# 7.2 Red Black Trees

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**Induction on the height of $v$.**

**base case** ($\text{height}(v) = 0$)

- If $\text{height}(v)$ (maximum distance btw. $v$ and a node in the sub-tree rooted at $v$) is 0 then $v$ is a leaf.
- The black height of $v$ is 0.
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# 7.2 Red Black Trees

## Proof (cont.)

**induction step**

$h \to h+1$

for every vertex $x$ with $\text{height}(x) \le h$

$\#\text{internal}(T_x) \ge 2^{\text{bh}(x)} - 1$

$h+1$

**Proof (cont.)**

**induction step**

- Supose $v$ is a node with $\text{height}(v) > 0$.
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- These children $(c_1, c_2)$ either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
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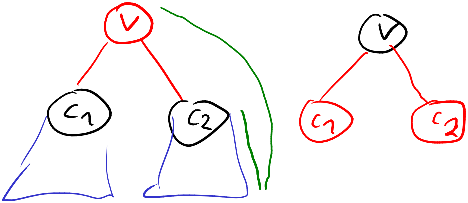


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**Proof of Lemma 13.**

Let $h$ denote the height of the red-black tree, and let $P$ denote a path from the root to the furthest leaf.

At least half of the node on $P$ must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \le n$.

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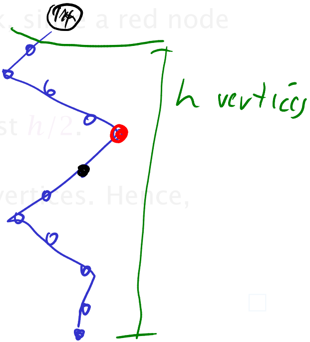


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# 7.2 Red Black Trees

**Definition 1**

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

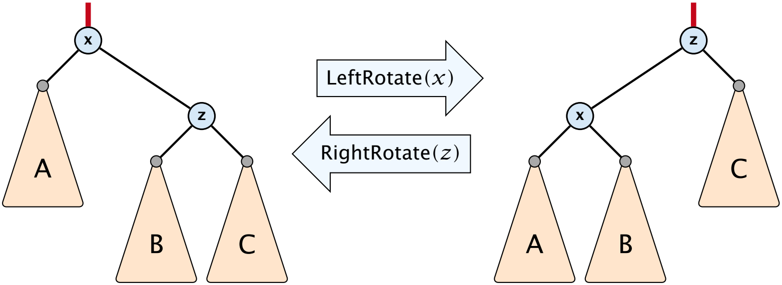
The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

# 7.2 Red Black Trees

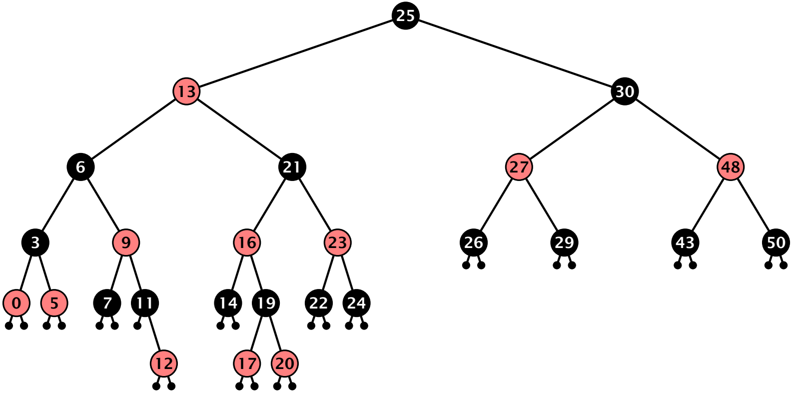
We need to adapt the insert and delete operations so that the red black properties are maintained.

# Rotations

The properties will be maintained through rotations:



# Red Black Trees: Insert

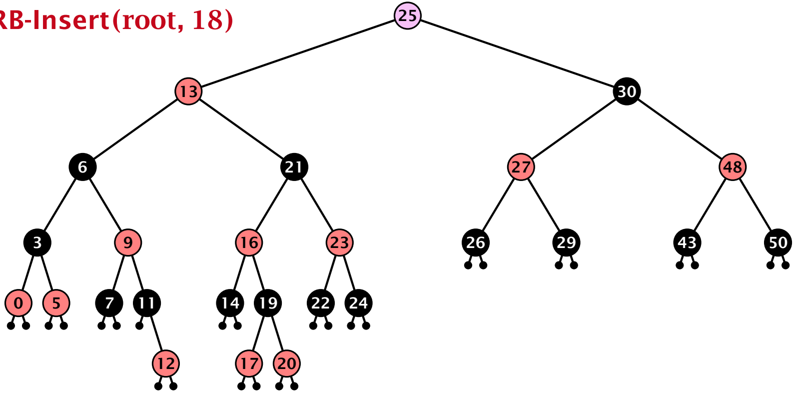



**Insert:**

- first make a normal insert into a binary search tree
- then fix red-black properties

# Red Black Trees: Insert

**RB-Insert(root, 18)**

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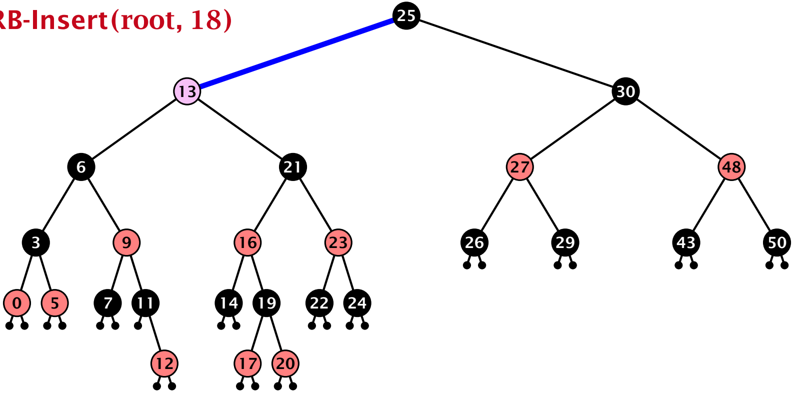


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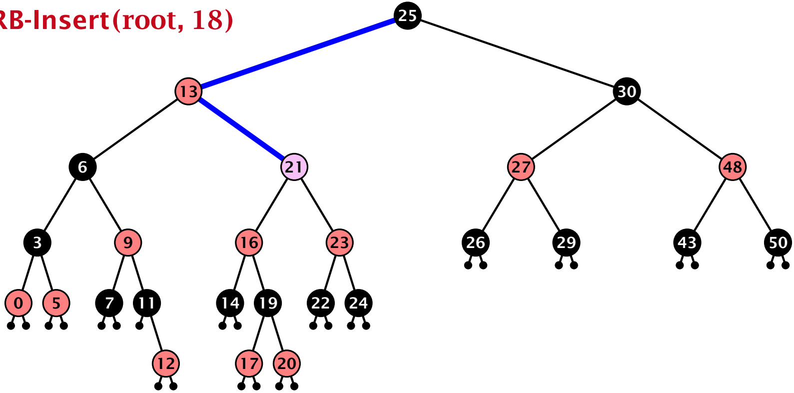


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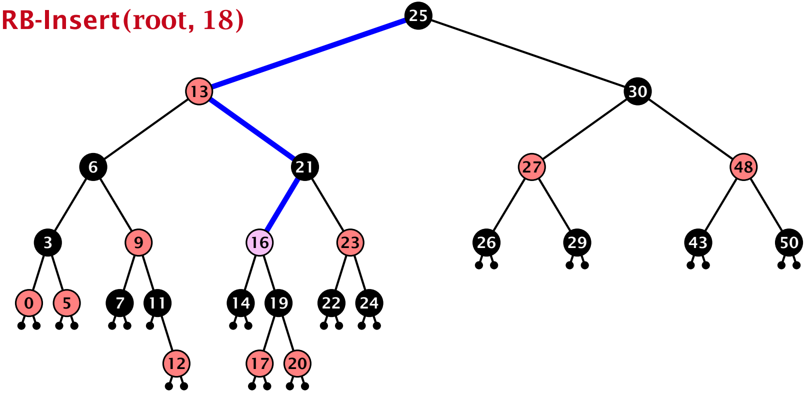


