

Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after access, an element is moved to the root (splay op)
- repeated accesses are faster
- only amortized guarantee
- read-only operations change the tree

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- only amortized guarantees
- local rotations change the tree

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Splay Trees:

When accessing an element is moved to the root (splay op)

Repeated accesses are faster

Very simple data structure

Can be used to dynamically change the tree

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Splay Trees:

What happens if elements are inserted in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 10?

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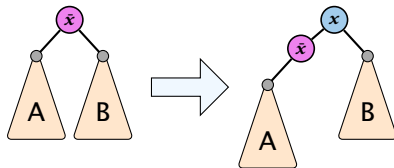
find(x)

- ▶ search for x according to a search tree
- ▶ let \tilde{x} be last element on search-path
- ▶ splay(\tilde{x})

Splay Trees

insert(x)

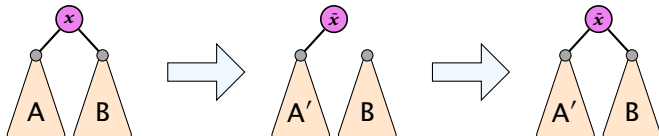
- ▶ search for x ; \bar{x} is last visited element during search (successor or predecessor of x)
- ▶ splay(\bar{x}) moves \bar{x} to the root
- ▶ insert x as new root



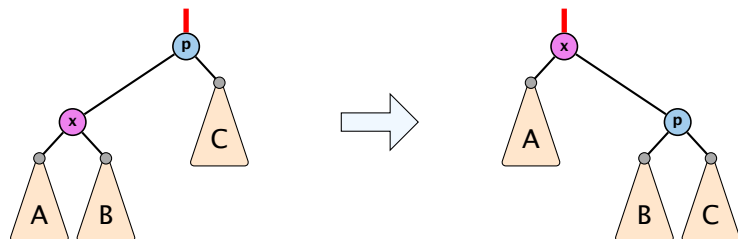
Splay Trees

delete(x)

- ▶ search for x ; splay(x); remove x
- ▶ search largest element \bar{x} in A
- ▶ splay(\bar{x}) (on subtree A)
- ▶ connect root of B as right child of \bar{x}



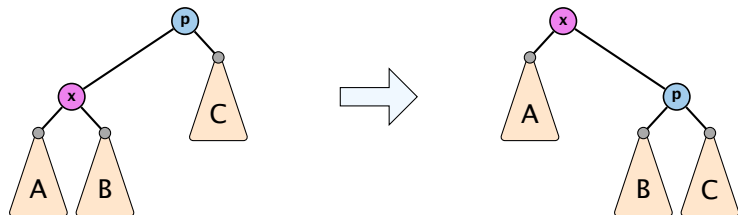
Move to Root



How to bring element to root?

- ▶ one (bad) option: `moveToRoot(x)`
- ▶ iteratively do rotation around parent of x until x is root
- ▶ if x is left child do right rotation otw. left rotation

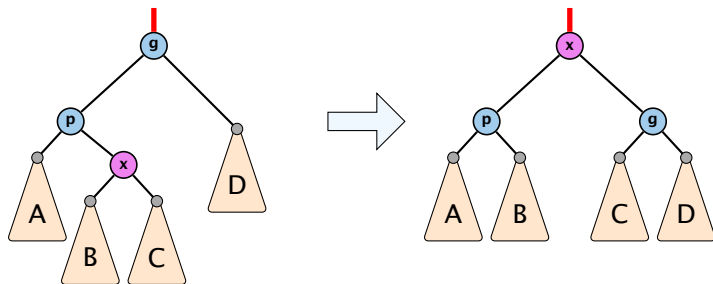
Splay: Zig Case



better option splay(x):

- ▶ zig case: if x is child of root do left rotation or right rotation around parent

