9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- **$\mathcal{P}$. makeset(x):** Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.
- **$\mathcal{P}$. find(x):** Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
- **$\mathcal{P}$. union(x, y):** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

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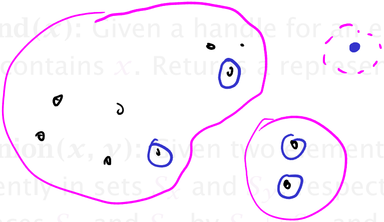

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Applications:

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- Kruskals Minimum Spanning Tree Algorithm

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9 Union Find

Algorithm 16 Kruskal-MST($G = (V, E), w$)

```
A ← ∅;
for all v ∈ V do
    v.set ← P.makeset(v.label)                    n
sort edges in non-decreasing order of weight w    (O(m log m))
for all (u, v) ∈ E in non-decreasing order do
    if P.find(u.set) ≠ P.find(v.set) then         |E| = m
        A ← A ∪ {(u, v)}                          ← n-1
        P.union(u.set, v.set)                     ← n-1
```

List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.

- $\text{makeset}(x)$ can be performed in constant time.
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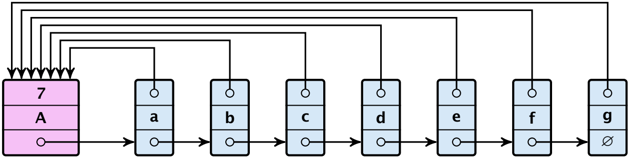

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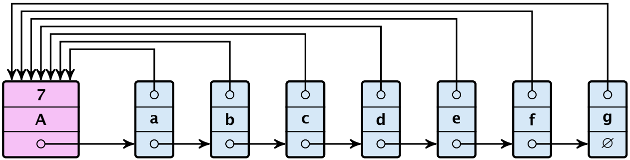

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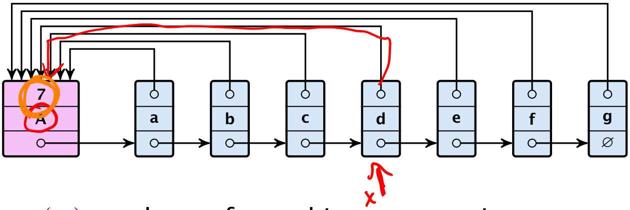

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List Implementation

union(x, y)

- Determine sets S_x and S_y.
- Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- Insert list S_y at the head of S_x.
- Adjust the size-field of list S_x.
- Time: $\min\{|S_x|, |S_y|\}$.

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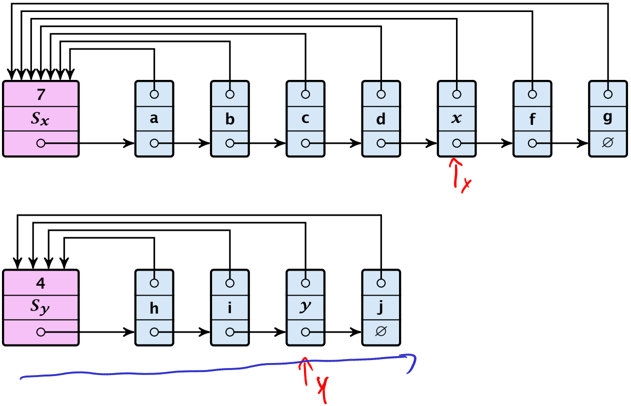

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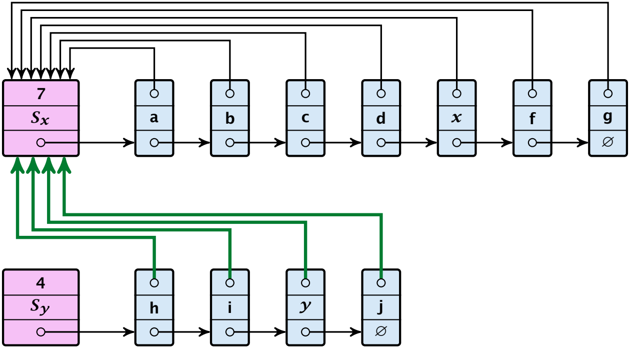