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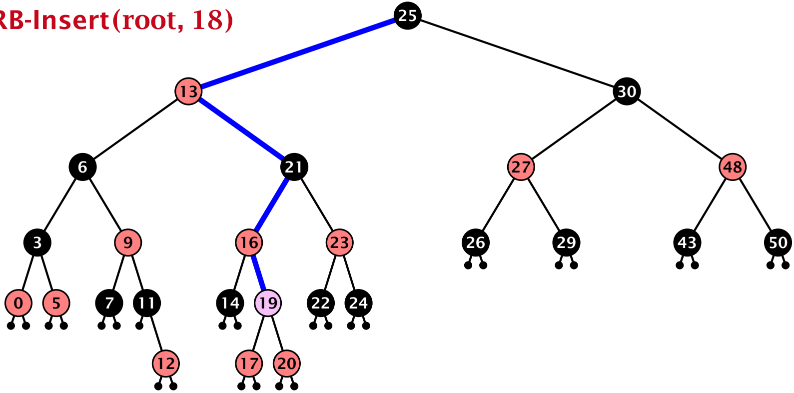


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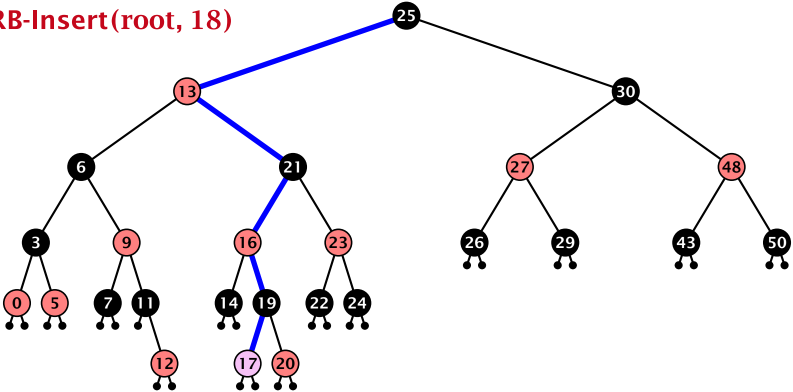


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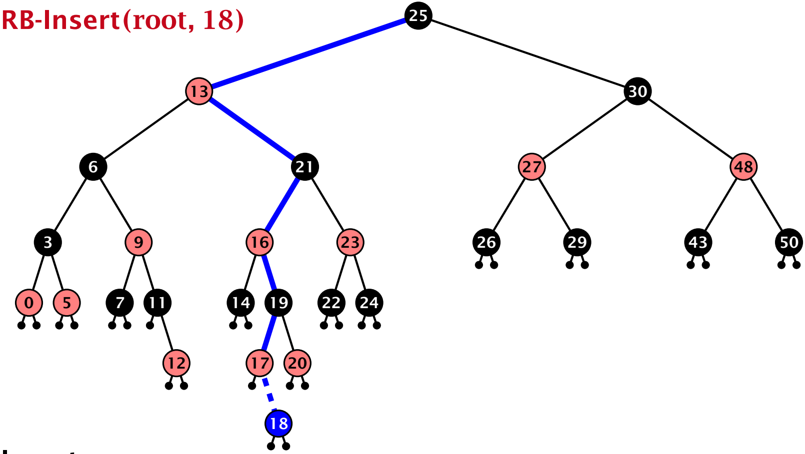


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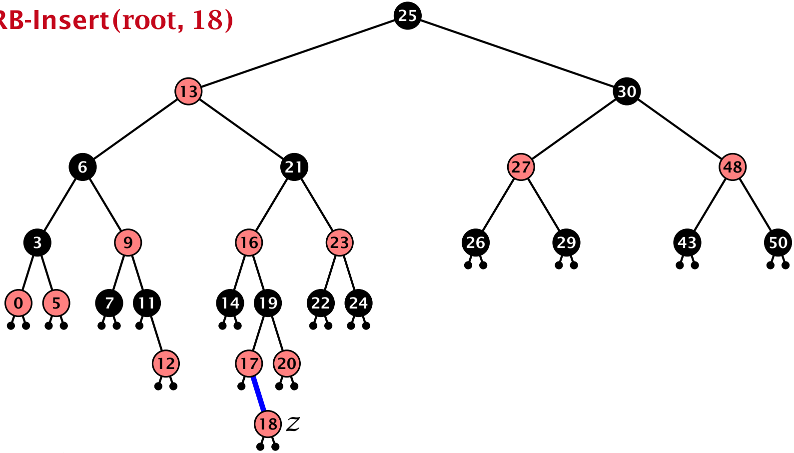


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**Invariant of the fix-up algorithm:**

- $z$ is a red node
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  - either both of them are red (most important case)
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**Algorithm 10** InsertFix($z$)

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while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
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    else same as then-clause but right and left exchanged
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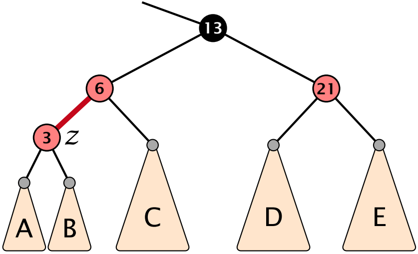
# Case 1: Red Uncle



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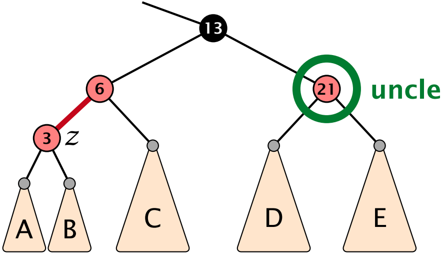

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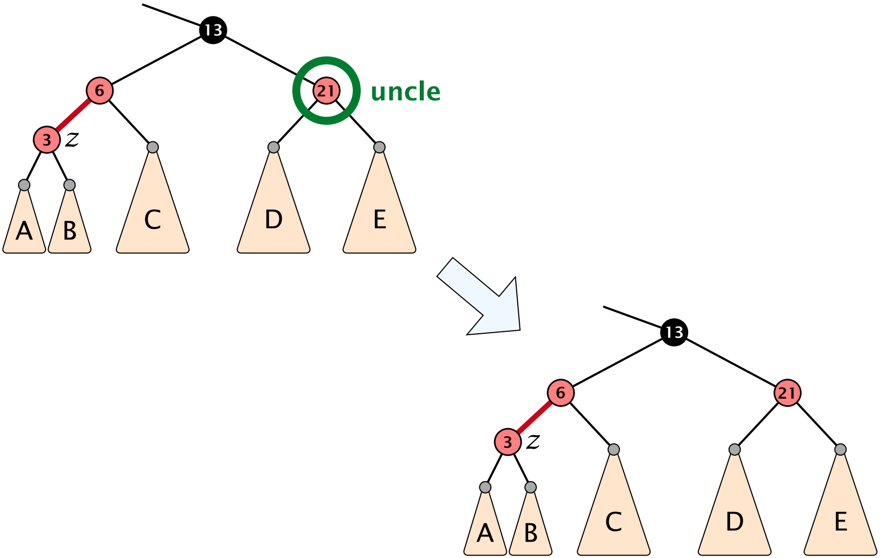