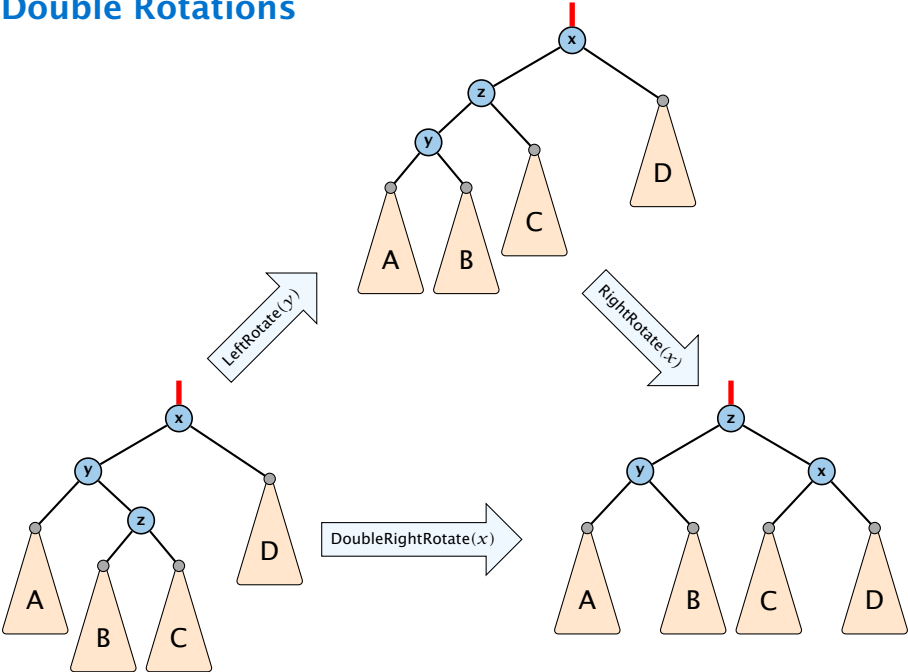
Splay: Zigzag Case



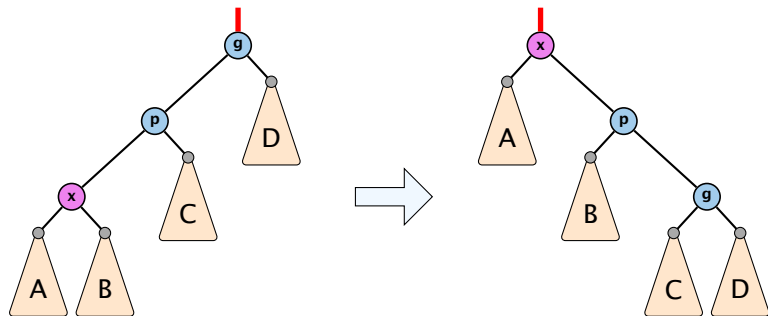
better option $\text{splay}(x)$:

- ▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



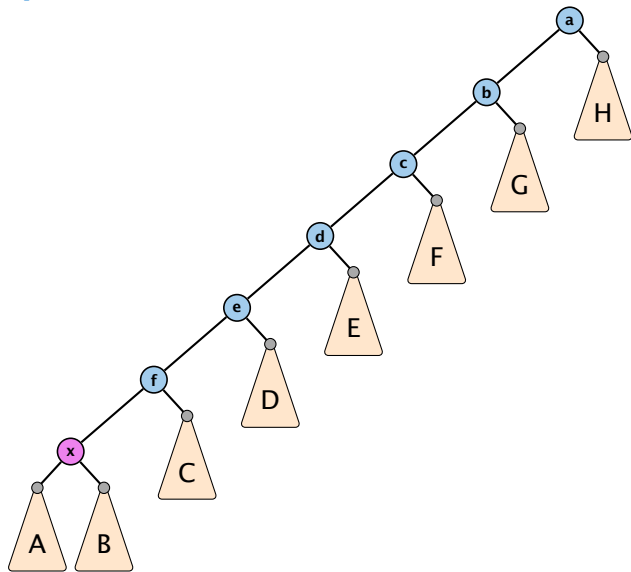
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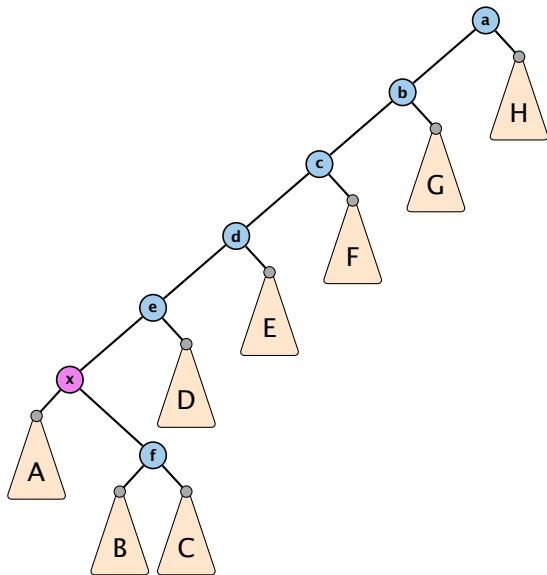
better option $\text{splay}(x)$:

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- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

