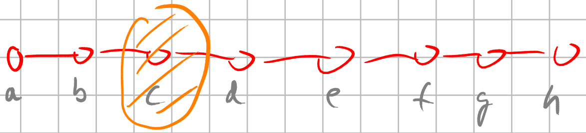
16 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set F and capacities w that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f_G(s, t)$ in G is equal to $f_T(s, t)$.
2. **Cut Property:** A minimum s-t cut in T is also a minimum cut in G.

Here, $f(s, t)$ is the value of a maximum s-t flow in G, and $f_T(s, t)$ is the corresponding value in T.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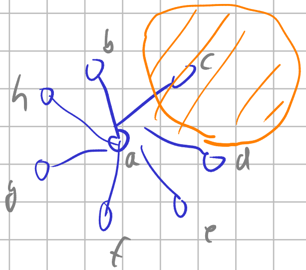
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Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \dots, S_t$), and a spanning tree T on the vertex set $\{S_1, \dots, S_t\}$.

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In the end this gives a tree on the vertex set V.

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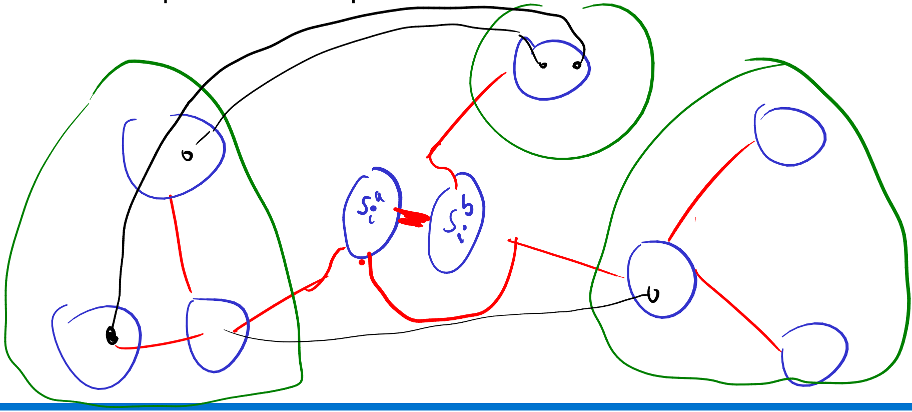

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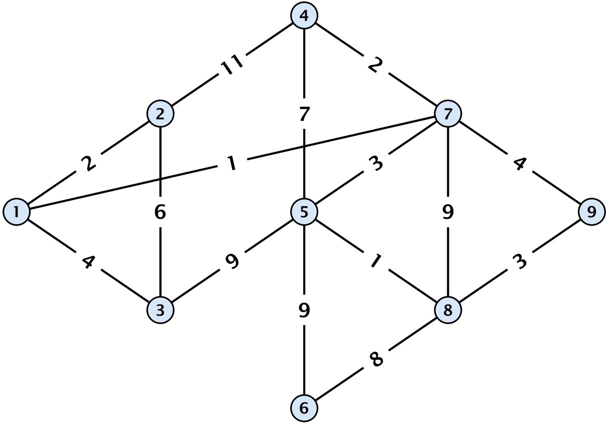
- Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction



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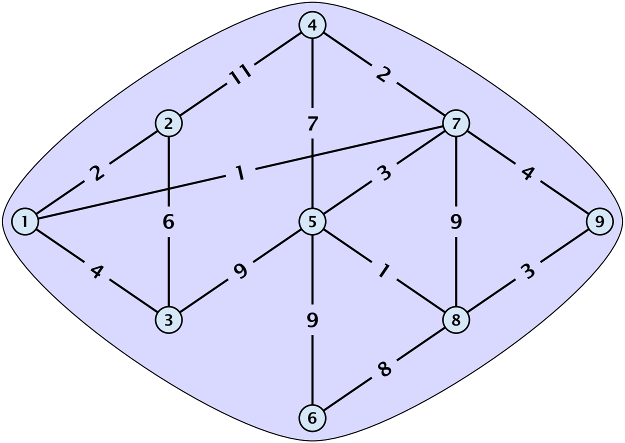

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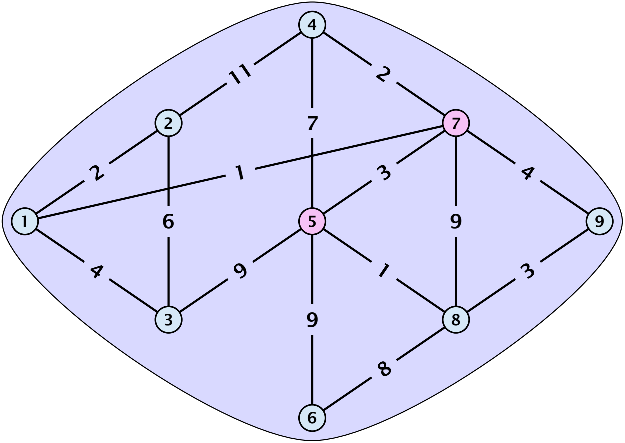

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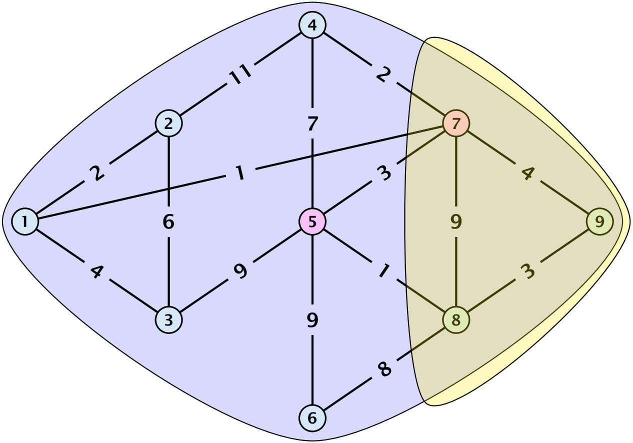

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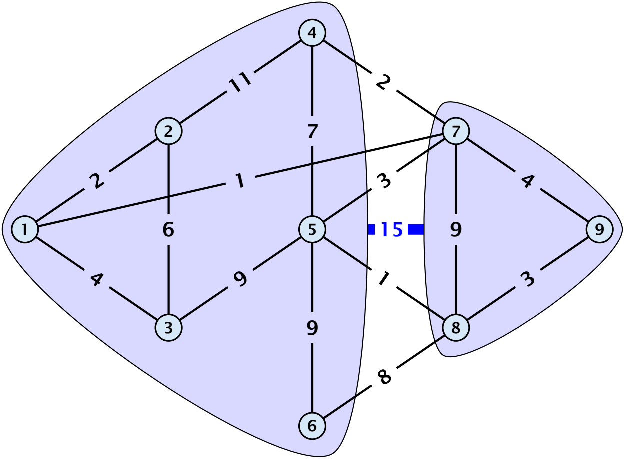

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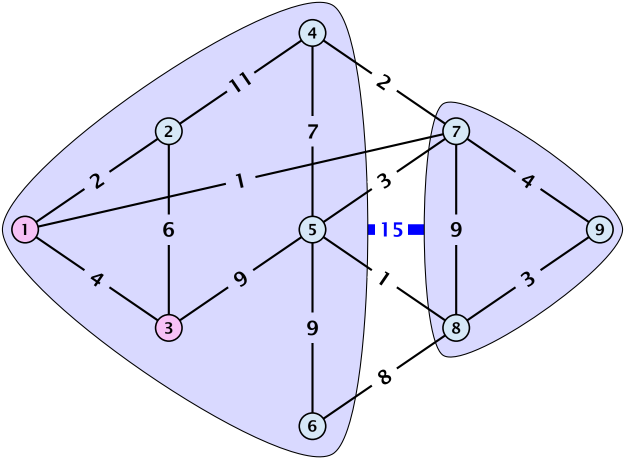

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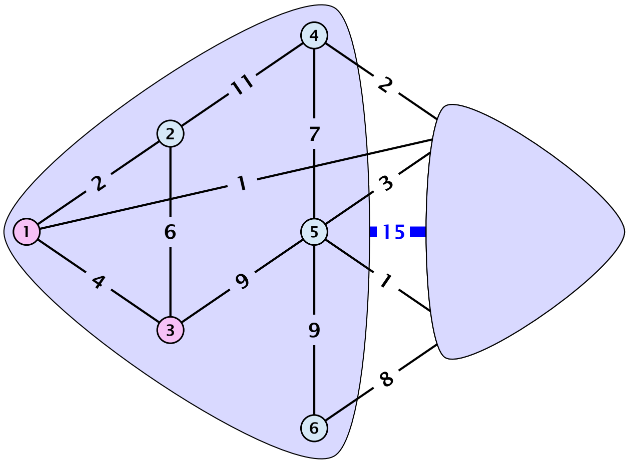

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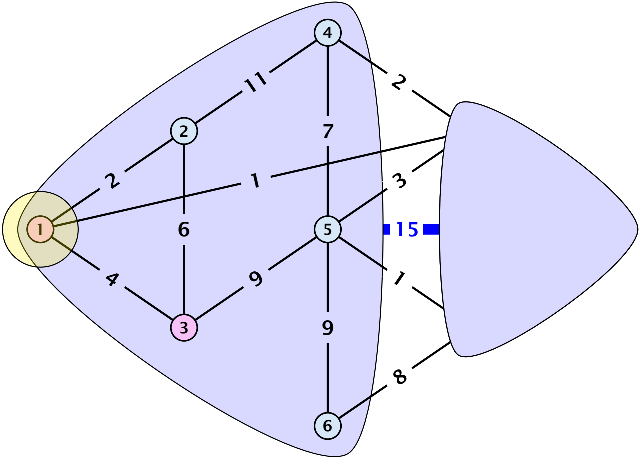