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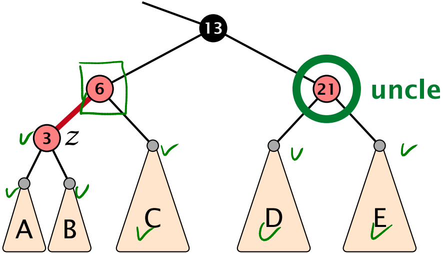
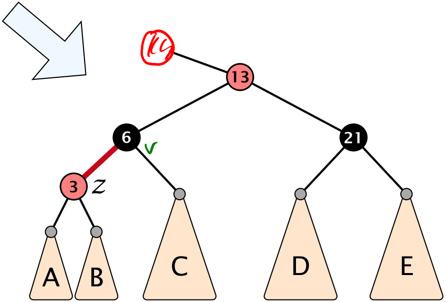



1. recolour



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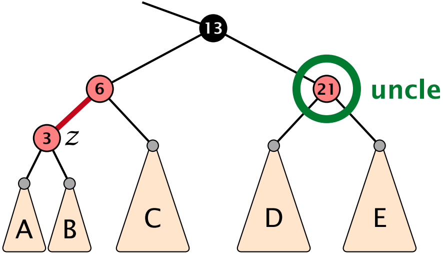


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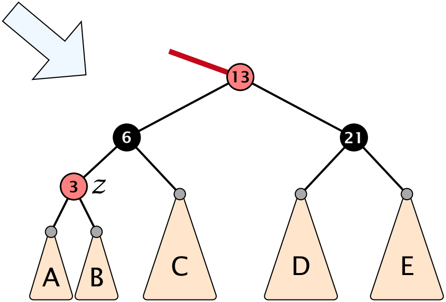

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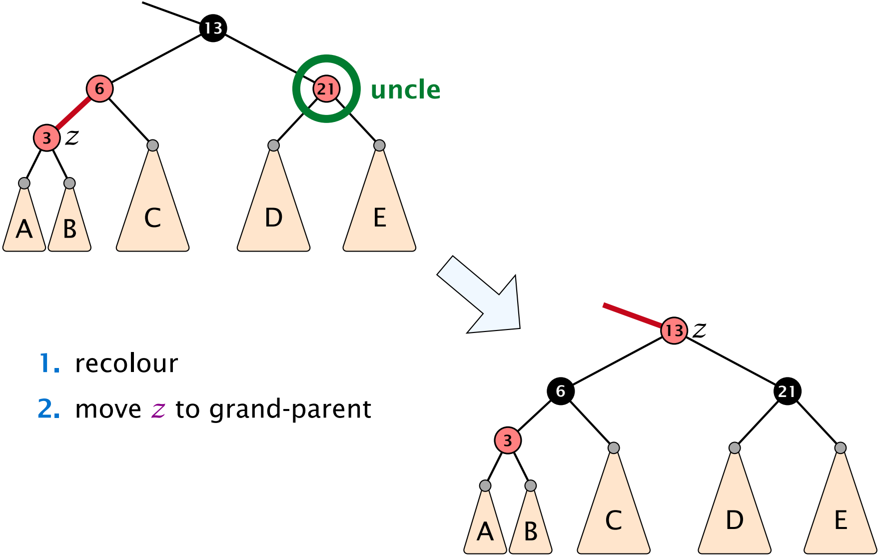


1. recolour
2. move $z$ to grand-parent

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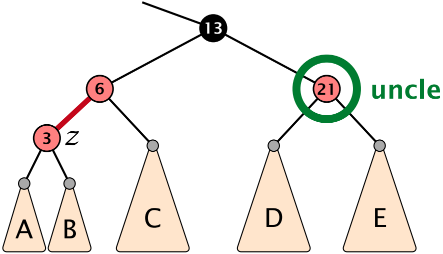


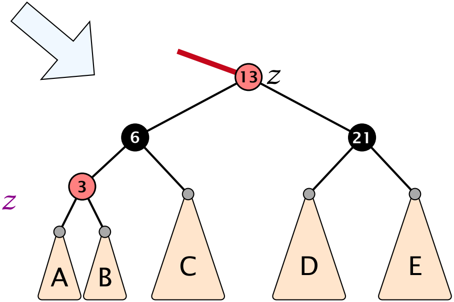


1. recolour
2. move $z$ to grand-parent
3. invariant is fulfilled for new $z$

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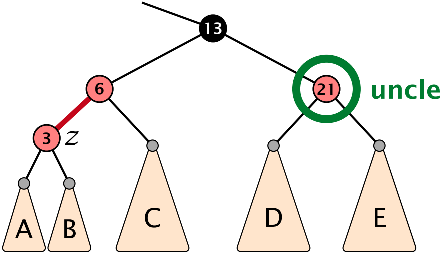
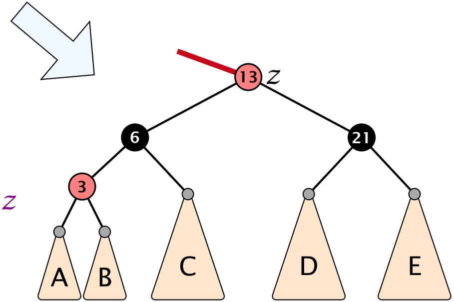


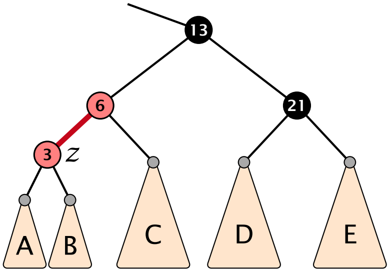




1. recolour
2. move $z$ to grand-parent
3. invariant is fulfilled for new $z$
4. you made progress

# Case 2b: Black uncle and $z$ is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree

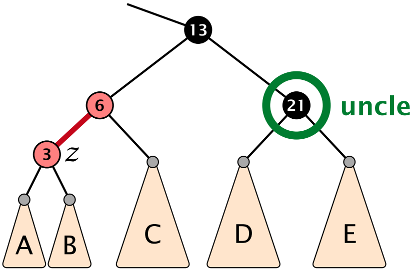


# Case 2b: Black uncle and $z$ is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



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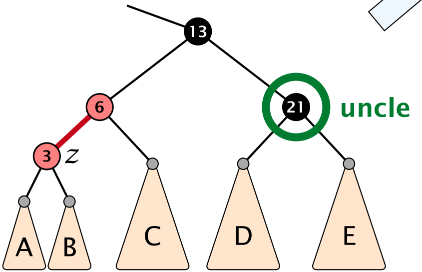


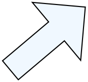

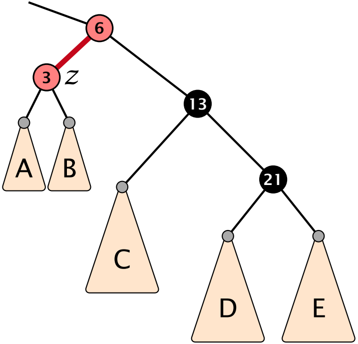

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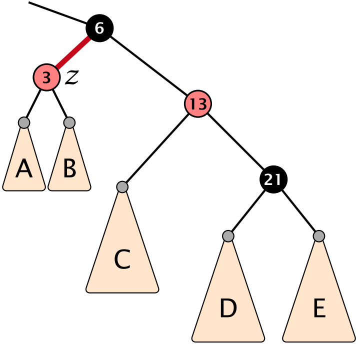


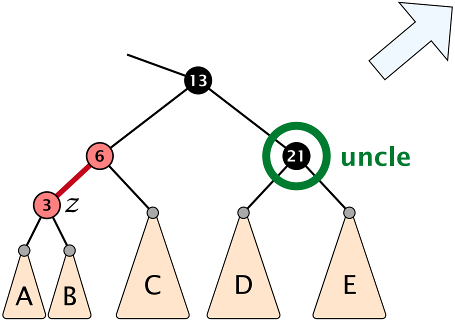

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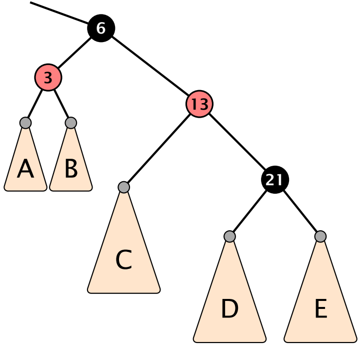


