Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

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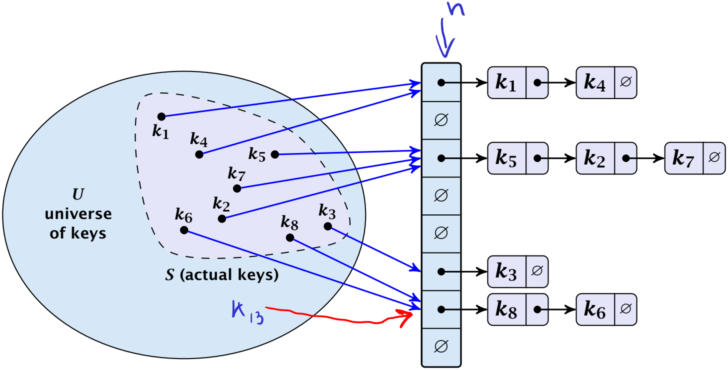
Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\mathrm{key}[x]$.
- Insert: insert at the front of the list.



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Let A denote a strategy for resolving collisions. We use the following notation:

- A^+ denotes the average time for a **successful** search when using A;
- A^- denotes the average time for an **unsuccessful** search when using A;
- We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

We assume uniform hashing for the following analysis.

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$$A^- = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\Bigg[\frac{1}{m}\sum_{i=1}^{m}\Bigg(1 + \sum_{j=i+1}^{m} X_{ij}\Bigg)\Bigg]$$

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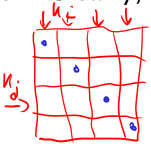

The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\underbrace{\left(1+\sum_{j=i+1}^{m}X_{ij}\right)}_{\text{cost for key } k_i}\right]$$

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$$=\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\frac{1}{n}\right)$$

Hashing with Chaining

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&= 1+\frac{m-1}{2n}
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$-\frac{1}{2n}$

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Hence, the expected cost for a successful search is $A^{+} \leq 1+\frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0)$,. . . ,$h(k, n - 1)$ must form a permutation of $0, \ldots, n - 1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, ...

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

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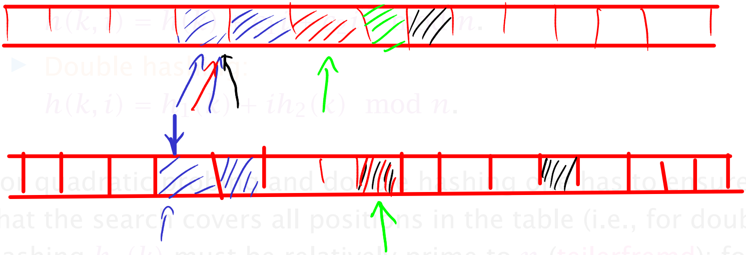
Open Addressing

1 + 2 4 8 16

Choices for $h(k,j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).
- Quadratic probing:
  $h(k,i) = h(k) + c_1 i + c_2 i^2 \mod n$.
- Double hashing:
  $h(k,i) = h_1(k) + ih_2(k) \mod n$.



For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.
- Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 21

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

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- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

Lemma 22

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

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Double Hashing

- Any probe into the hash-table usually creates a cache-miss.

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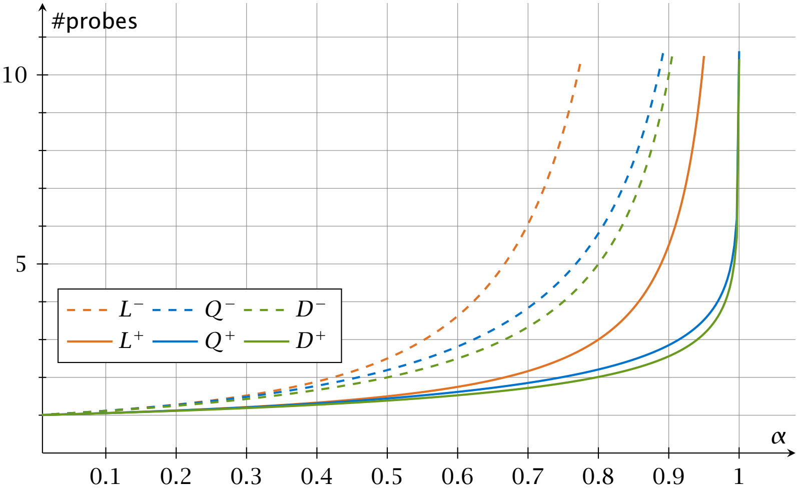
Open Addressing

Some values:

α	Linear Probing		Quadratic Probing		Double Hashing	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n-1 \rangle$.