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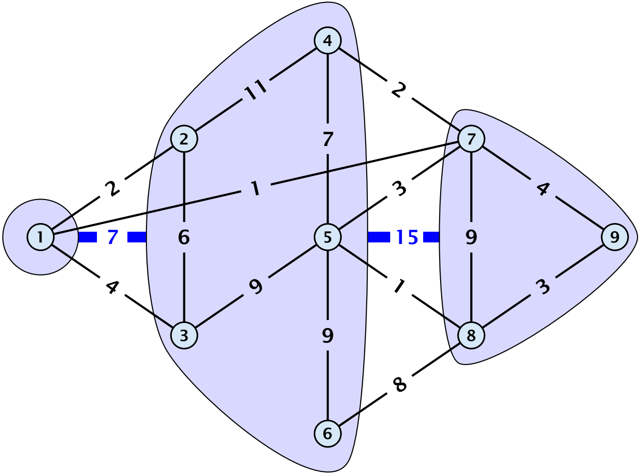

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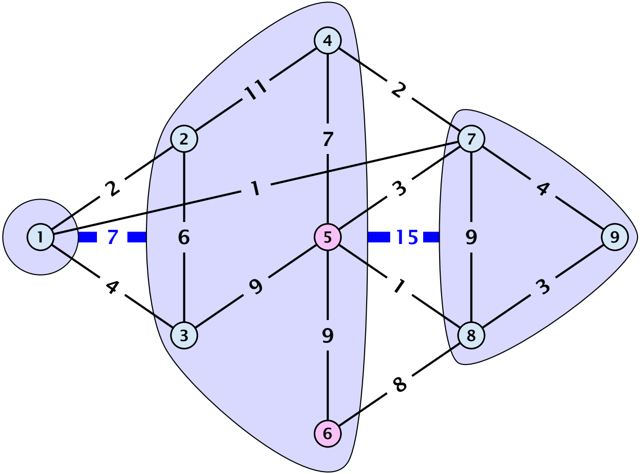

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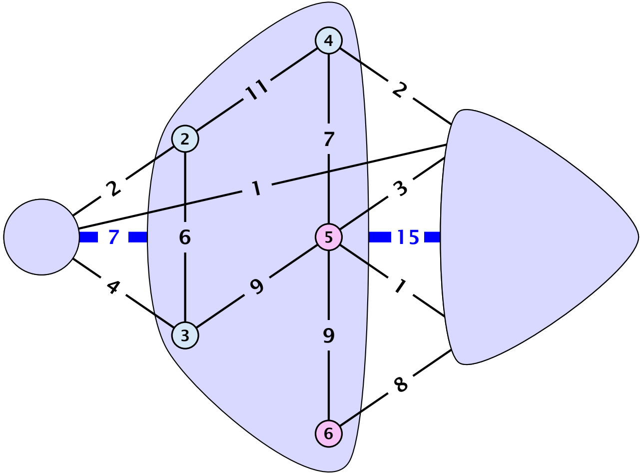

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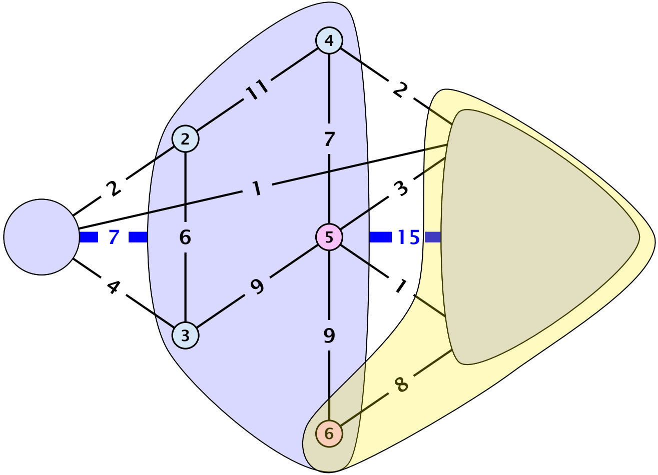

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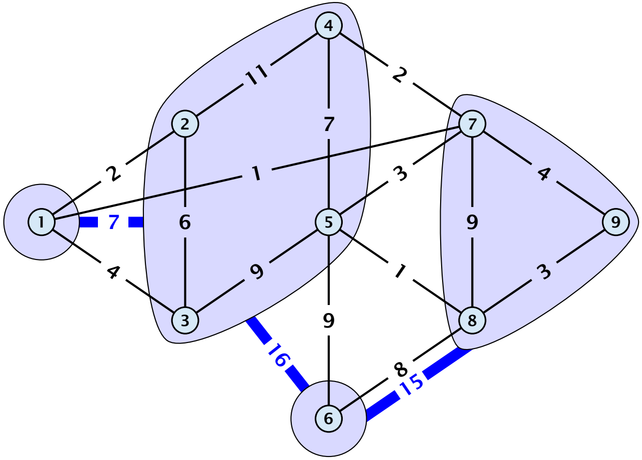

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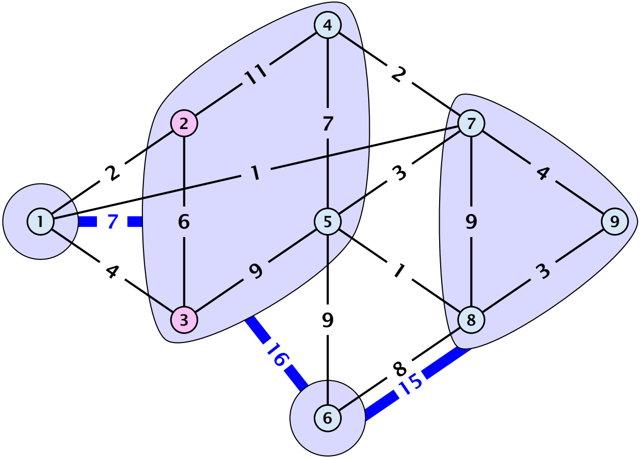

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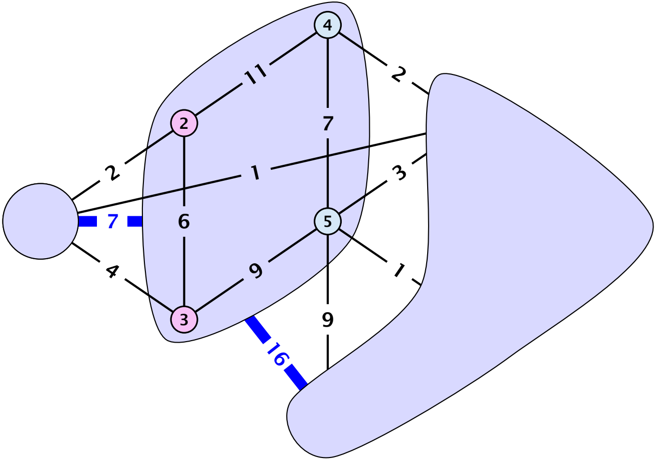

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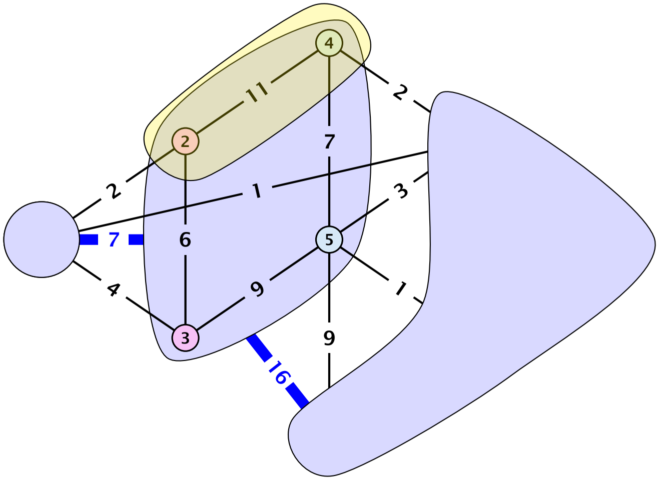

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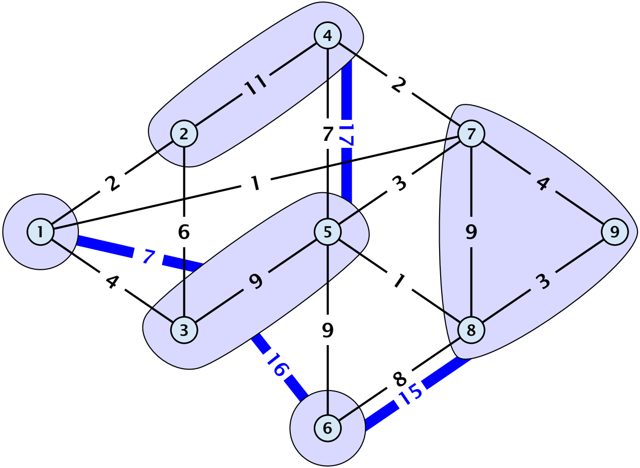

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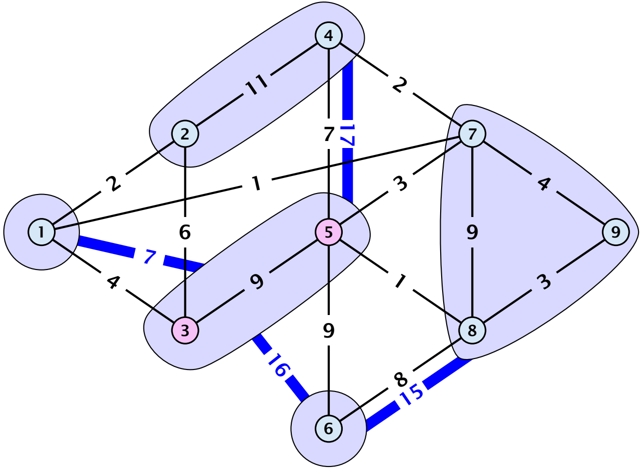

