Technische Universität München
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Summer term 2019
Assignment 7
June 3, 2019

Efficient Algorithms and Data Structures II

*Deadline: June 17, 2019, 10:15 am in the **Efficient Algorithms** folder.*

Homework 1 (5 Points)

Consider the (very simple) LP:

$$\begin{array}{llcrcl} \text{min.} & -x & & & & \\ \text{s.t.} & x & + & y & \leq & 1 \\ & -x & & & \leq & 0 \\ & & & -y & \leq & 0 \ . \end{array}$$

1. Compute the logarithmic barrier function for the LP, as well as its gradient.
2. Compute the analytic center and the trajectory of the central path (using some computer algebra system, e.g. Mathematica.
3. Plot the barrier function together with the central path for $t \in (0, 100]$.

Homework 2 (5 Points)

A sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$ is *self-concordant* if

$$|f'''(x)| \leq 2f''(x)^{3/2}$$

An n-dimensional function $f : U \to \mathbb{R}$ is *self-concordant* if the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$, $\tilde{f}(t) = f(x + tv)$ is self-concordant for all x in the domain of f and all $v \in \mathbb{R}^n$.
Show that any logarithmic barrier function Φ (as defined in the lecture) is self-concordant.

Homework 3 (7 Points)

Recall that any point $x^*(t)$ on the central path yields a dual feasible point $z^*(t)$ with

$$z_i^*(t) = \frac{1}{ts_i(x^*(t))} \quad .$$

Now suppose that $x^*(t)$ is only calculated approximately using the Newton Method. In this exercise, we show that we can still obtain a dual feasible point.
Assume that $x(t)$ is strictly feasible, let Δx_{nt} be the Newton step at $x(t)$ and D_x the diagonal matrix of inverse slacks $\text{diag}(d_x)$ with $d_x = (1/s_1(x(t)), \ldots, 1/s_m(x(t)))^T$.
Show that if the Newton decrement $\lambda(x) = \|D_x A \Delta x_{\text{nt}}\|_2$ is at most 1, then the point

$$z = (d_x + D_x^2 A \Delta x_{\text{nt}})/t$$

is dual feasible (i.e. $A^T z + c = 0$ and $z \geq 0$).

Homework 4 (5 Points)

Let $\mathcal{P}$ be a bounded polyhedron defined by the inequalities $Ax \leq b$. Let $s_i(x)$ denote the ith slack function, defined as $b_i - a_i^T x$. Let $\hat{x}$ be the analytic center with respect to the barrier function $\Phi(x)$, as defined in the lecture.
Show that for every point $x \in P^\circ$ it holds that

$$\sum_{i=1}^{m} \frac{s_i(x)}{s_i(\hat{x})} = m \ .$$

Tutorial Exercise 1

Let $\mathcal{P}$ be a bounded polyhedron defined by the inequalities $Ax \leq b$. Let $\hat{x}$ be the analytic center with respect to the barrier function $\Phi(x)$, as defined in the lecture.
The expanded Dikin ellipsoid is

$$\mathcal{E} = \{y \mid (y - \hat{x})^T H_{\hat{x}}(y - \hat{x}) \leq m^2 - m\} \ .$$

Show that the polyhedron $\mathcal{P}$ is completely contained in $\mathcal{E}$.
Hint: Use the result from Homework 2.

Tutorial Exercise 2

1. Given a directed graph $G = (V, E)$, pick a maximum cardinality set of edges from E so that the resulting subgraph is acyclic. Find a 1/2-approximation algorithm for this problem.

2. A maximal matching is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching.
   Pick a minimum cardinality maximal matching.

   Find a 2-approximation algorithm for this problem!

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Geometry was invented that we might expeditiously
avoid, by drawing Lines, the Tediousness of Computa-
tion.
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- I. Newton