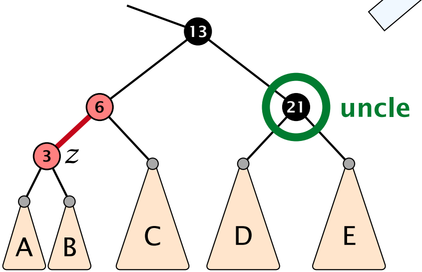

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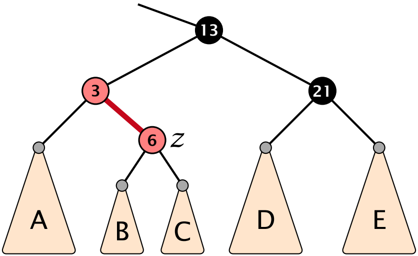

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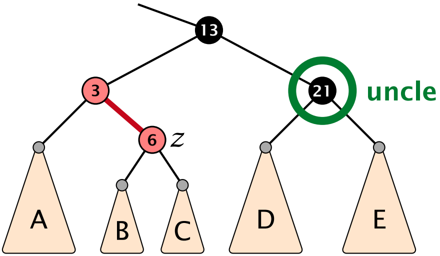

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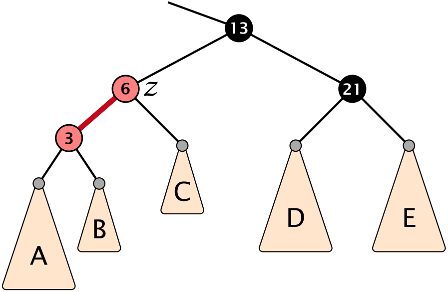


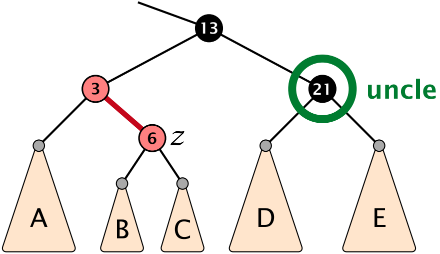

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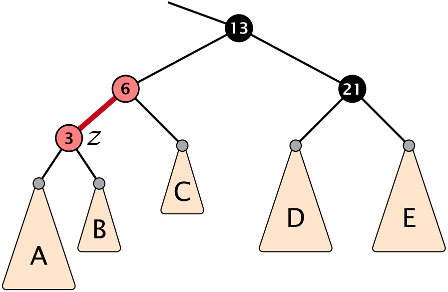


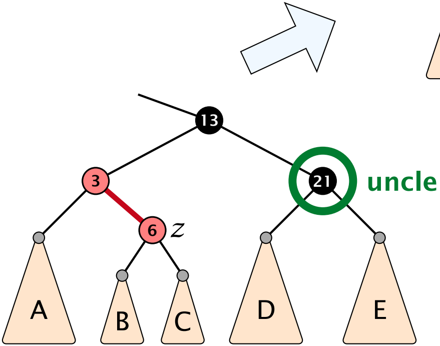

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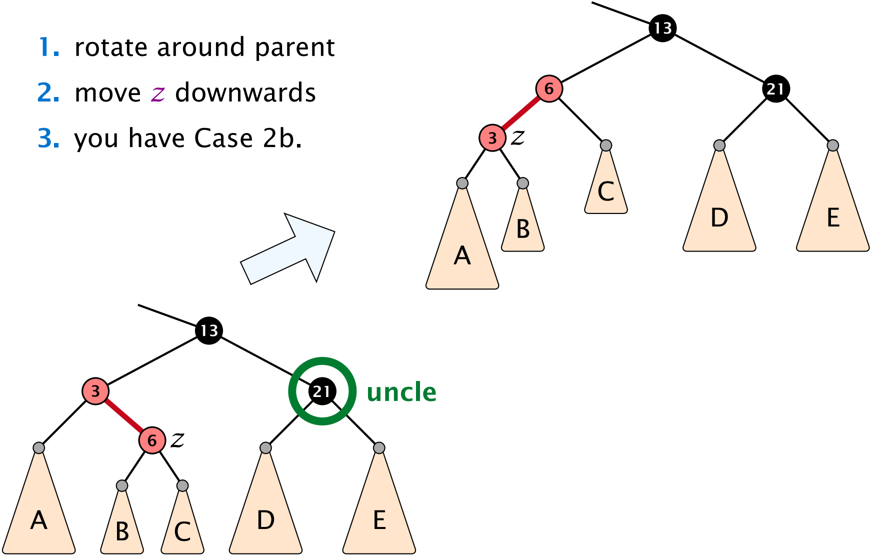

# Red Black Trees: Insert

**Running time:**

- Only Case 1 may repeat; but only $h/2$ many steps, where $h$ is the height of the tree.
- Case 2a → Case 2b → red-black tree
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Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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# Red Black Trees: Delete

First do a standard delete.

If the spliced out node $x$ was red everything is fine.

If it was black there may be the following problems.

- Parent and child of $x$ were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of $x$ to a descendant leaf of $x$ changes the number of black nodes. Black height property might be violated.

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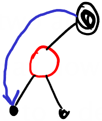

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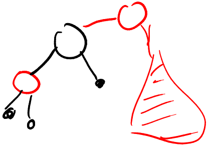

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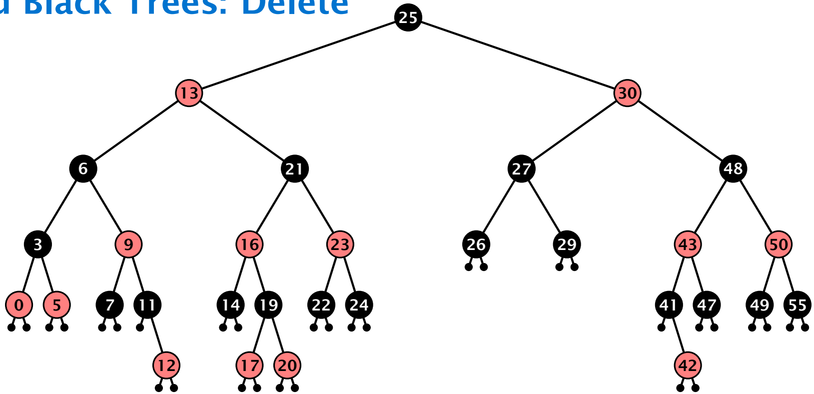

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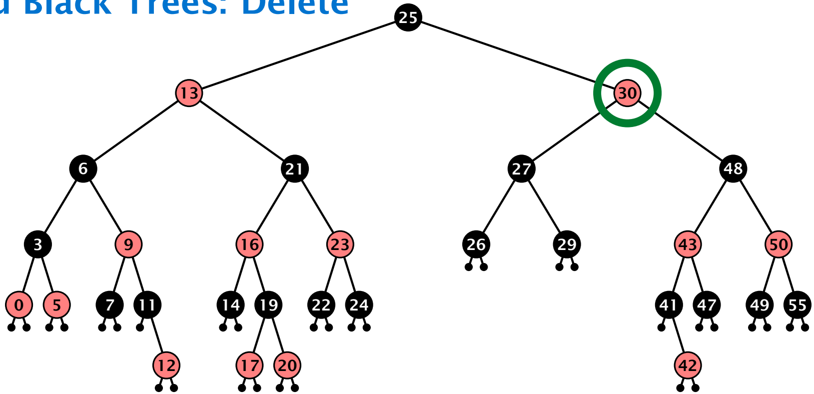


