

Overview: Shortest Augmenting Paths

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The length of the shortest augmenting path never decreases.

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These two lemmas give the following theorem:

Theorem 3

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.

We can find the shortest augmenting paths in time $\mathcal{O}(m)$.

Thus,

the number of augmentations for paths of strictly increasing edges

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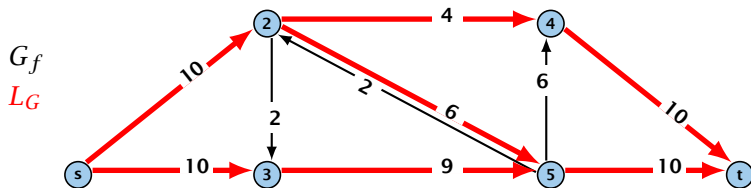
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A path P is a shortest s - t path in G_f if it is an s - t path in L_G .



In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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These changes cannot decrease the distance between s and t .

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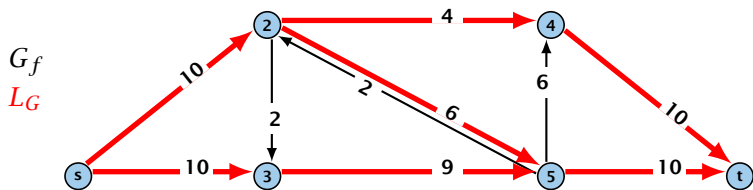
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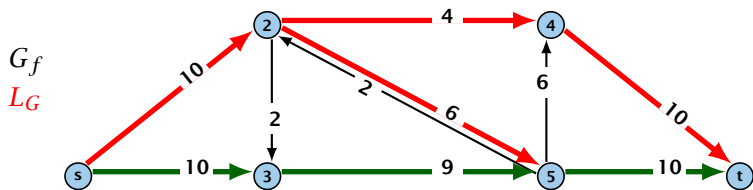
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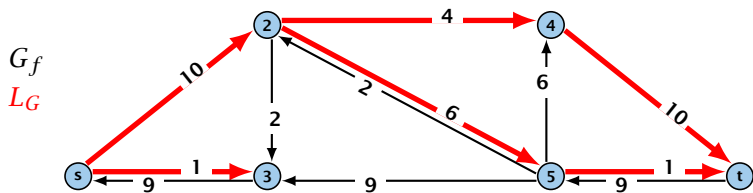
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An s - t path in G_f that uses edges not in E_L has length larger than k , even when considering edges added to G_f during the round.

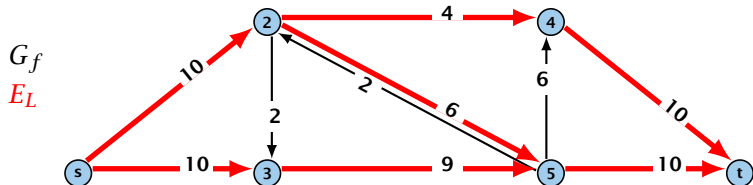
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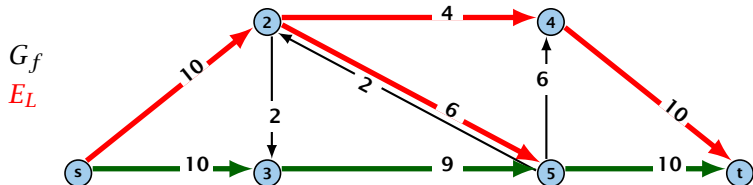
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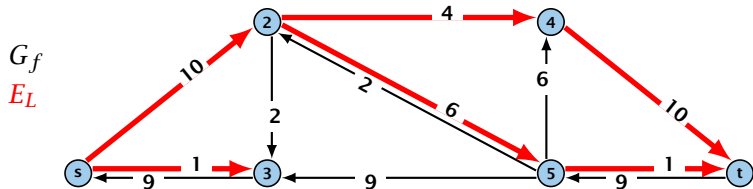
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Theorem 4

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 5 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

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We maintain a subset E_L of the edges of G_f with the guarantee that a shortest $s-t$ path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L .

When E_L does not contain an $s-t$ path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

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Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a **DFS search** to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation **during** a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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