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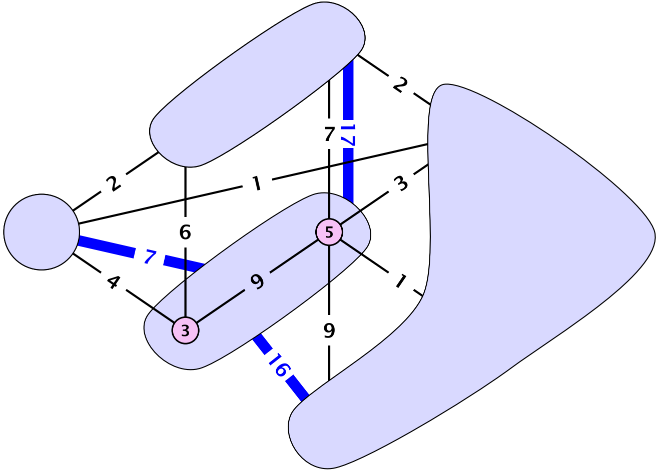

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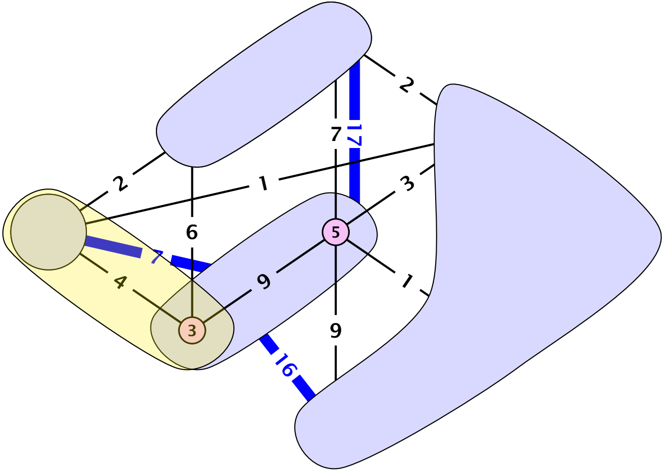

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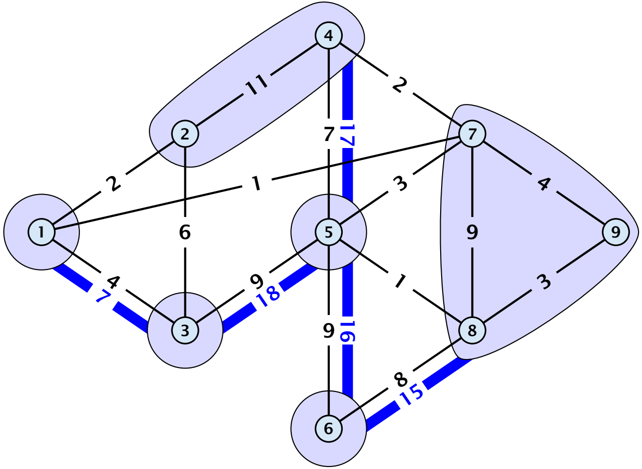

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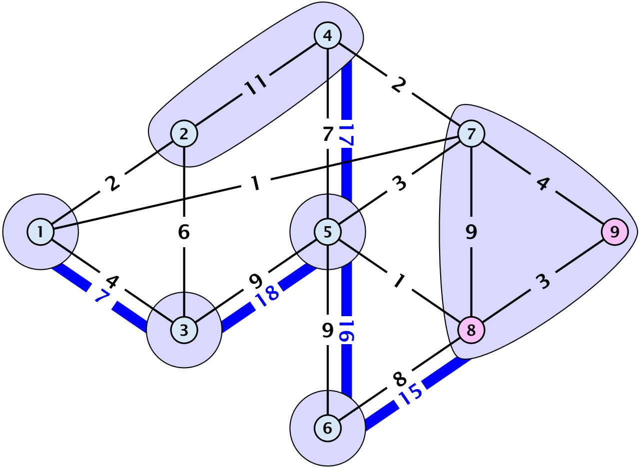

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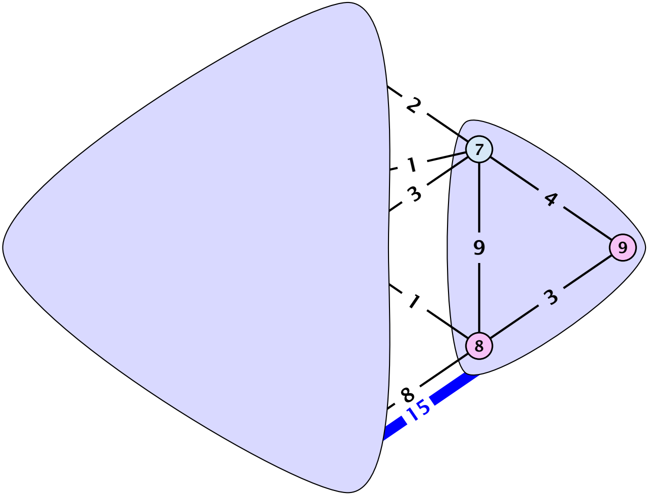

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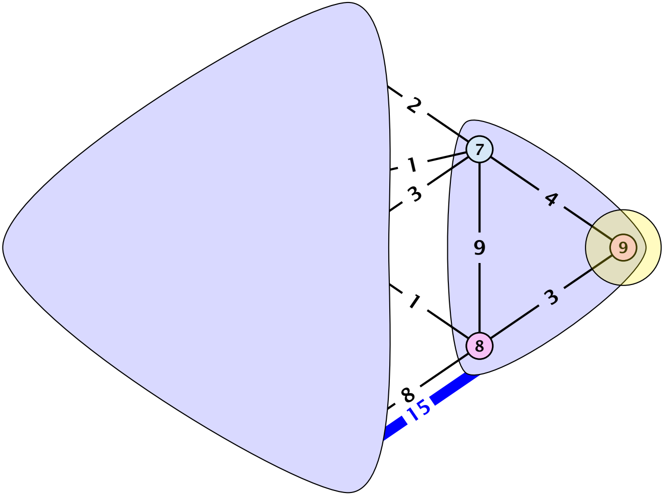

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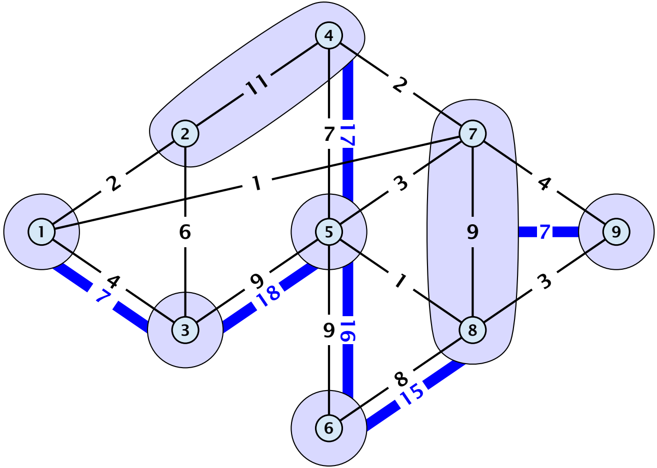

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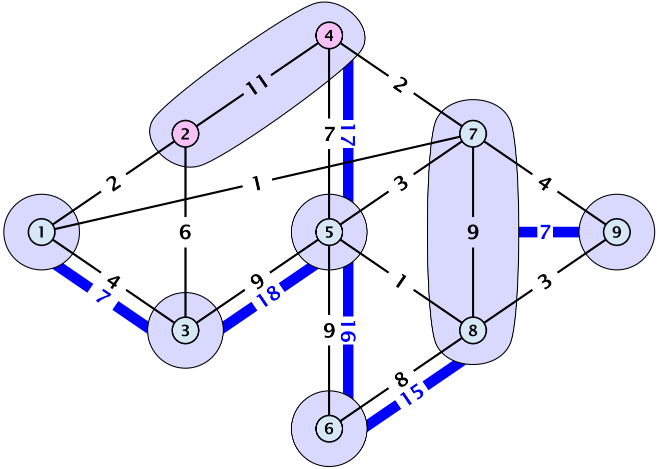

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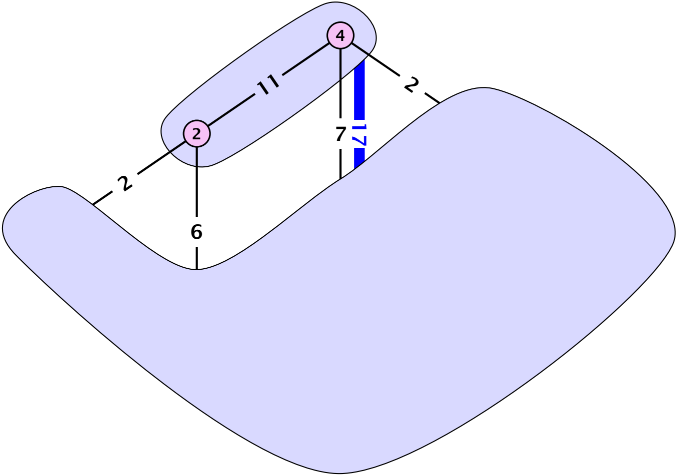

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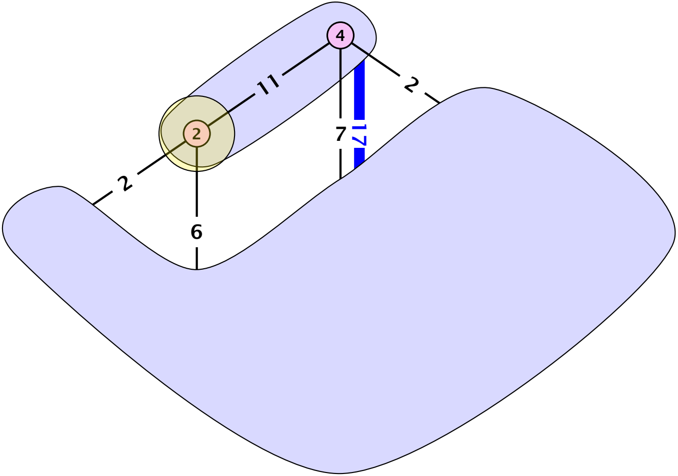