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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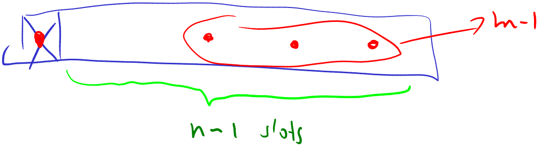
$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$
$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]$$



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$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]& \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\qquad \ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]\end{aligned}$$

$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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&\qquad = \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\
&\qquad\qquad \ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]
\end{aligned}$$

$$\begin{aligned}
\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2} \\
&\leq \left(\frac{m}{n}\right)^{i-1}
\end{aligned}$$

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]& \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]\end{aligned}$$

$$\begin{aligned}\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2} \\ &\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} \ .\end{aligned}$$

Analysis of Idealized Open Address Hashing

$\mathrm{E}[X]$

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$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

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Analysis of Idealized Open Address Hashing

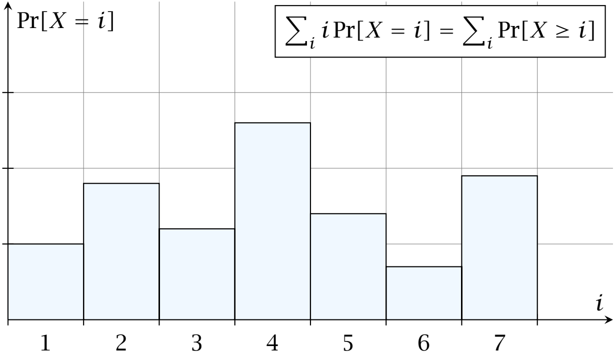
$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^{i} = \frac{1}{1-\alpha} \enspace .$$

Analysis of Idealized Open Address Hashing

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^{i} = \frac{1}{1-\alpha} \ .$$

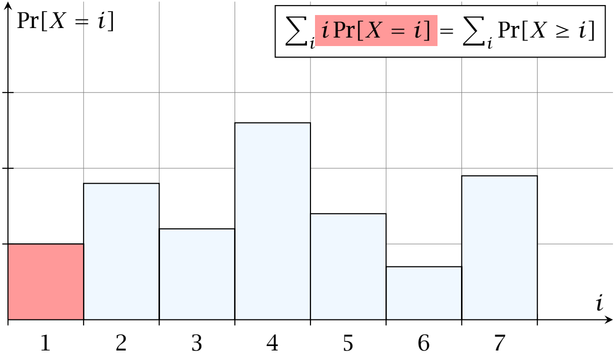
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



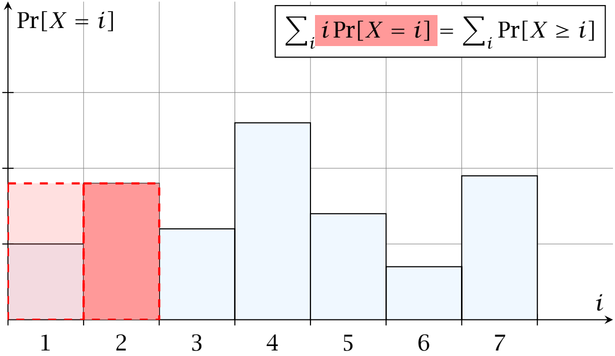
Analysis of Idealized Open Address Hashing

$i = 1$



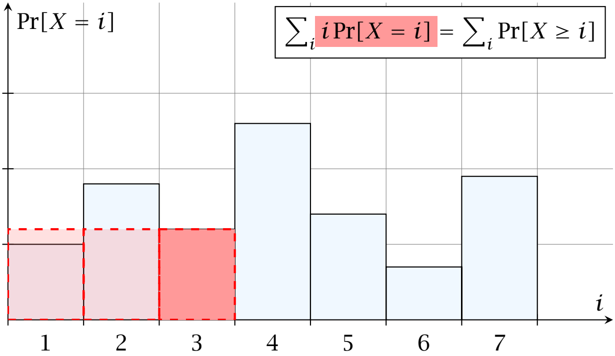
Analysis of Idealized Open Address Hashing

$i = 2$



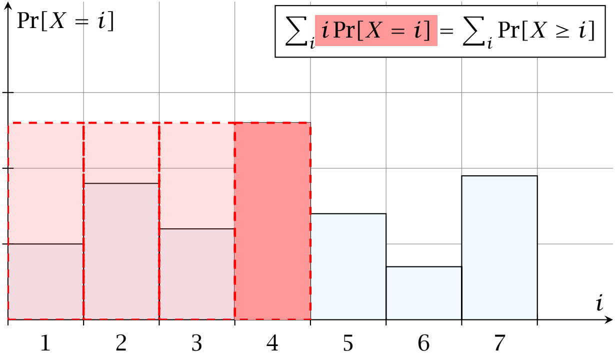
Analysis of Idealized Open Address Hashing