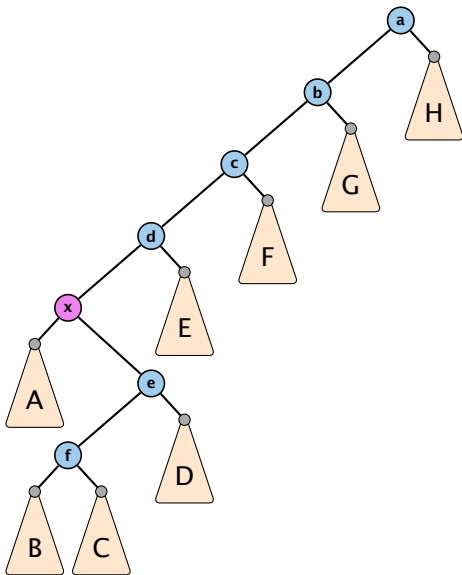
Splay vs. Move to Root



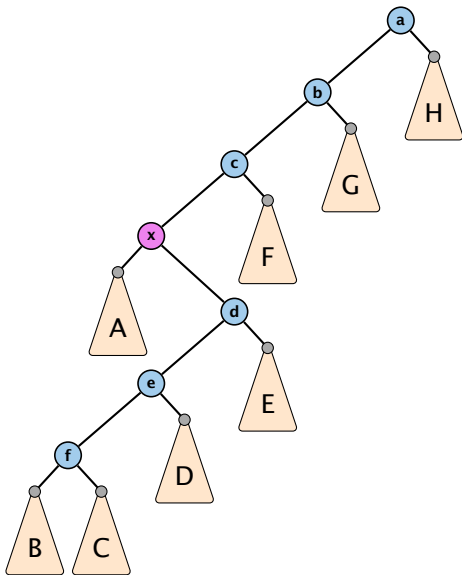
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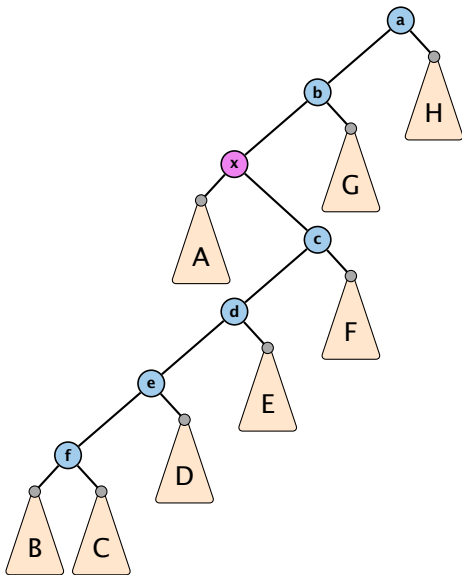
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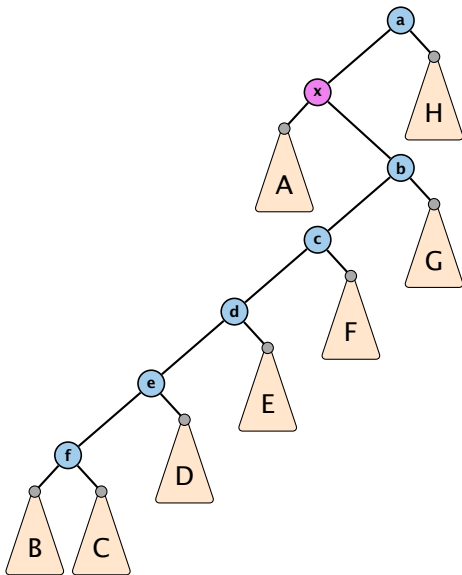
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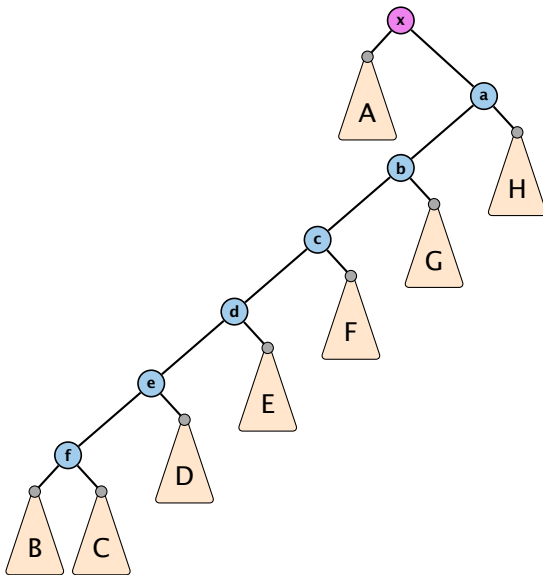
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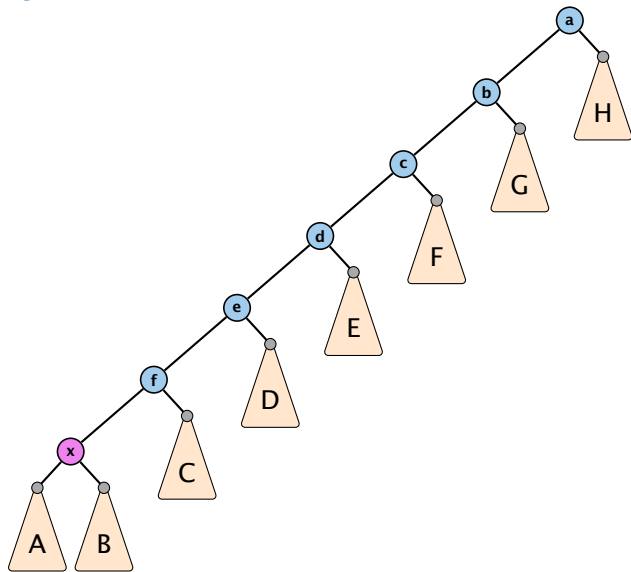
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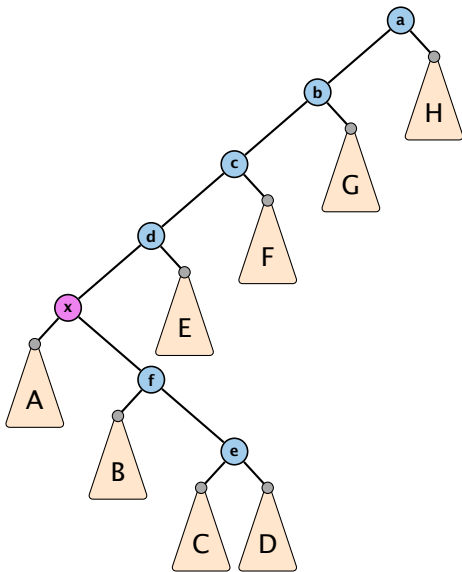
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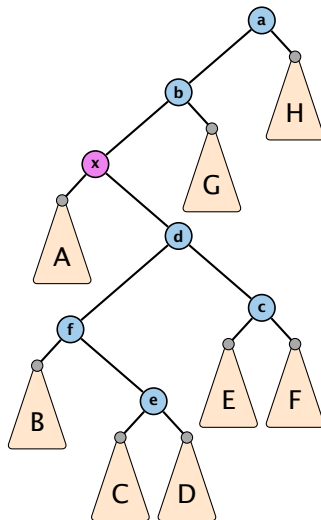
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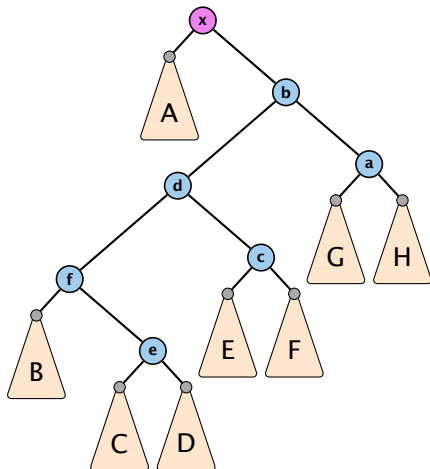
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Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a **static** search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where T_{\min} is an **optimal static search tree**.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \text{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:

A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S .

Lemma 1

*Splay Trees have an **amortized** running time of $\mathcal{O}(\log n)$ for all operations.*

Amortized Analysis

Definition 2

A data structure with operations $\text{op}_1(), \dots, \text{op}_k()$ has amortized running times t_1, \dots, t_k for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most n elements, and let k_i denote the number of occurrences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

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Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ $S.\text{push}()$
- ▶ $S.\text{pop}()$
- ▶ $S.\text{mulpop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and mulpop do not generate an underflow.

Actual cost:

- ▶ $S.\text{push}()$: cost 1.
- ▶ $S.\text{pop}()$: cost 1.
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- ▶ $S.\text{multipop}(k)$: cost

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Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n -bit binary counter may require to examine n -bits, and maybe change them.

Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
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Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x .

Amortized cost:

Let c denote the number of consecutive ones in the least significant bit positions. An increment involves c operations, and one $\text{O}(1)$ operation.

Hence, the amortized cost is

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- ▶ **Increment:** Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k (1 \rightarrow 0)-operations, and one (0 \rightarrow 1)-operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.

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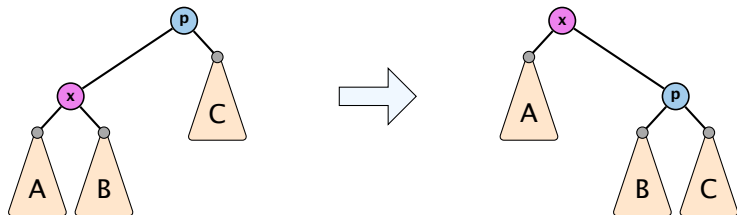
potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

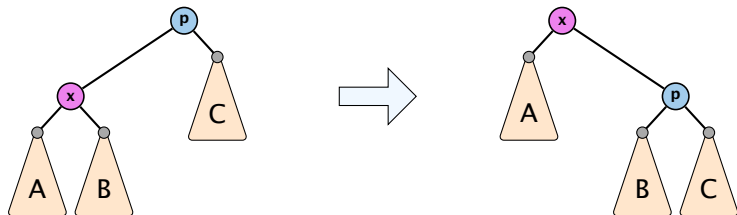
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$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