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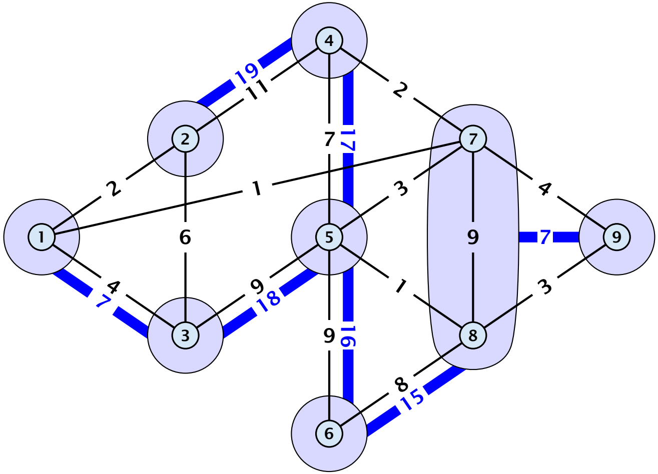

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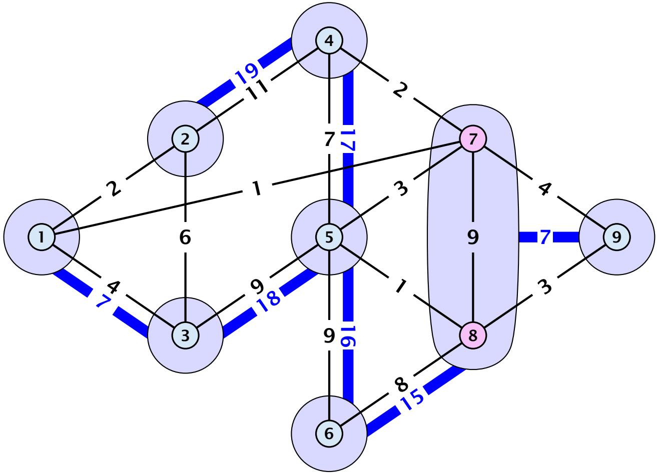

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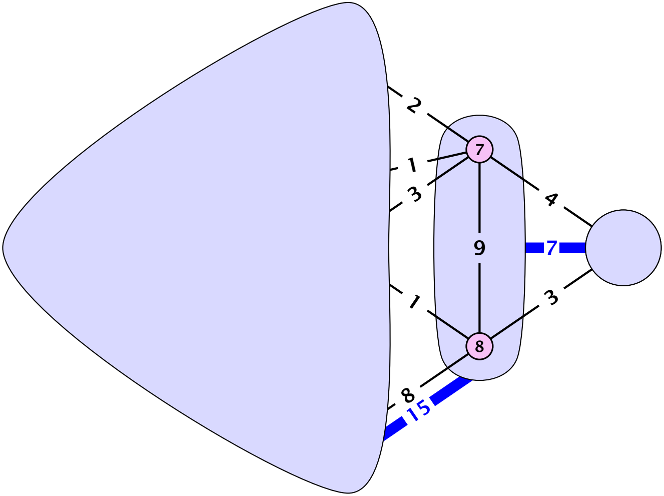

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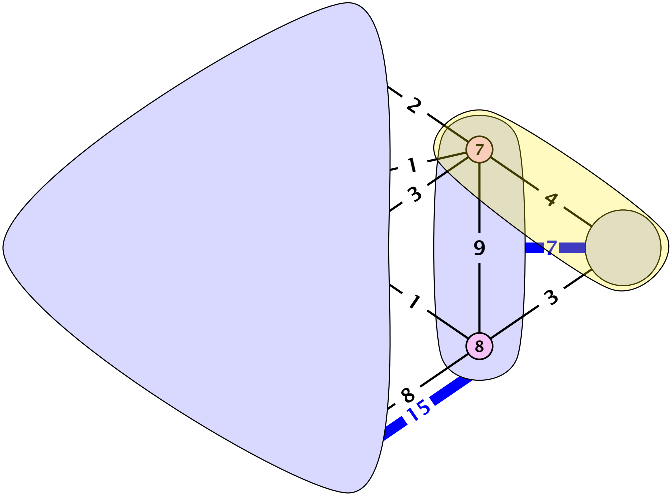

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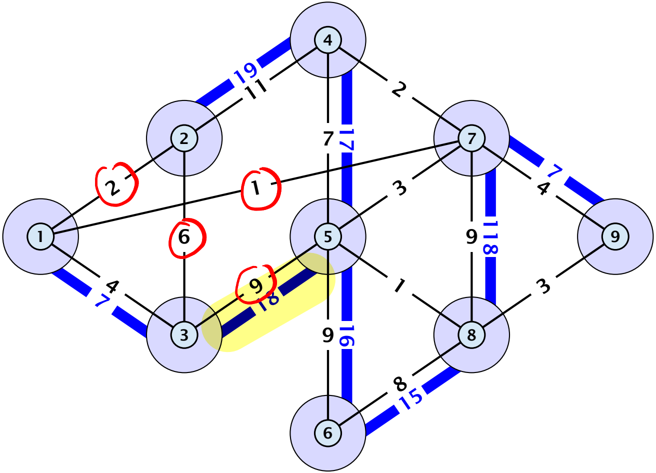

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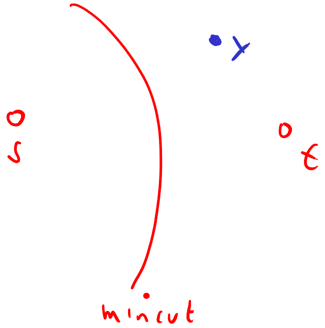
For nodes $s, t, x \in V$ *we have* $f(s,t) \geq \min\{f(s,x), f(x,t)\}$



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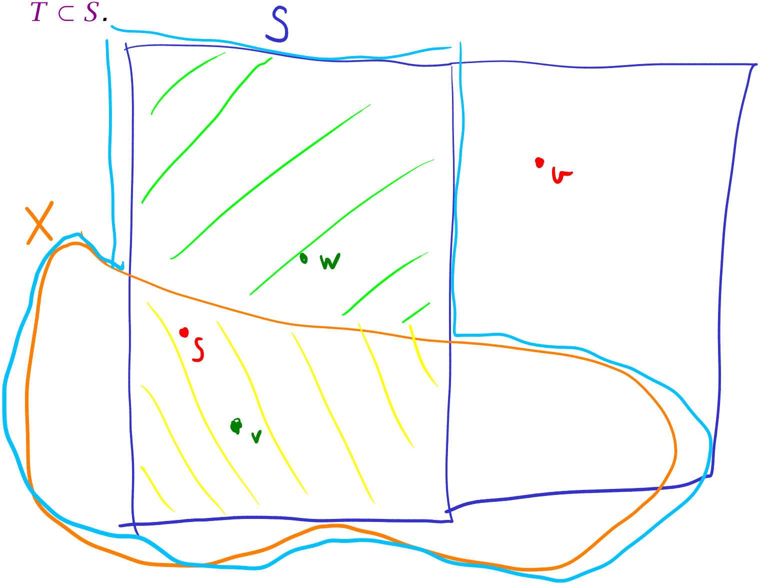

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- $\mathrm{cap}(X \setminus S) + \mathrm{cap}(S \setminus X) \leq \mathrm{cap}(S) + \mathrm{cap}(X)$.
- $\mathrm{cap}(X \setminus S) \geq \mathrm{cap}(S)$ because $X \setminus S$ is an r-s cut.
- This gives $\mathrm{cap}(S \setminus X) \leq \mathrm{cap}(X)$.

Second case $r \notin X$.

- $\mathrm{cap}(X \cup S) + \mathrm{cap}(S \cap X) \leq \mathrm{cap}(S) + \mathrm{cap}(X)$.
- $\mathrm{cap}(X \cup S) \geq \mathrm{cap}(S)$ because $X \cup S$ is an r-s cut.

Lemma 91

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

We may assume w.l.o.g. $s \in X$.

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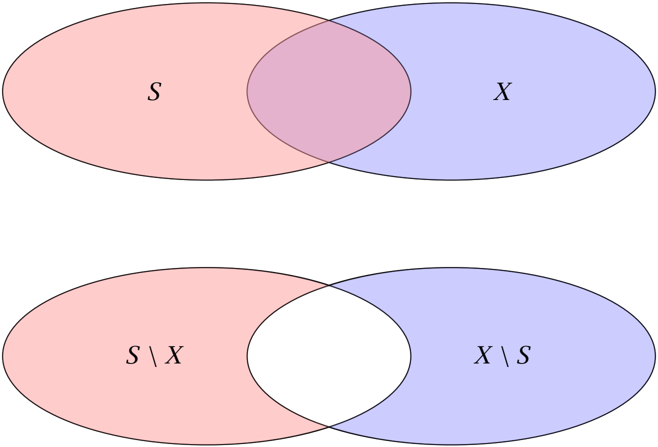

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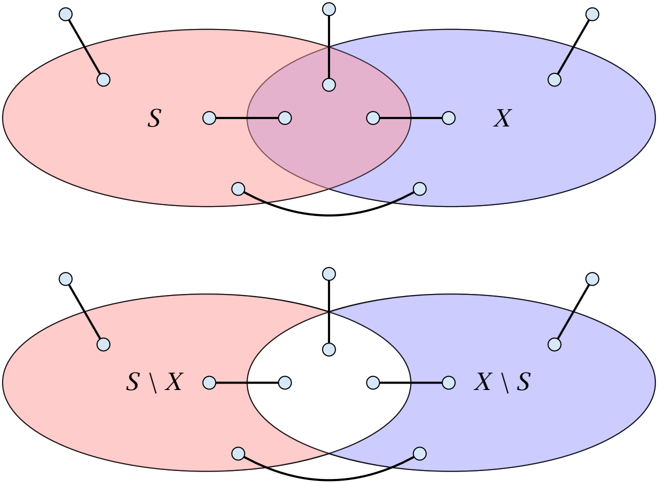

