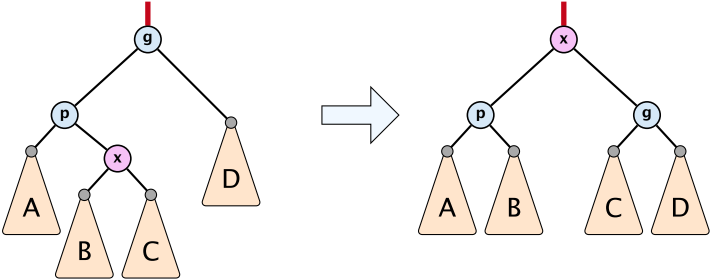
Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\le r'(p) + r'(g) - r(x) - r(x)\\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)\\ &\le -2 + 2(r'(x) - r(x)) \qquad \Rightarrow \mathrm{cost}_{\mathrm{zigzag}} \le 3(r'(x) - r(x))\end{aligned}$$

Splay: Zigzag Case



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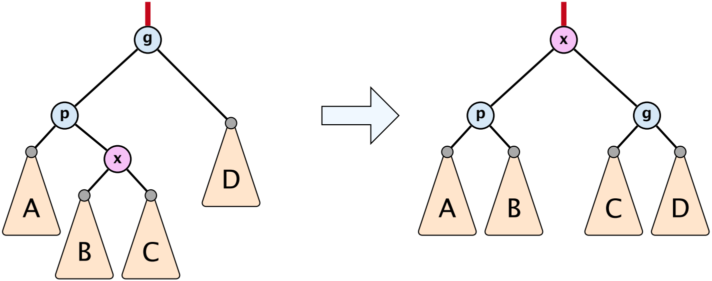

$$\frac{1}{2}\Big(r'(p)+r'(g)-2r'(x)\Big)$$

$$=\frac{1}{2}\Big(\log(s'(p))+\log(s'(g))-2\log(s'(x))\Big)$$

$$\leq\log\Big(\frac{1}{2}\frac{s'(p)}{s'(x)}+\frac{1}{2}\frac{s'(g)}{s'(x)}\Big)\leq\log\Big(\frac{1}{2}\Big)=-1$$

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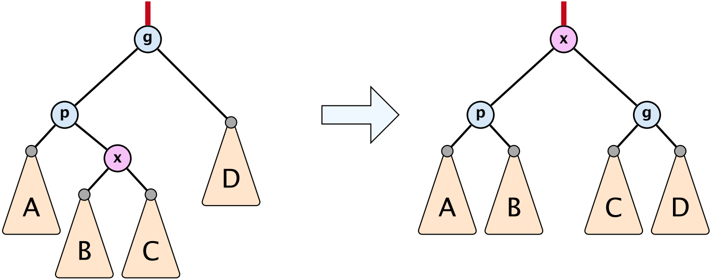

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

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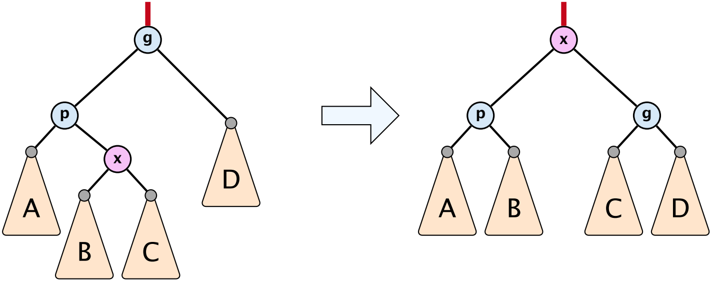
Splay: Zigzag Case



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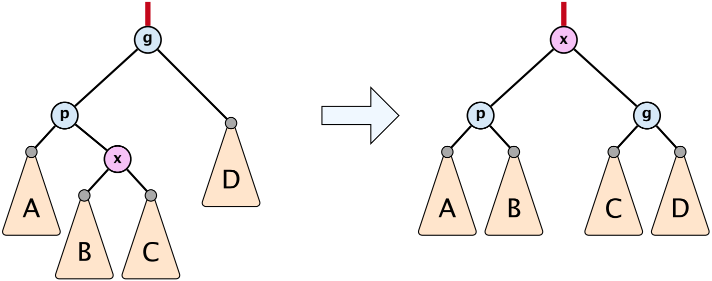

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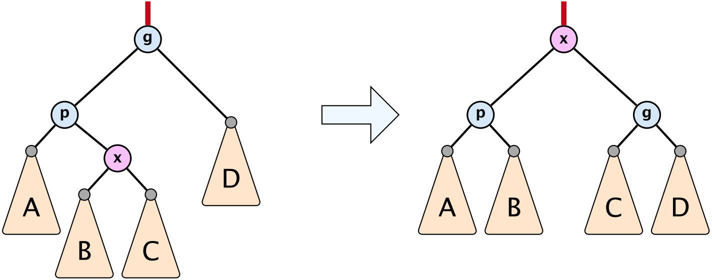

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Amortized cost of the whole splay operation:

$$\begin{aligned}
&\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
&= 2 + 3(r(\text{root}) - r_0(x)) \\
&\leq \mathcal{O}(\log n)
\end{aligned}$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return “error” if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v.
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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