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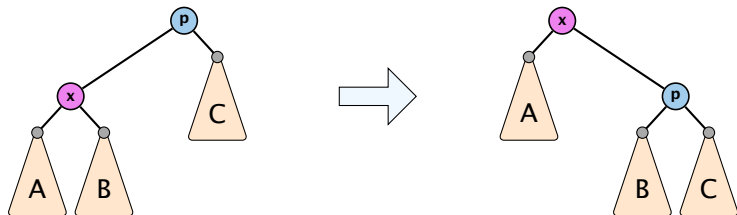
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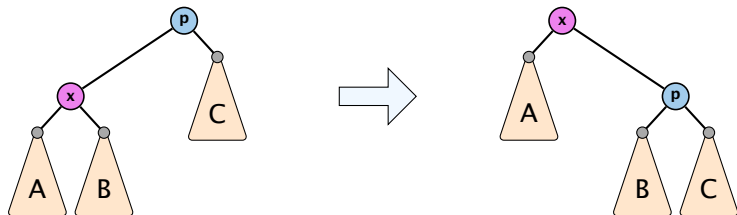
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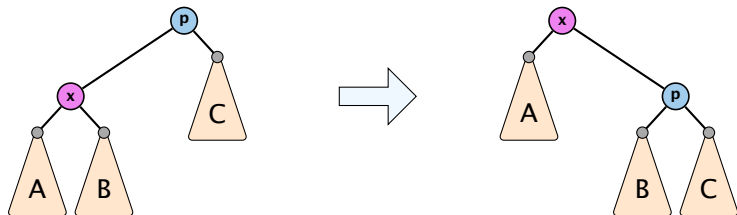
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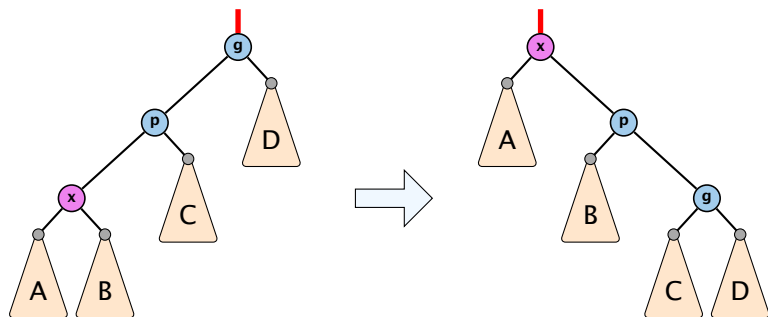
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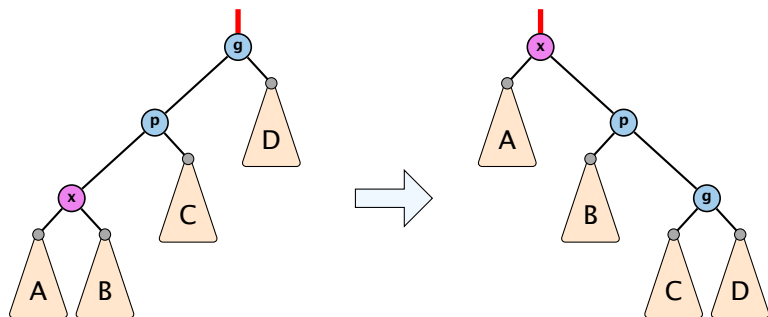
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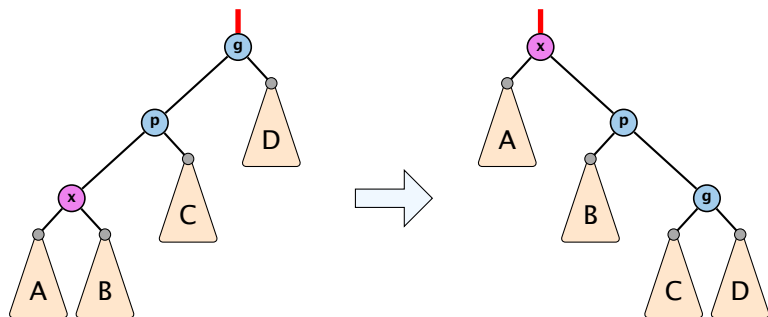
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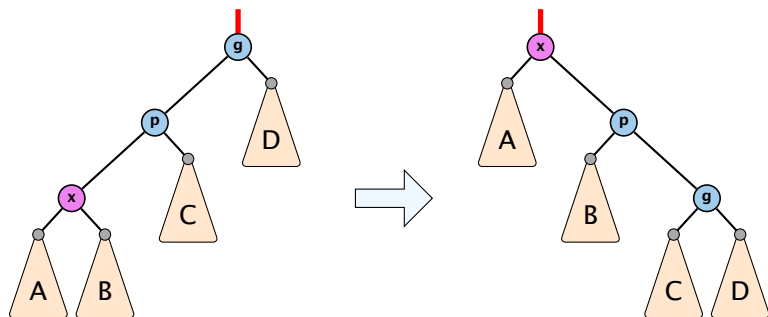
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Splay: Zigzig Case



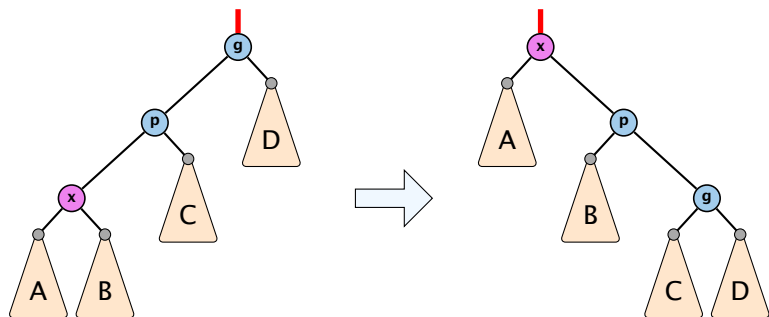
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x))\end{aligned}$$