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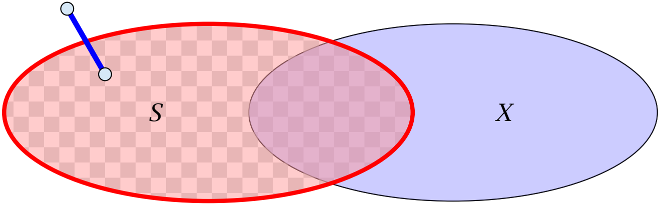

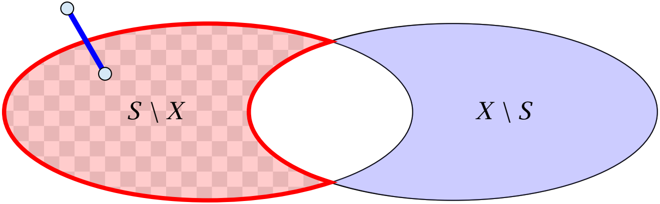

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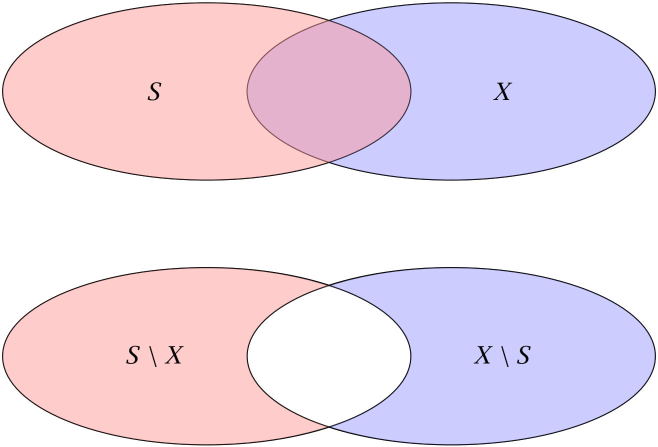

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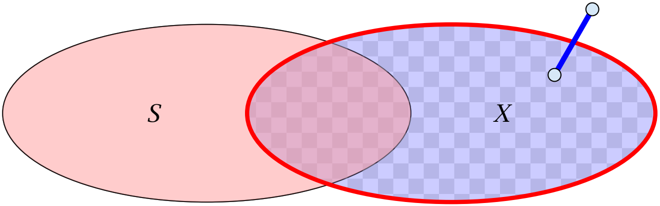

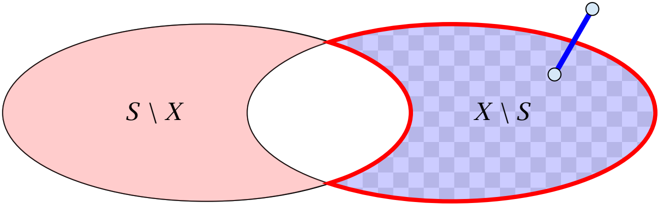

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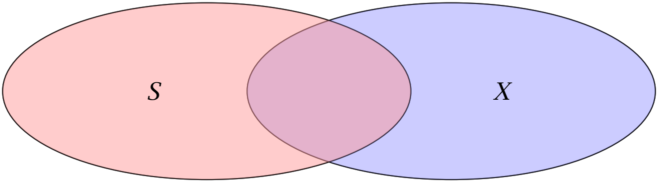

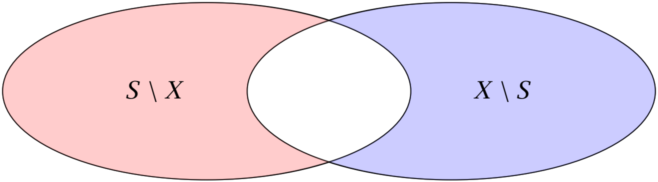

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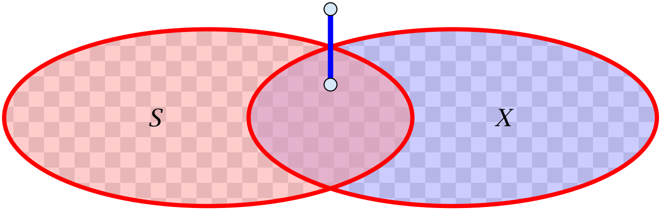

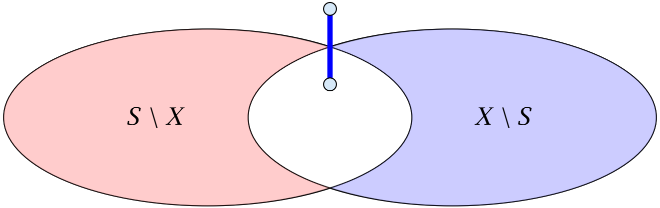

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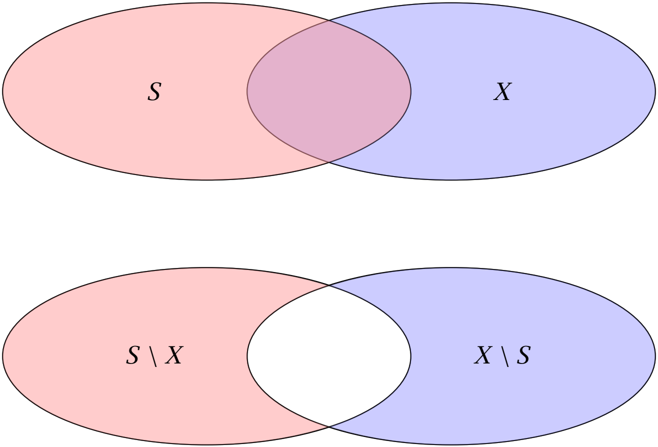

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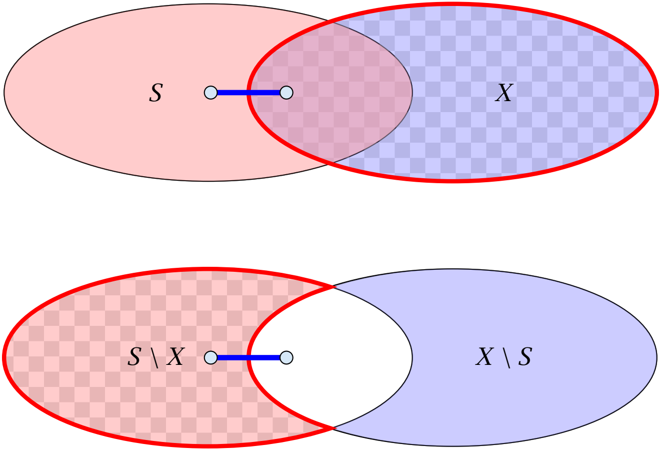

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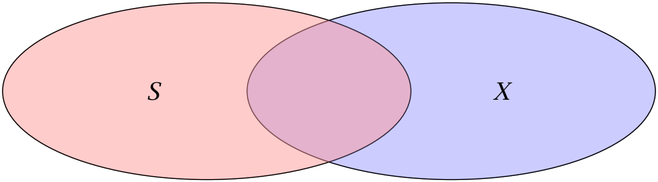

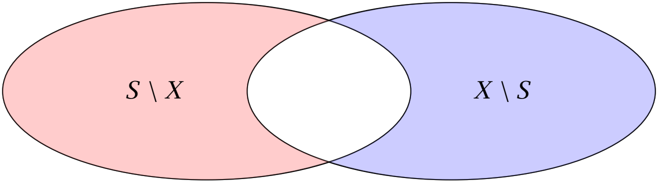

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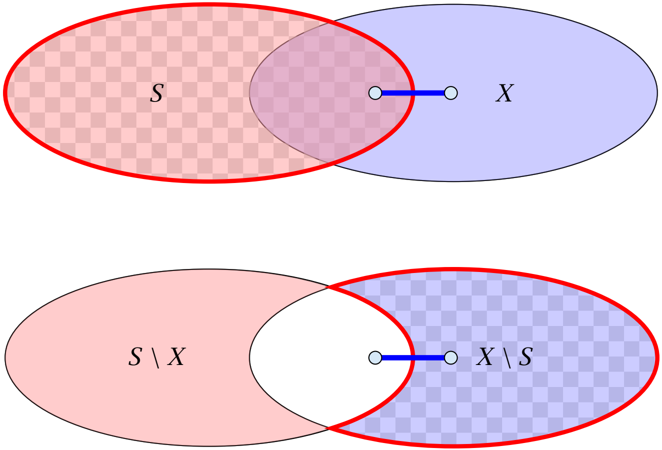

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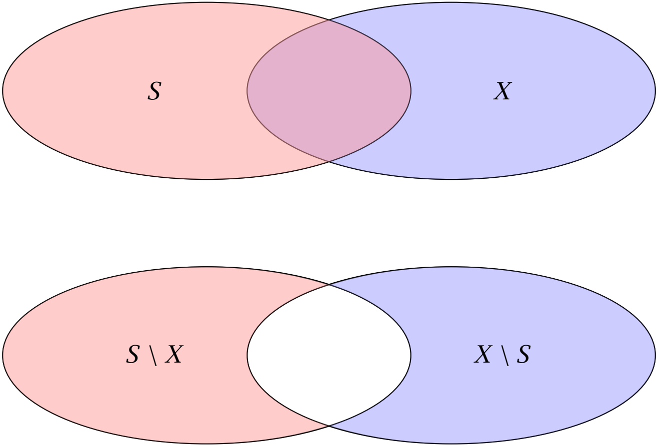