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Element has two children

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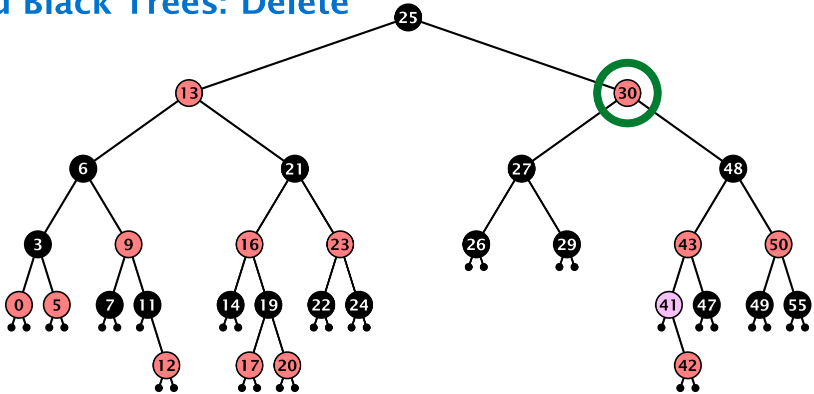


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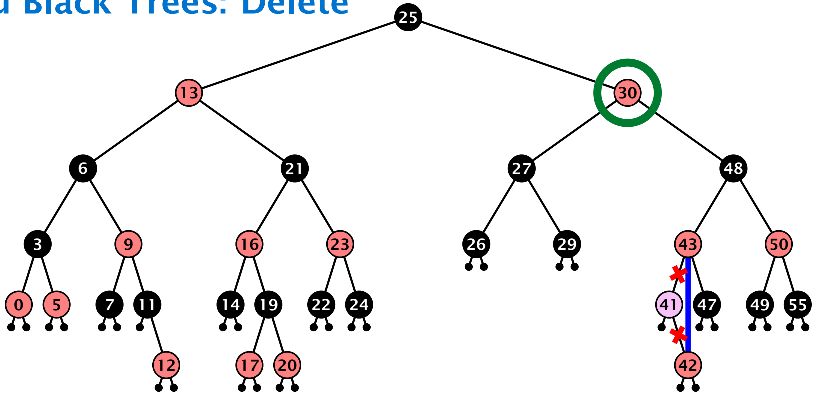


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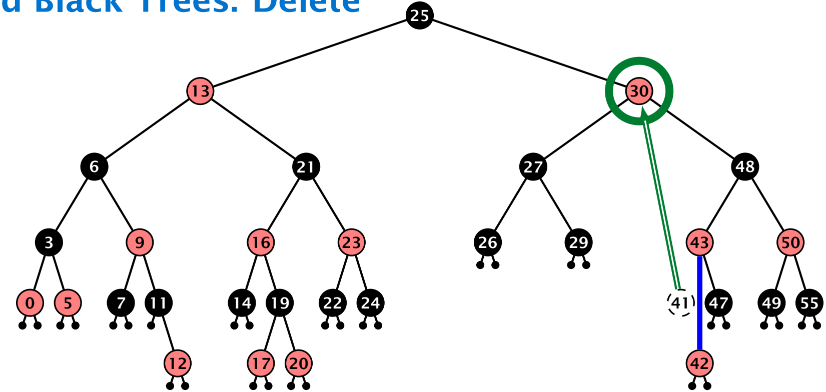


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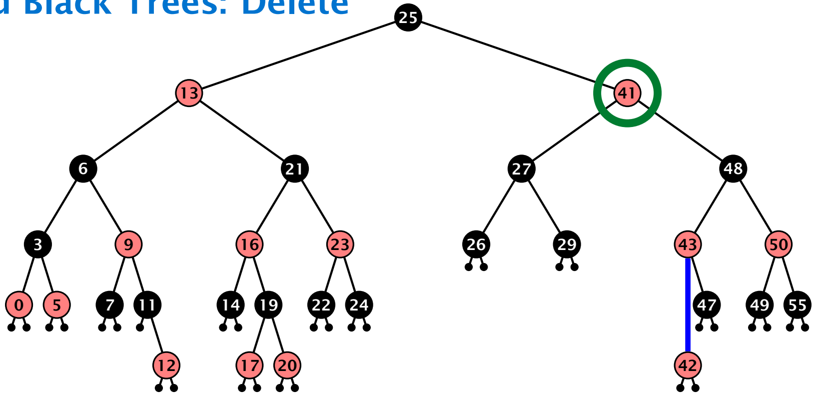


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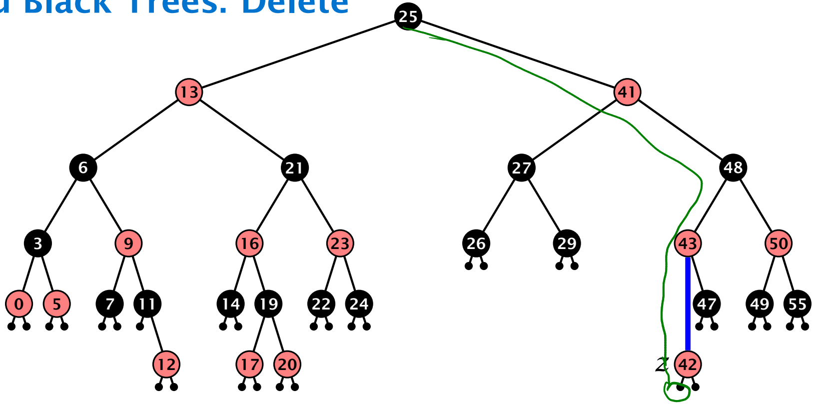


Delete:

- deleting black node messes up black-height property
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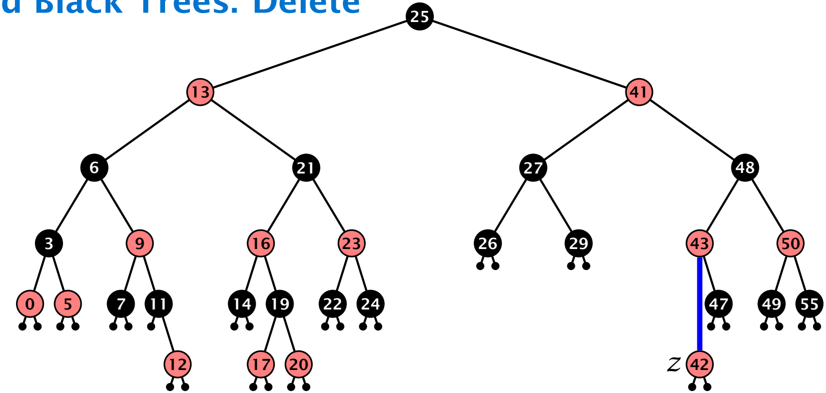


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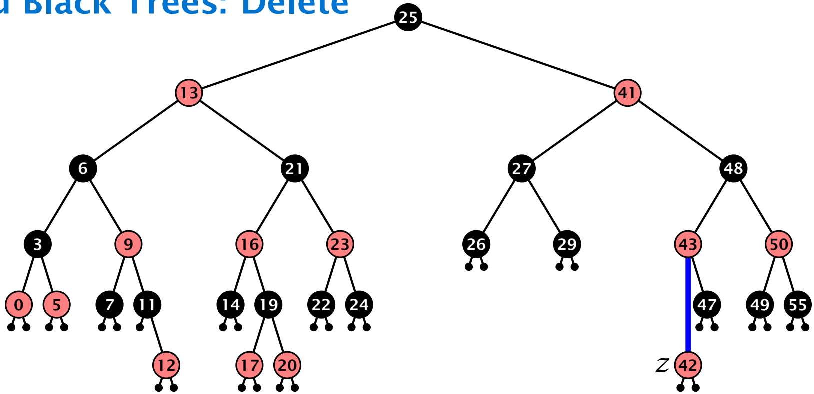


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**Invariant of the fix-up algorithm**

- the node $z$ is black
- if we “assign” a fake black unit to the edge from $z$ to its parent then the black-height property is fulfilled

