There are different types of complexity bounds:

- amortized complexity:
  The average cost of data structure operations over a worst case sequence of operations.
- randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

- We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.
- An exact analysis (e.g. *exactly* counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.
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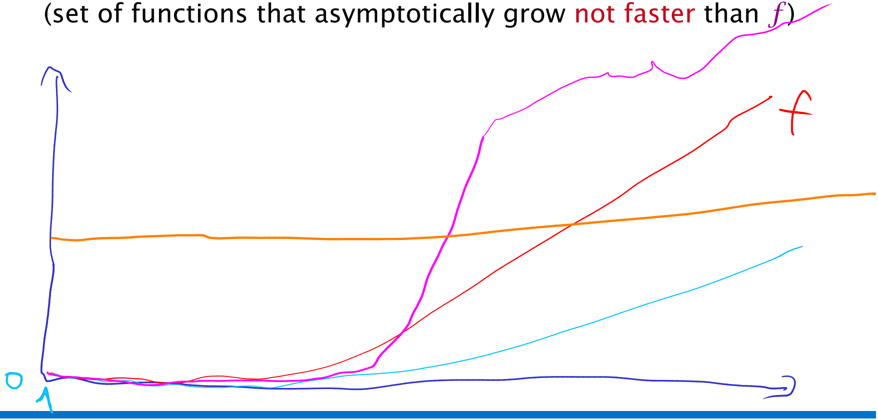
Asymptotic Notation

$f = n$ $g = n^2$

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \ge n_0 : [g(n) \le c \cdot f(n)]\}$
  (set of functions that asymptotically grow not faster than f)



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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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Abuse of notation

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2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.
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$\mathcal{O}(n) = \mathcal{O}(n^2)$

Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an anonymous function in the set $\Theta(n)$ that makes the expression true.

Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Careful!

"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

Asymptotic Notation in Equations

How do we interpret an expression like:

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$f \in \Theta(n) \Rightarrow \exists c:\ f \le c \cdot n$

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e.g. $f(n) = n$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} f(i) = \Theta(n^2)$

$\sum_{i=1}^{n} f(i) \le c \sum_{i=1}^{n} i \le c \cdot \frac{n(n+1)}{2} = \mathcal{O}(n^2)$

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Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)$$
$$\text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \Big\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

$n = \Theta(n^2)$

represents

$\{n\} \subseteq \Theta(n^2)$

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

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Lemma 3

Let f, g *be functions with the property* $\exists n_0 > 0\, \forall n \geq n_0 : f(n) > 0$ *(the same for* g*). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant* c
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
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The expressions also hold for Ω*. Note that this means that* $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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Asymptotic Notation

$func \in \Theta(1)$

```
for i = 1 to n do
    func(i)
```

Comments

- Do not use asymptotic notation within induction proofs.
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

Induction: for n

Start: $n = 1$ ✓

Step: $n \rightarrow n+1$

$\Theta(1)$ [for $i = 1$ to $n-1$ do func(i)]

$\Theta(1)$ [func(n)]

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$\Theta(n^{\log_2 3} \cdot \log n)$

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.
- However, suppose that I have two algorithms:
  - Algorithm A. Running time $f(n) = 1000 \log n = \mathcal{O}(\log n)$.
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Multiple Variables in Asymptotic Notation

Sometimes the input for an algorithm consists of several parameters (e.g., nodes and edges of a graph (n and m)).

If we want to make asympotic statements for $n \to \infty$ and $m \to \infty$ we have to extend the definition to multiple variables.

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Let f, g denote functions from $\mathbb{N}^d$ to $\mathbb{R}_0^+$.

Multiple Variables in Asymptotic Notation

Sometimes the input for an algorithm consists of several parameters (e.g., nodes and edges of a graph (n and m)).

If we want to make asympotic statements for $n \to \infty$ and $m \to \infty$ we have to extend the definition to multiple variables.

$f(n,m) = 1 \qquad g(n,m) = n-1$

$f \in \mathcal{O}(g)$

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- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists N \in \mathbb{N}_0\ \forall \vec{n}$ with $n_i \geq N$ for some i :
  $[g(\vec{n}) \leq c \cdot f(\vec{n})]\}$

  (set of functions that asymptotically grow not faster than f)

$\forall c, N \quad n = 1,\ m = N \qquad f(\vec{n}) \leq c \cdot g(\vec{n})$ ↯

Multiple Variables in Asymptotic Notation

$f \in \Theta(g)$??

Example 4

- $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $g(n,m) = n - 1$

Multiple Variables in Asymptotic Notation

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- $f : \mathbb{N}_0 \to \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N}_0 \to \mathbb{R}_0^+$, $g(n,m) = n$

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