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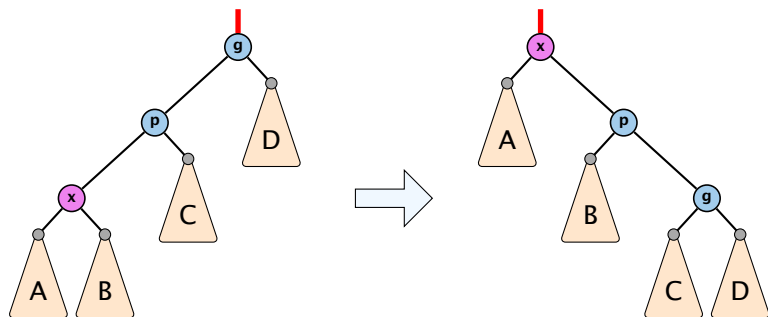
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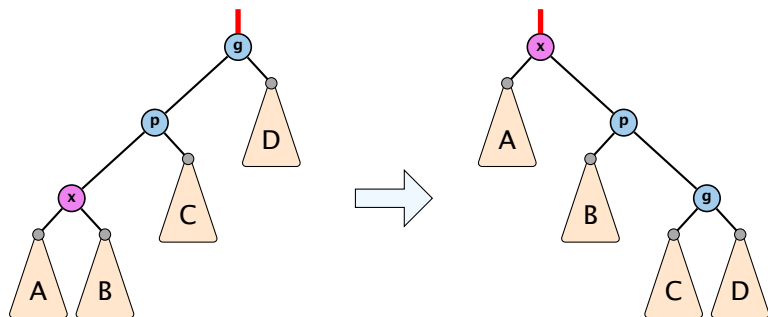
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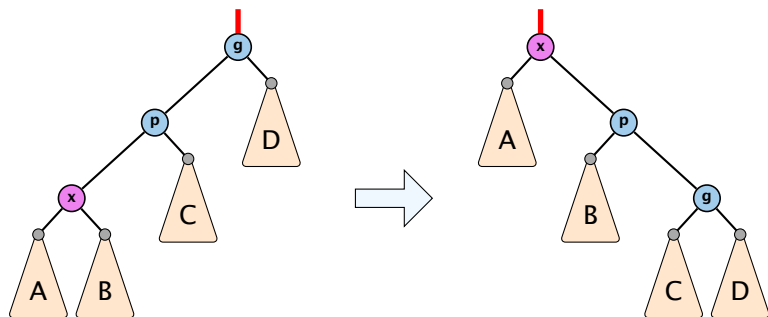
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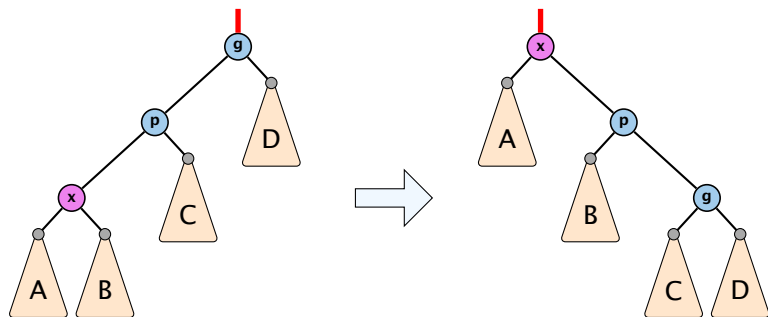
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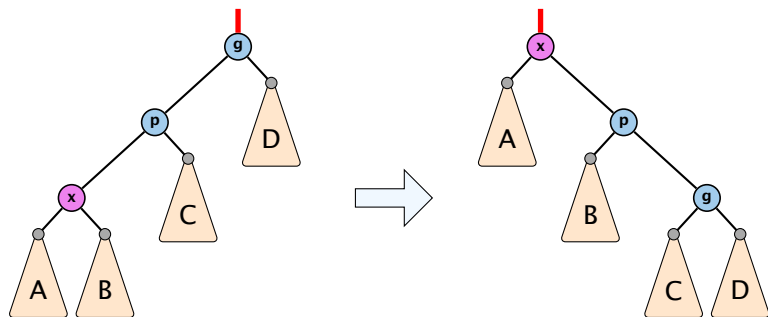
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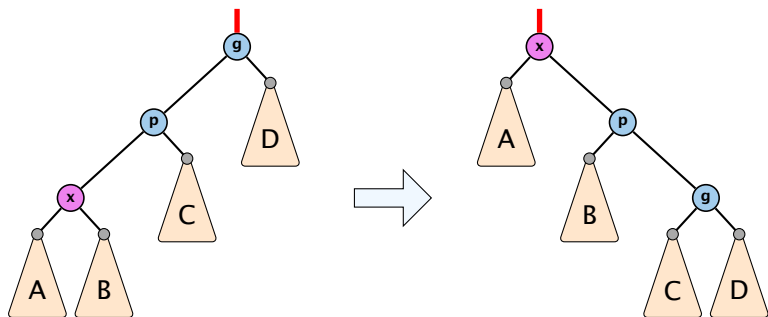
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2 \log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \\ &\leq \log\left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