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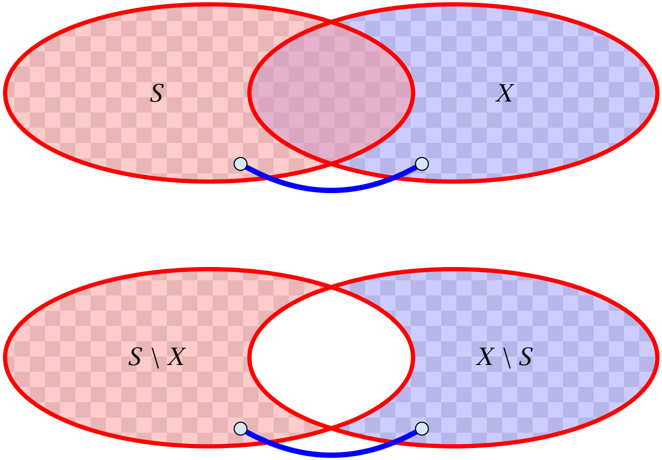

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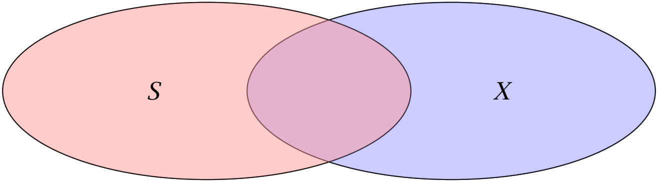

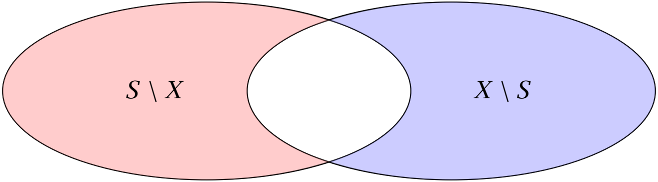

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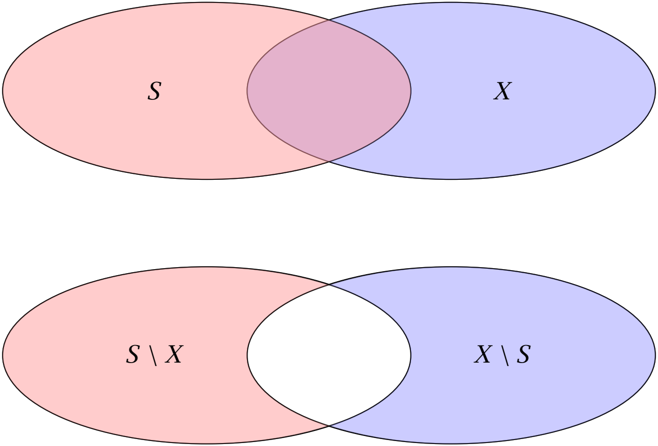

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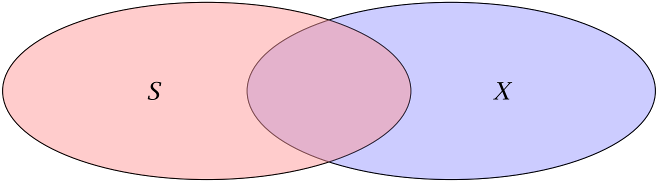

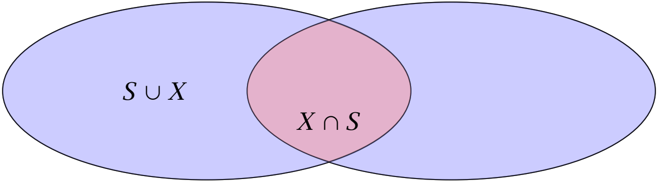

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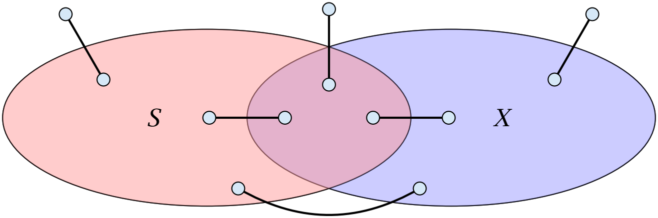

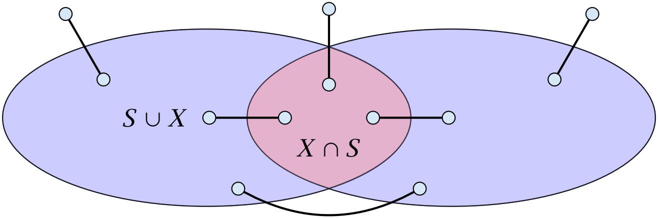

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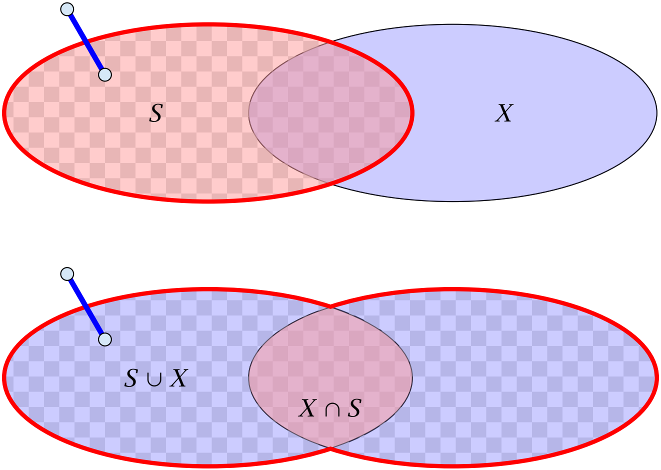

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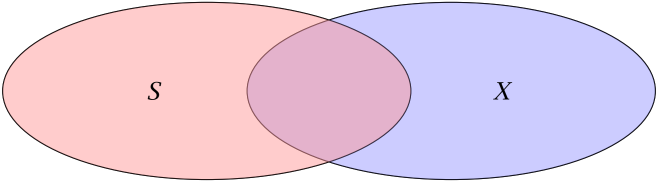

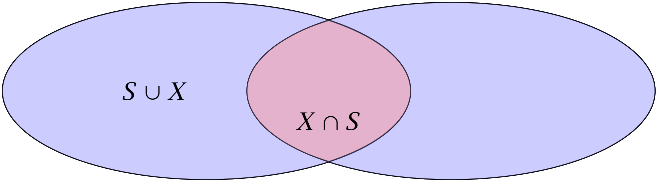

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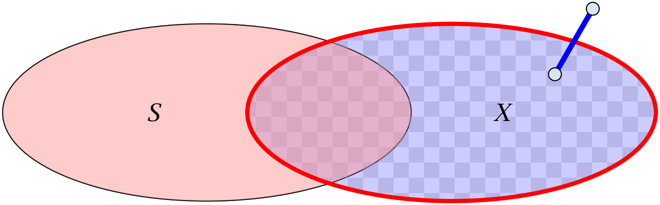

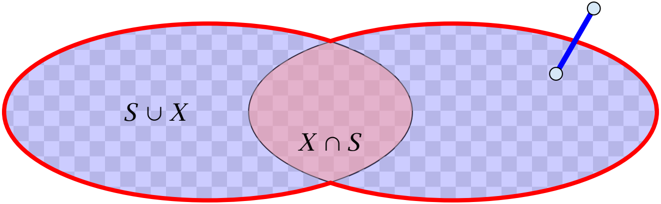

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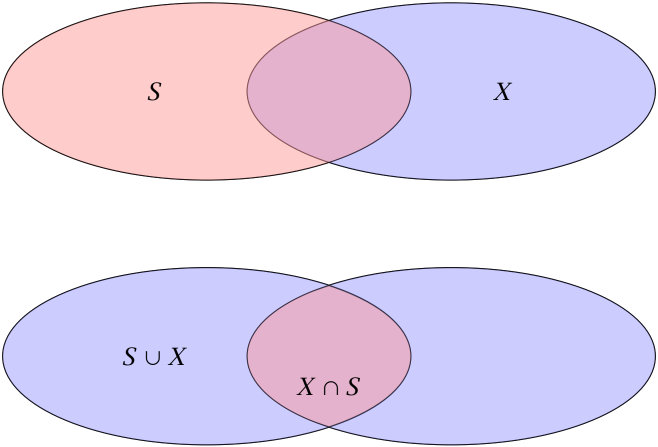

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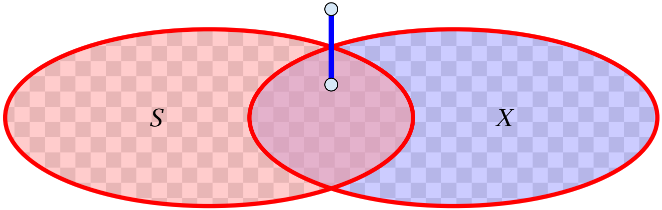

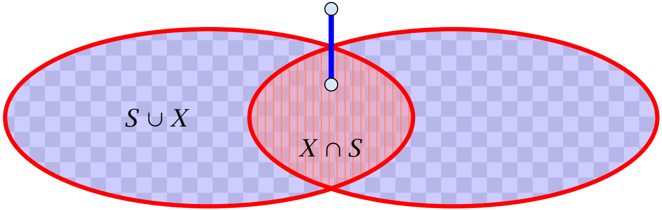

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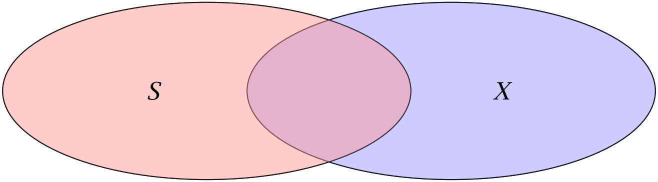