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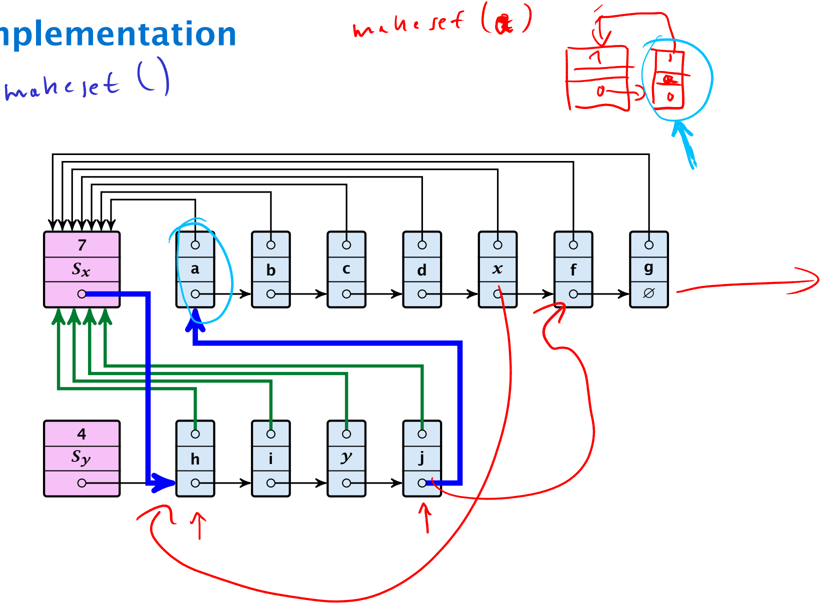

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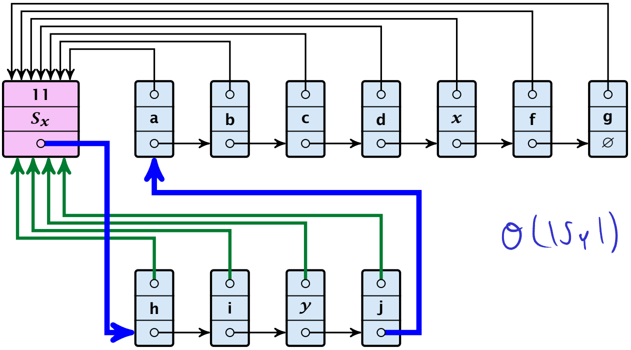

List Implementation

$n + 2|E| + (n-1)\,\mathcal{O}(n)$

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- O(1) ▶ find(x): $\mathcal{O}(1)$.
- O(1) ▶ makeset(x): $\mathcal{O}(\log n)$.
- O(n) ▶ union(x, y): $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
- Initially the balance on all accounts is zero.
- Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
- Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.

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makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

- If $S_x = S_y$ the cost is constant; no bank accounts change.
- Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.
- Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.
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Lemma 35

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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- Maintain nodes of a set in a tree.
- The root of the tree is the label of the set.
- Only pointer to parent exists; we cannot list all elements of a given set.
- Example:

Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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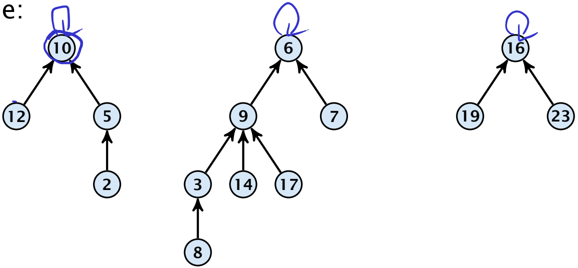

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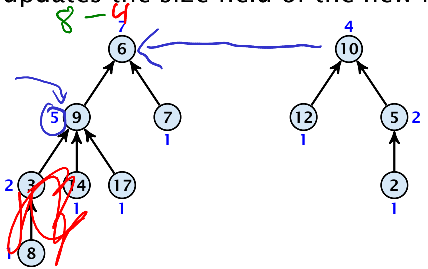

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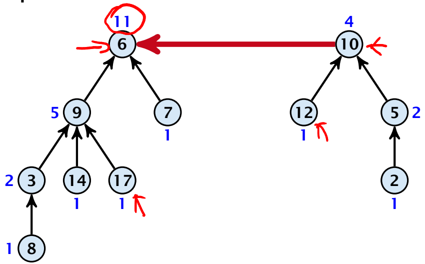

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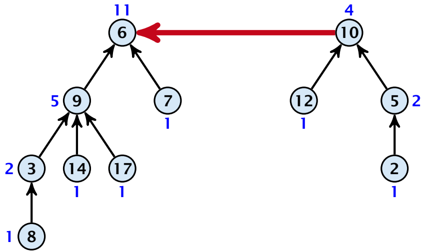
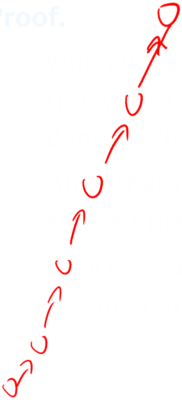


- Time: constant for $\text{link}(a, b)$ plus two find-operations.