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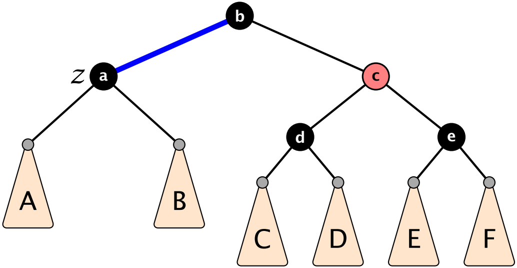


1. left-rotate around parent of $z$
2. recolor nodes $b$ and $c$
3. the new sibling is black
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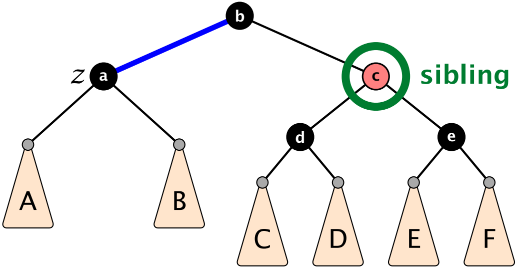


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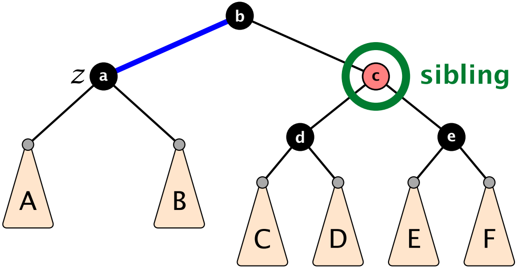


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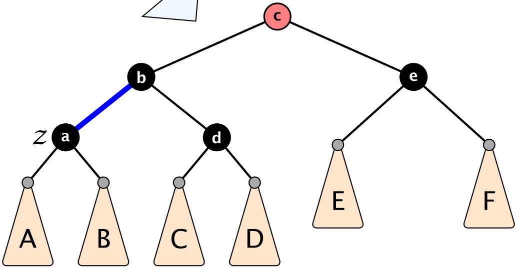

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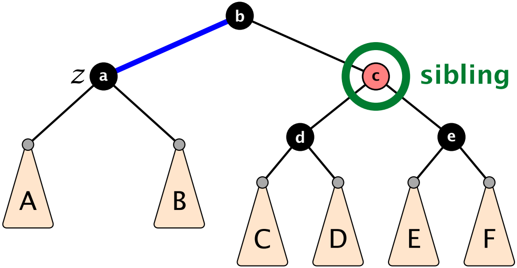


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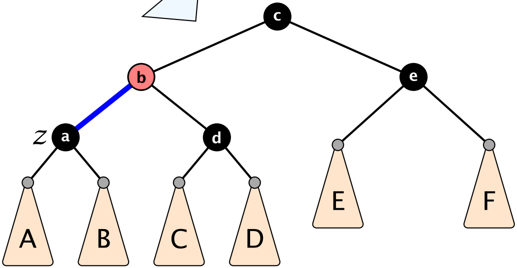

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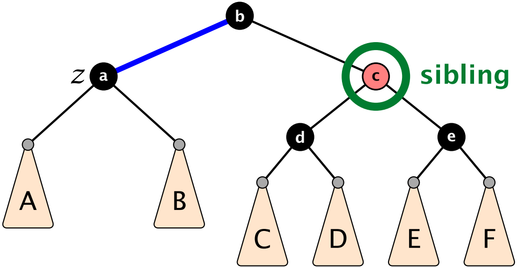


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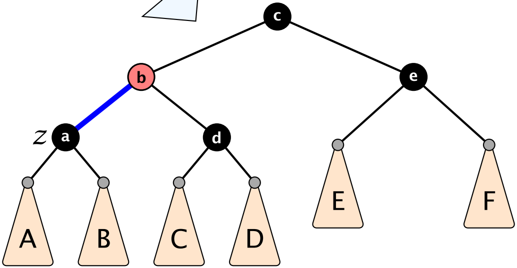

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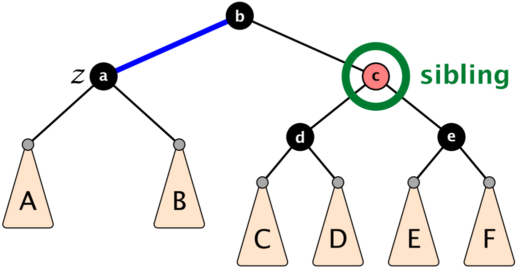


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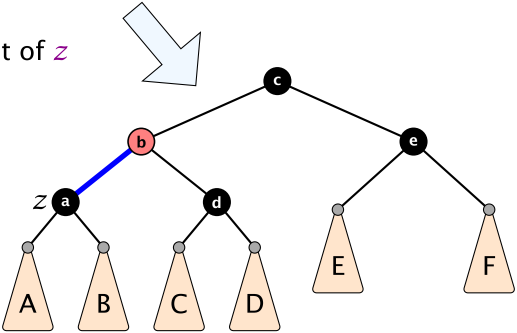

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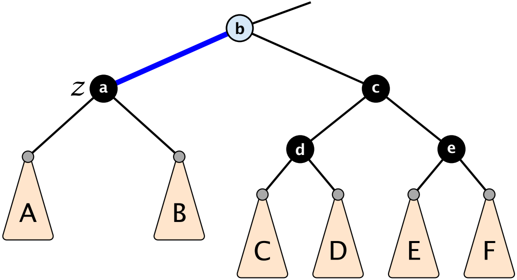


1. re-color node $c$
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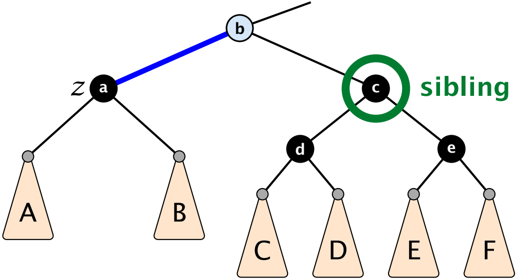


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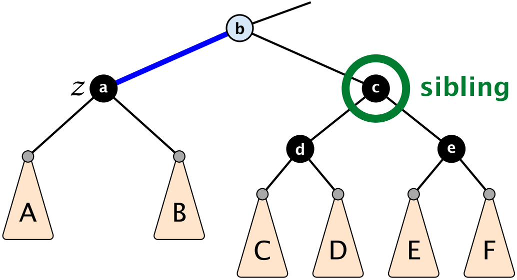


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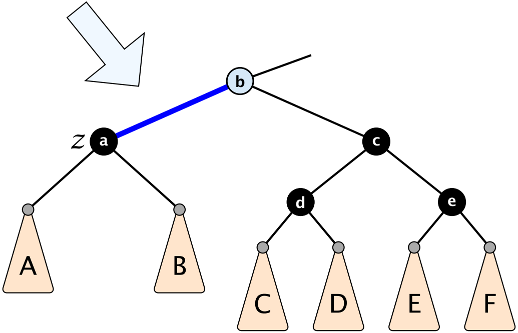

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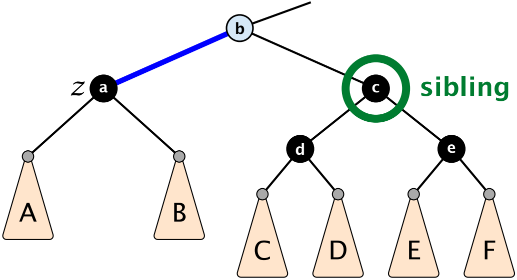


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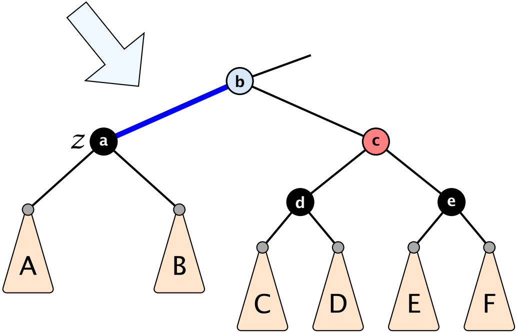