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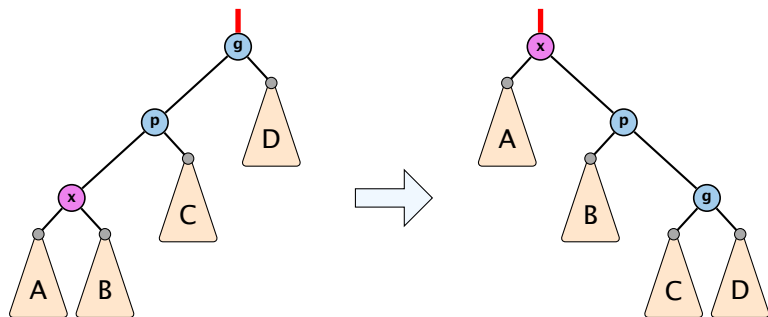
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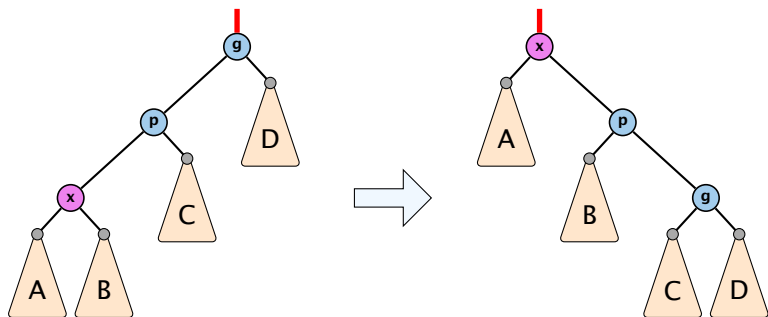
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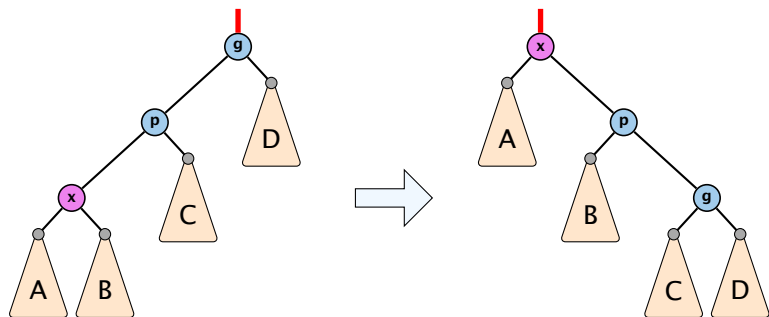
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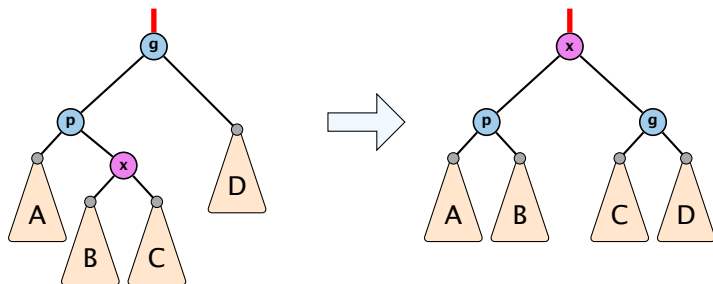
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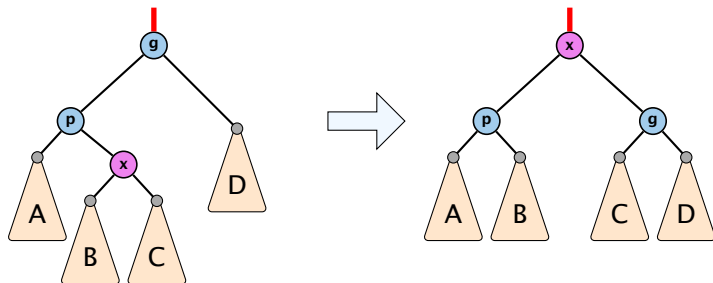
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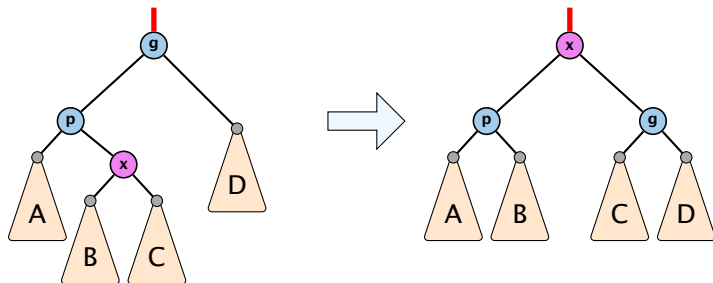
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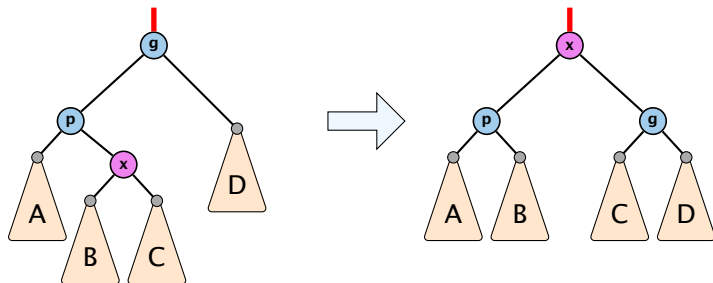
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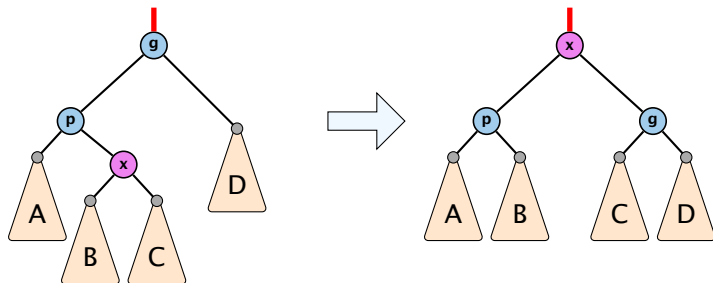