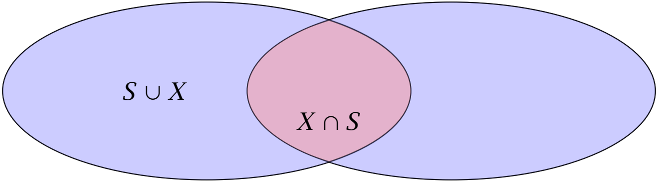

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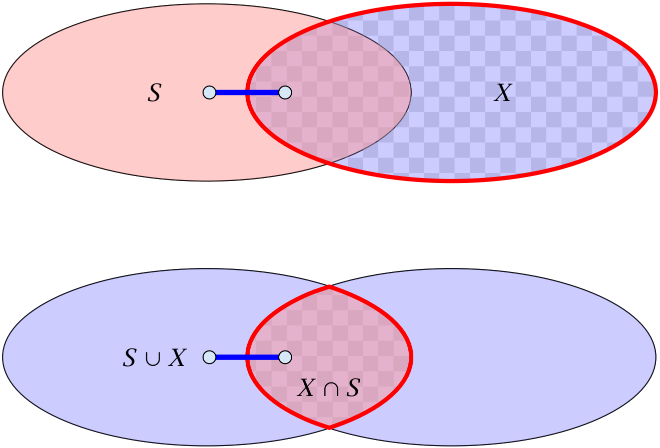

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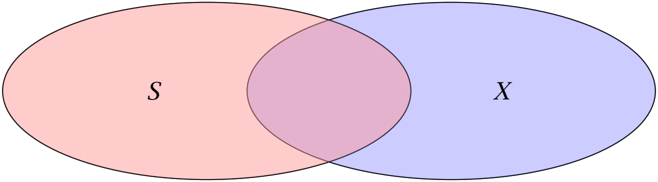

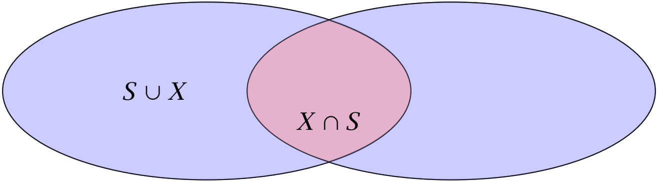

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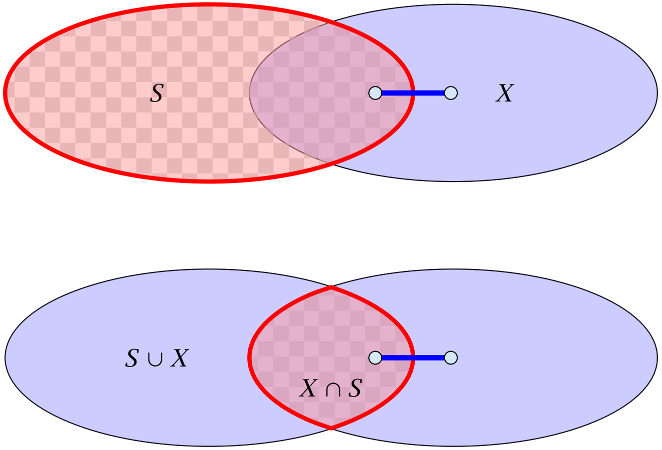

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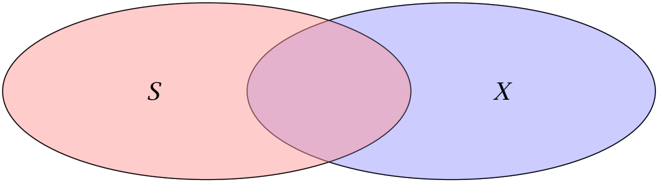

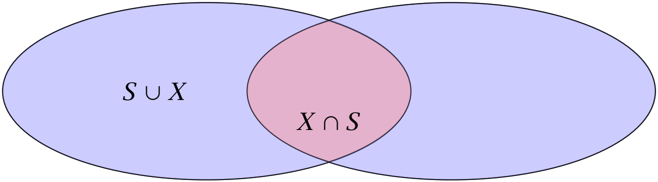

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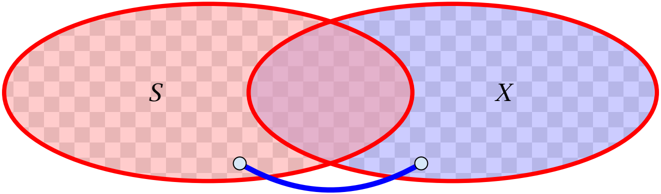

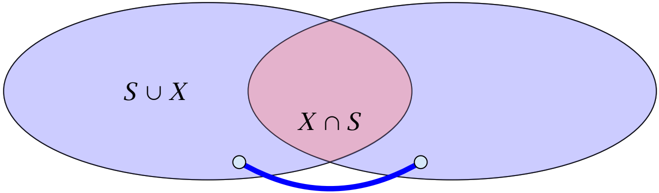

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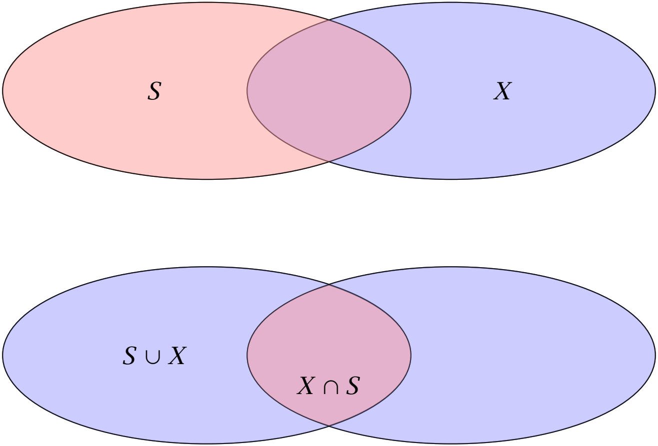

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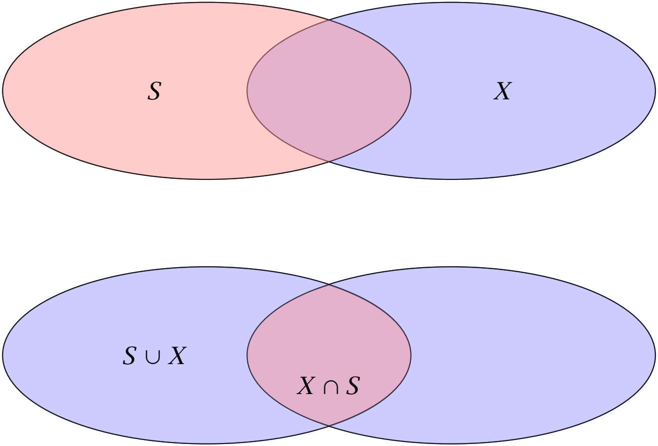

Analysis

Lemma 91 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum s-t mincut in graph H.

Analysis

Invariant [existence of representatives]:
For any edge $\{S_i, S_j\}$ in T, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a-b cut in G.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

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- Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

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$$f_T(s,t)$$

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- Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.
- Since by the invariant this edge induces an s-t cut with capacity $f(x_j, x_{j+1})$ we get $f(s,t) \le f(x_j, x_{j+1}) = f_T(s,t)$.

Analysis

- Hence, $f_T(s,t) = f(s,t)$ (flow equivalence).

Analysis

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- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).
- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t, this is an s-t mincut (cut property).

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After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose a and b as representatives.

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If $s \in S_i^a$ we can keep x and s as representatives.

Otherwise, we choose x and a as representatives. We need to show that $f(x, a) = f(x, s)$.

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The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 91 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

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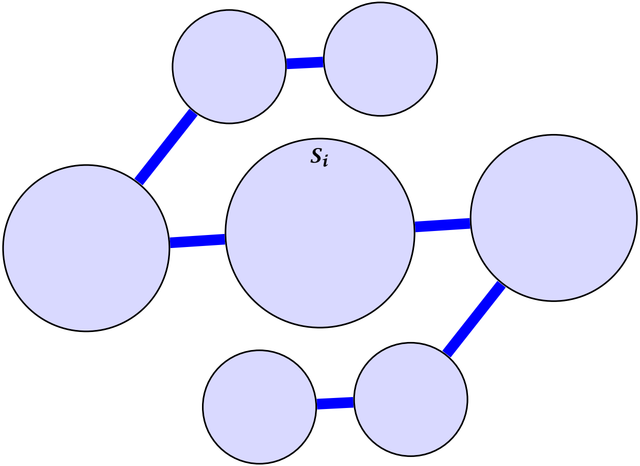
The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 91 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

We further have $f'(x,a) \geq \min\{f'(x,v_B), f'(v_B,a)\}$.

Since $s \in B$ we have $f'(v_B,x) \geq f(s,x)$.

Also, $f'(a,v_B) \geq f(a,b) \geq f(x,s)$ since the a-b cut that splits S_i into S_i^a and S_i^b also separates s and x.