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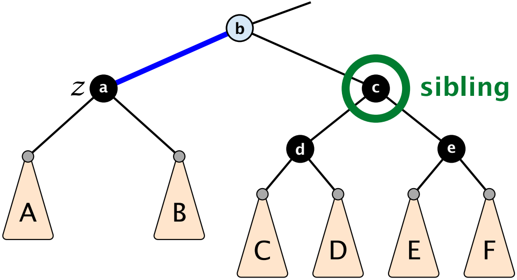


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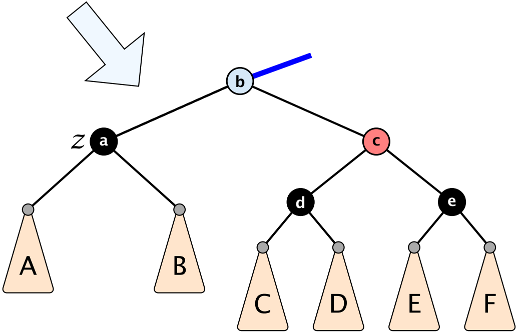

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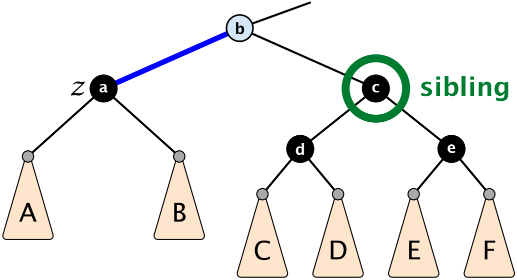


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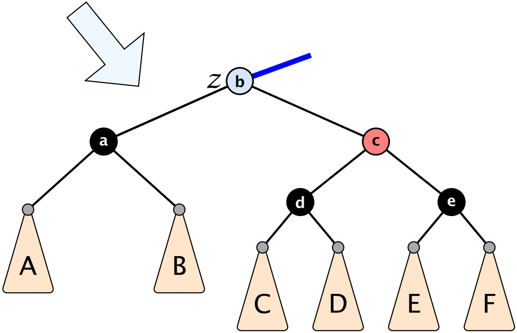
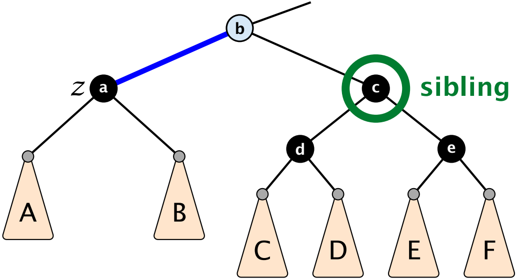
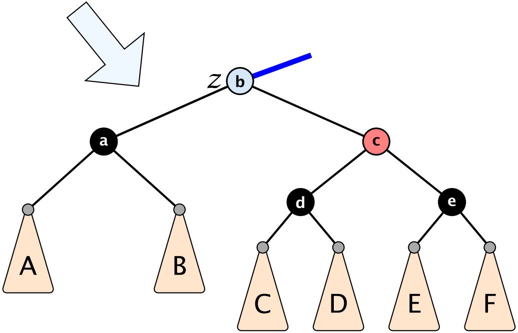


# Case 2: Sibling is black with two black children




1. re-color node $c$
2. move fake black unit upwards
3. move z upwards
4. we made progress
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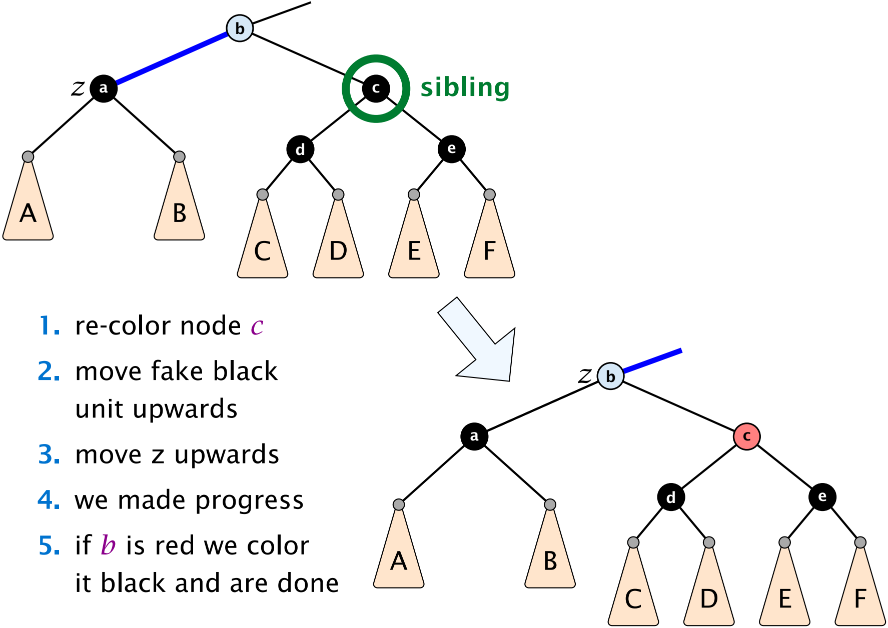


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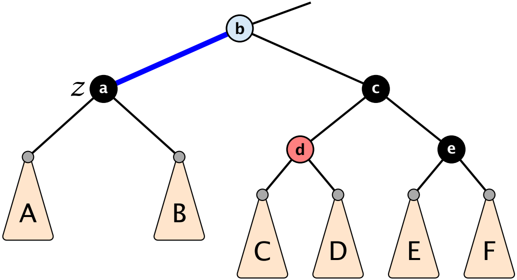
# Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor $c$ and $d$
3. new sibling is black with red right child (Case 4)



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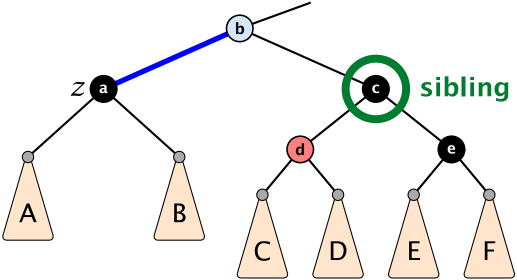

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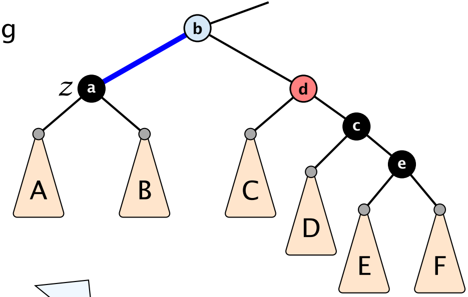


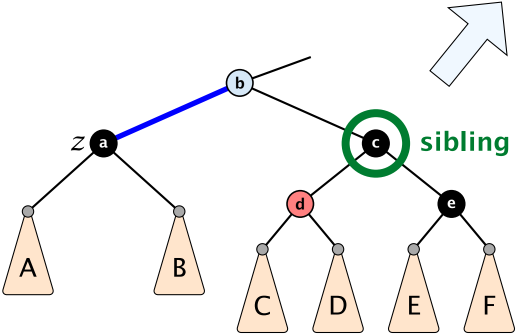

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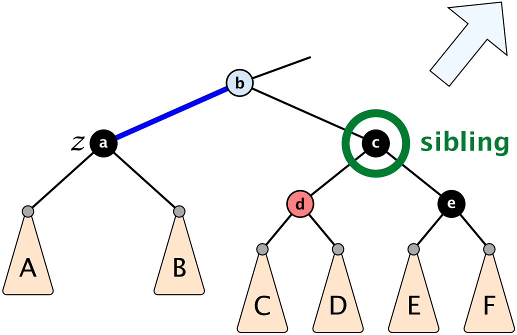