$i = 3$



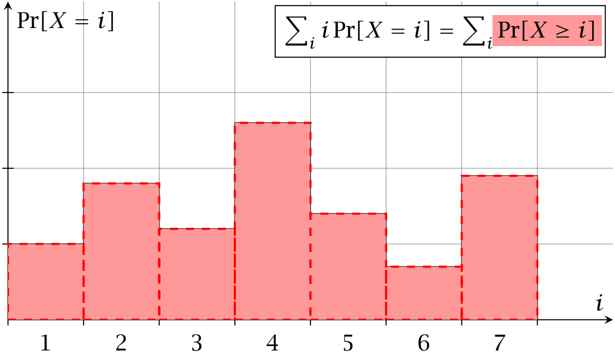
Analysis of Idealized Open Address Hashing

$i = 4$



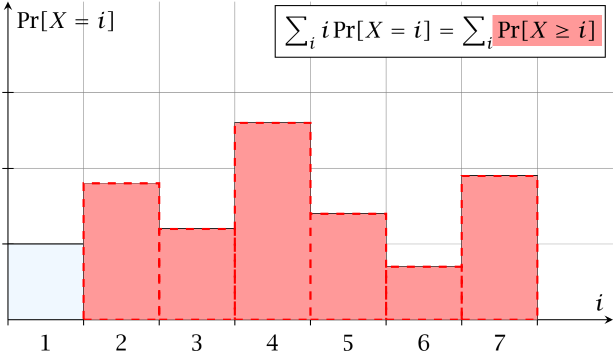
Analysis of Idealized Open Address Hashing

$i = 1$



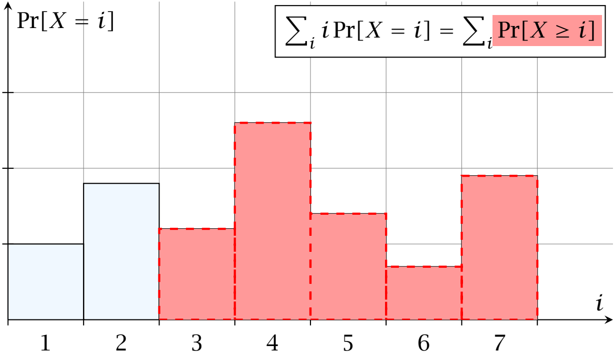
Analysis of Idealized Open Address Hashing

$i = 2$



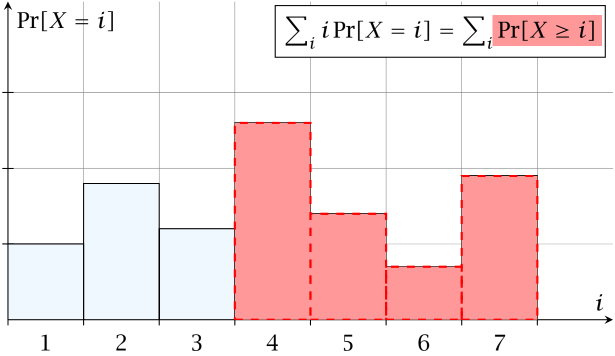
Analysis of Idealized Open Address Hashing

$i = 3$

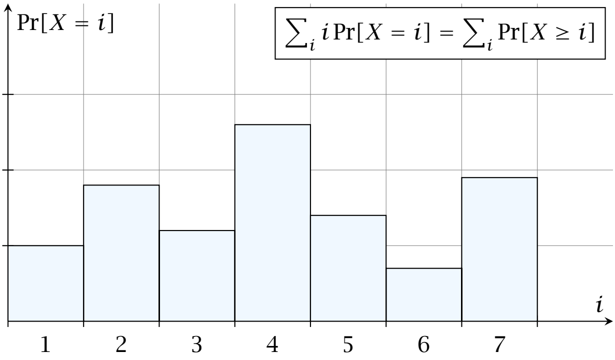


Analysis of Idealized Open Address Hashing

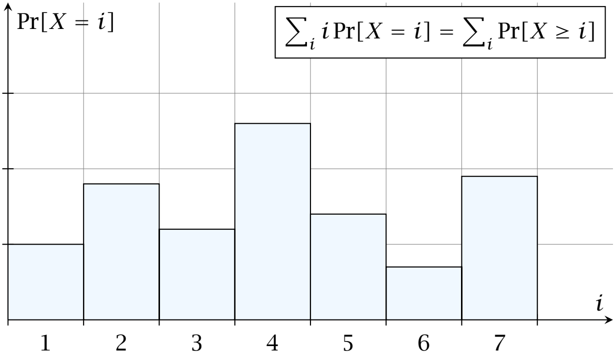
$i = 4$



Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing



The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)