Analysis



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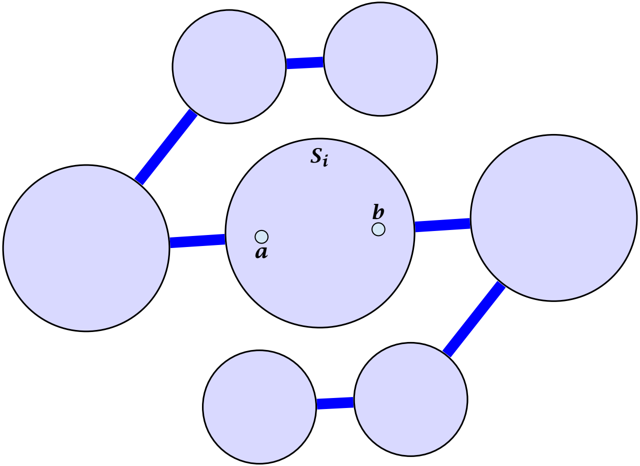

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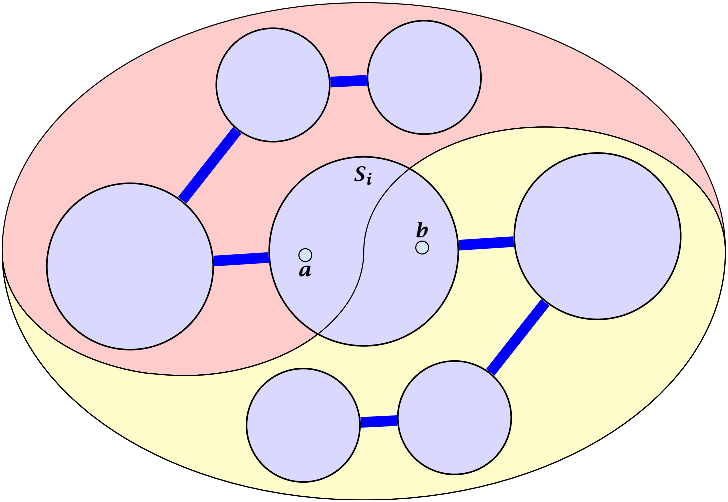

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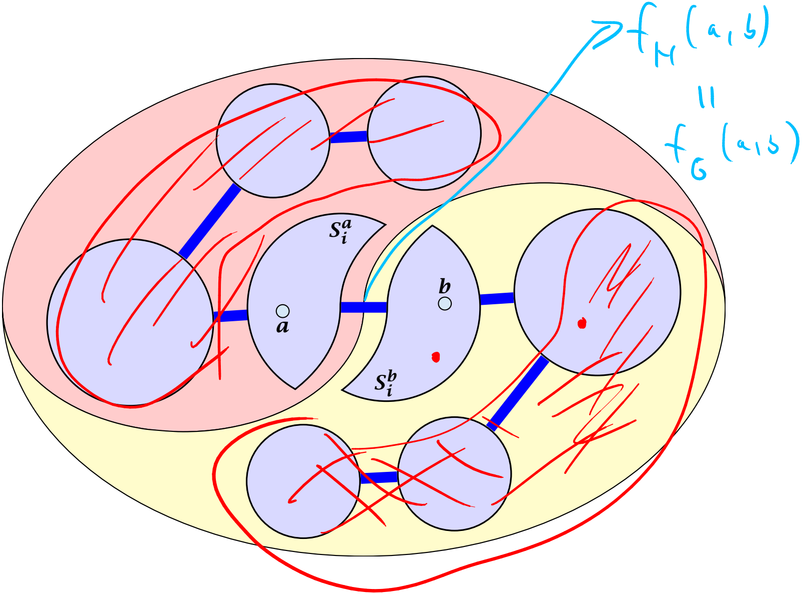

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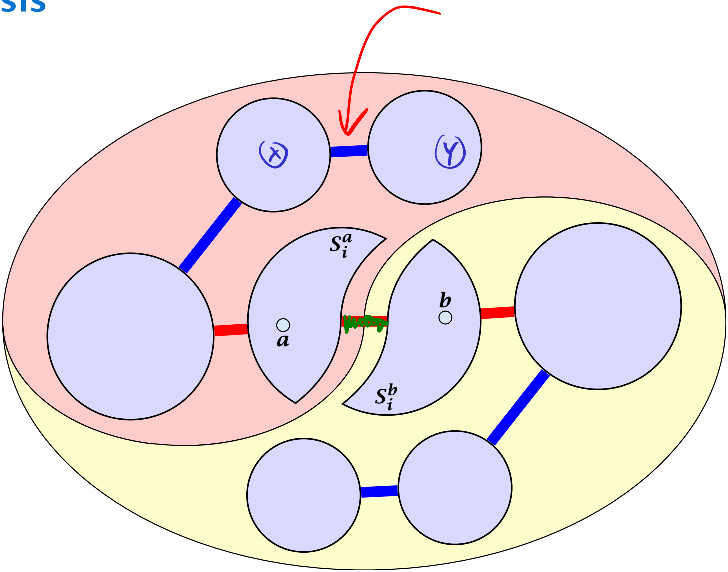

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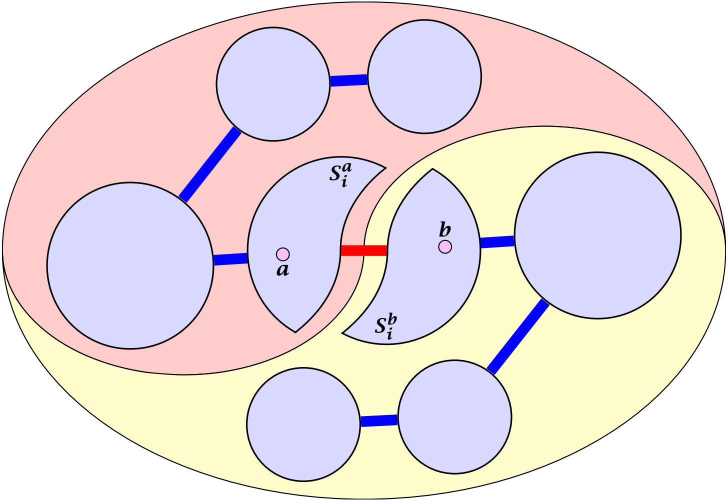

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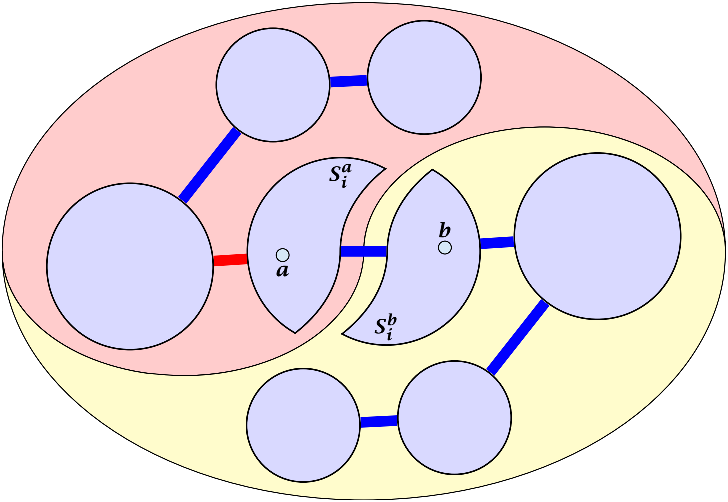

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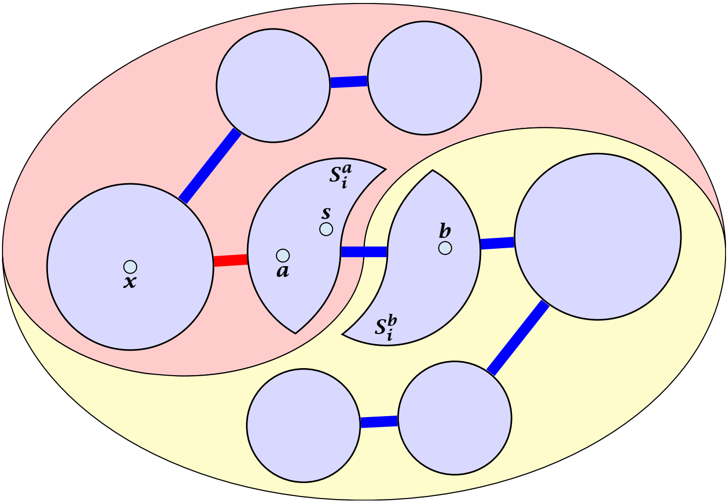

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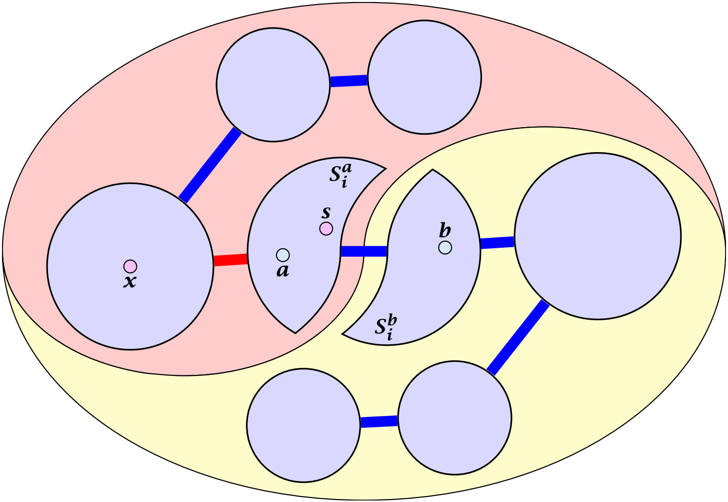

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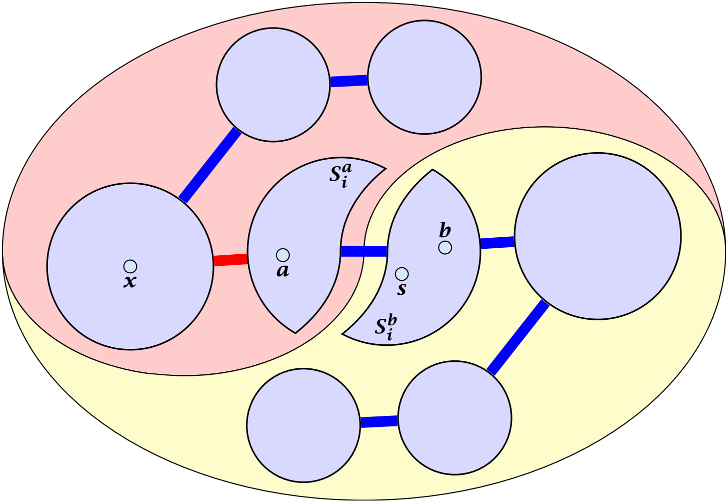

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$\rightarrow f(x,s)$