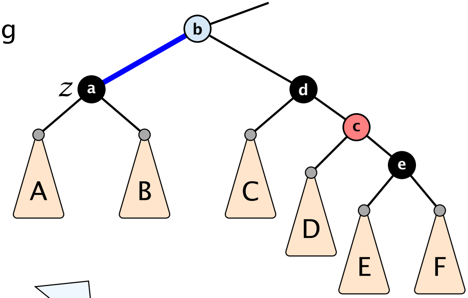

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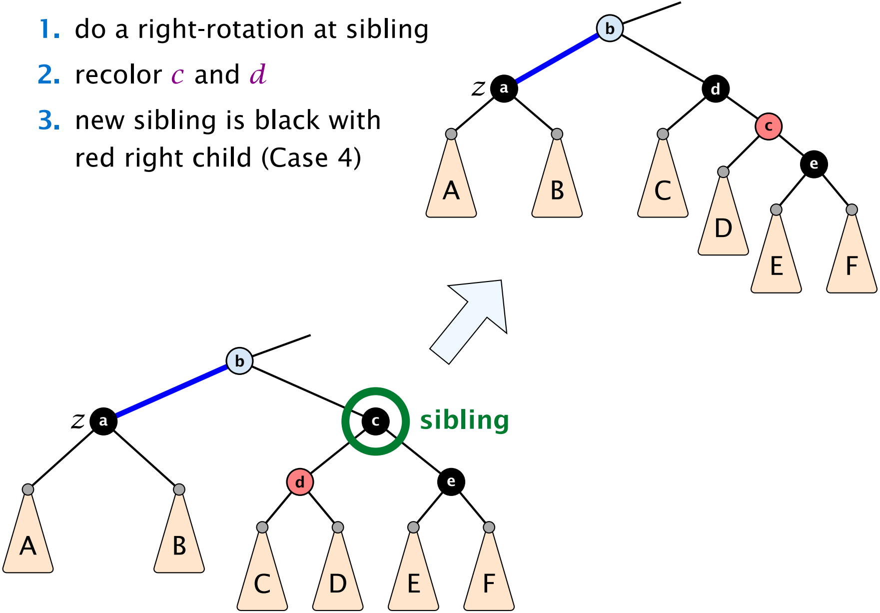
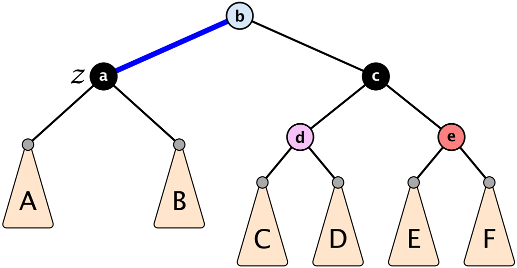
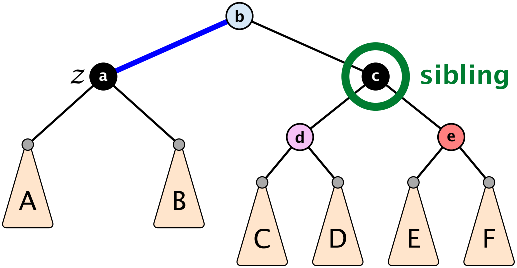


# Case 4: Sibling is black with red right child




1. left-rotate around $b$
2. remove the fake black unit
3. recolor nodes $b$, $c$, and $e$
4. you have a valid
   red black tree

# Case 4: Sibling is black with red right child

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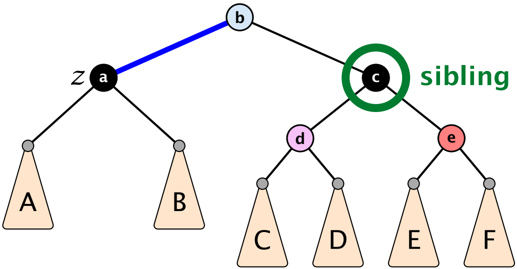


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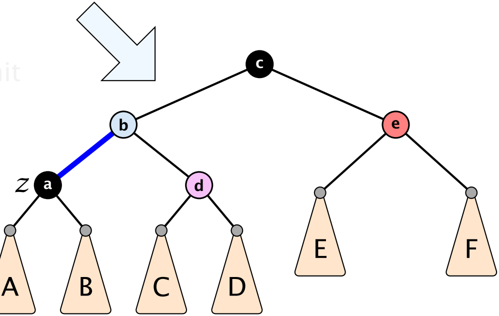

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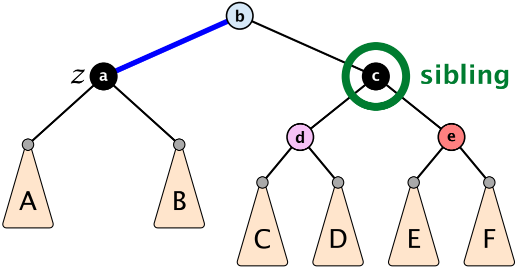


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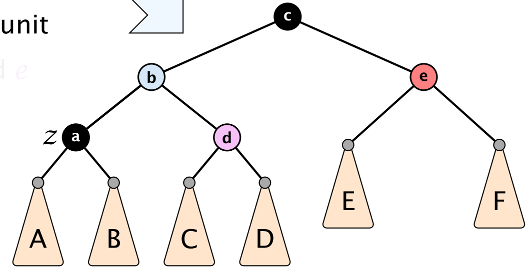

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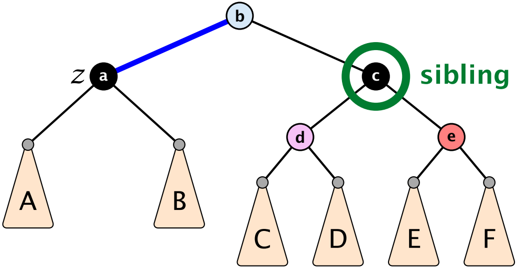


1. left-rotate around $b$
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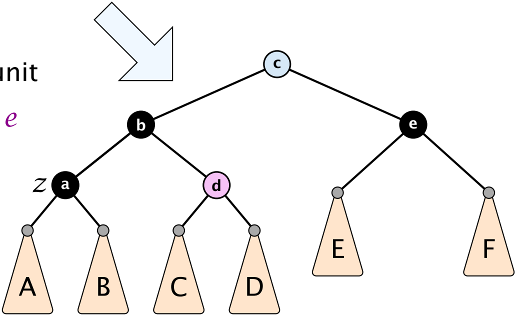

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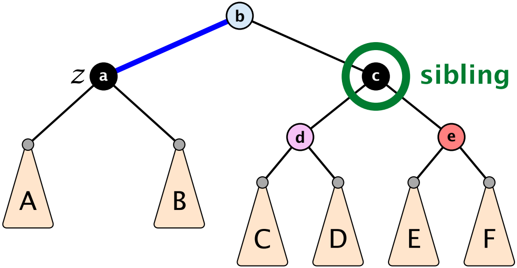


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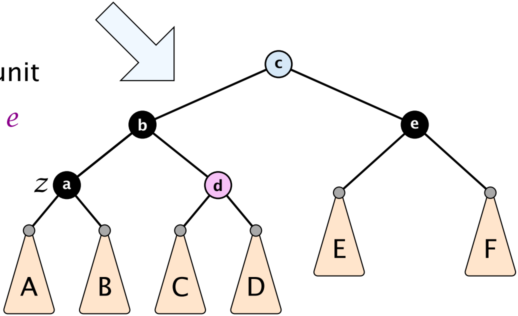

**Running time:**

- only Case 2 can repeat; but only $h$ many steps, where $h$ is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree
- Case 3 → Case 4 → red black tree
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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most $3$ rotations.

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