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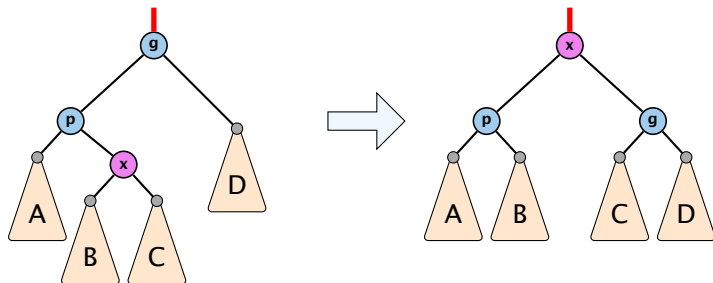
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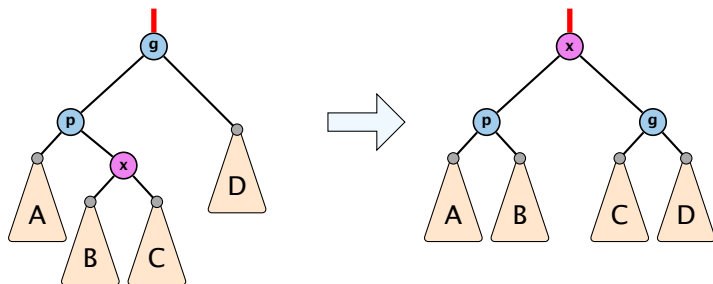
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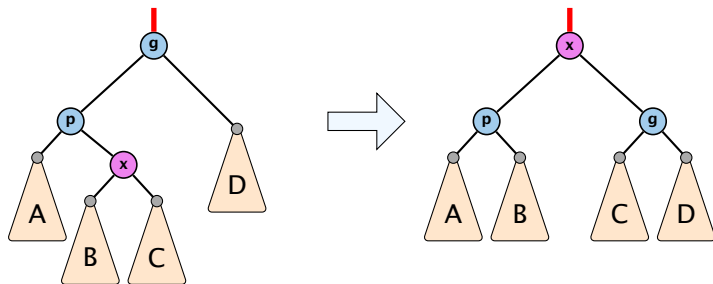
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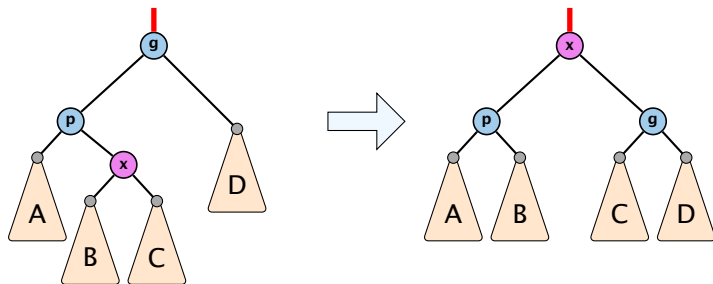
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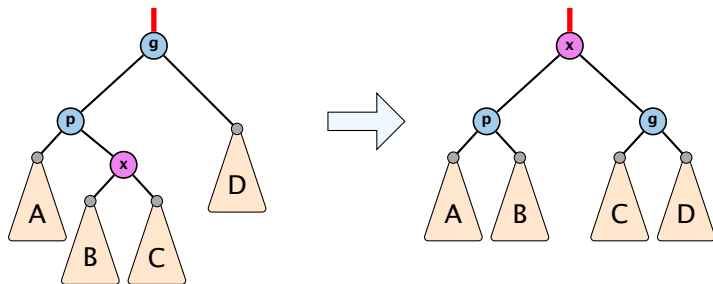
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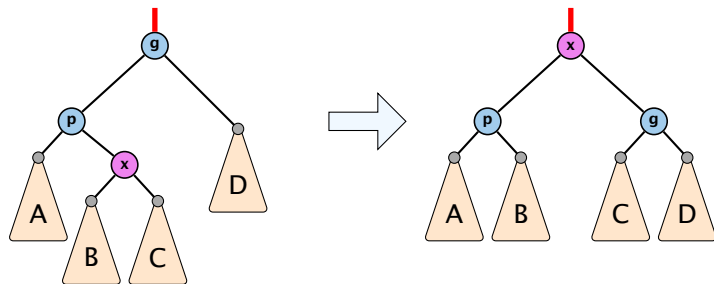
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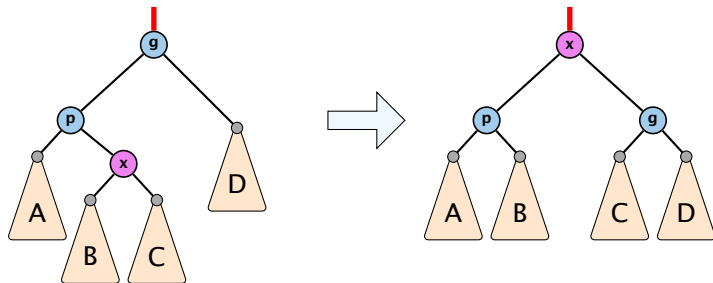
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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$