Implementation via Trees

Lemma 36

The running time (non-amortized!!!) for $\text{find}(x)$ *is* $\mathcal{O}(\log n)$*.*



Implementation via Trees

Lemma 36

The running time (non-amortized!!!) for $\text{find}(x)$ *is* $\mathcal{O}(\log n)$.

Proof.

- When we attach a tree with root c to become a child of a tree with root p, then $\text{size}(p) \geq 2\,\text{size}(c)$, where size denotes the value of the size-field right after the operation.
- After that the value of $\text{size}(c)$ stays fixed, while the value of $\text{size}(p)$ may still increase.
- Hence, at any point in time a tree fulfills $\text{size}(p) \geq 2\,\text{size}(c)$, for any pair of nodes (p, c), where p is a parent of c.

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Path Compression

find(x):

- Go upward until you find the root.
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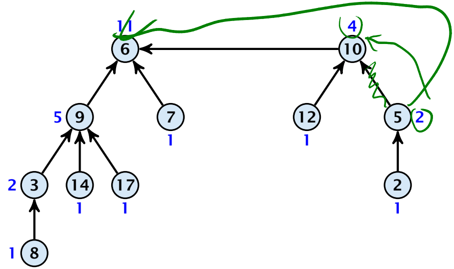

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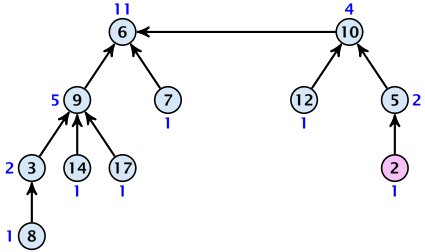

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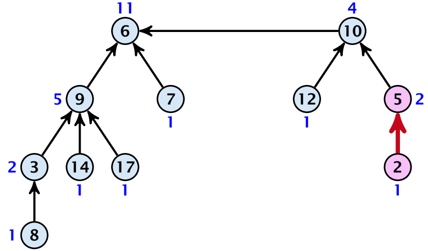

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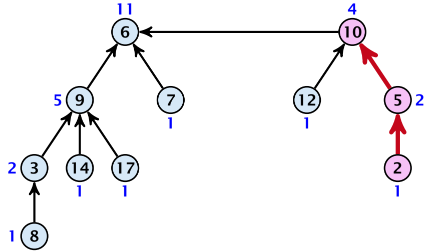

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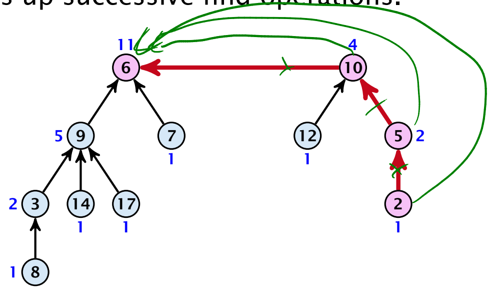

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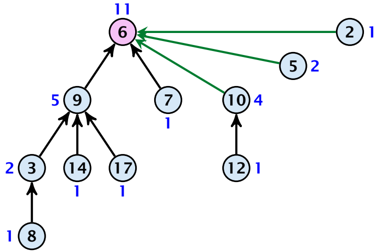

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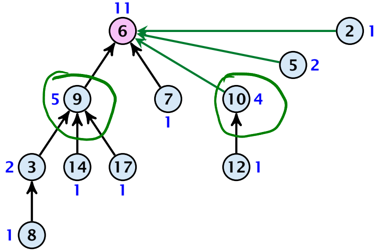

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Amortized Analysis

Definitions:

- $\text{size}(v)$ = the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find-operations.
- $\text{rank}(v) = \lfloor \log(\text{size}(v)) \rfloor$.
- $\Longrightarrow \text{size}(v) \geq 2^{\text{rank}(v)}$.

Lemma 37

The rank of a parent must be strictly larger than the rank of a child.

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There are at most $n/2^s$ nodes of rank s.

Proof.

- Let's say a node v sees node x if v is in x's sub tree at the time that x becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 39

Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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- The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.
- A rank group $g \geq 1$ contains ranks $\text{tow}(g-1)+1, \dots, \text{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
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Accounting Scheme:

- create an account for every find-operation
- create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
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