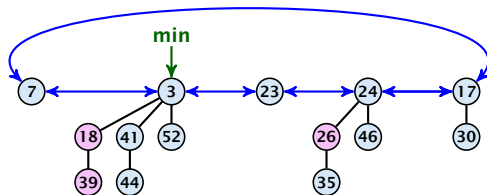


8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

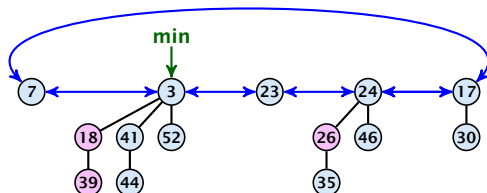
Additional implementation details:

- ▶ Every node x stores its degree in a field $x.degree$. Note that this can be updated in constant time when adding a child to x .
- ▶ Every node stores a boolean value $x.marked$ that specifies whether x is **marked** or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

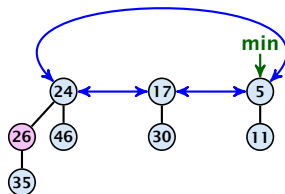
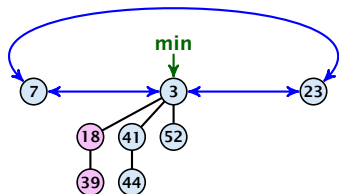
S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S . merge(S')

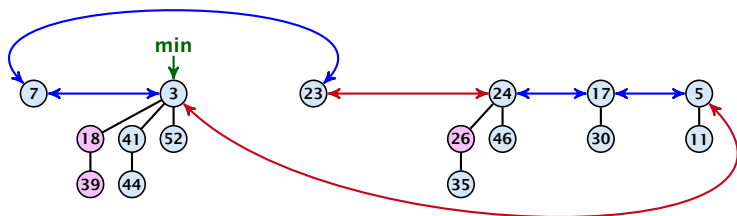
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



8.3 Fibonacci Heaps

S. merge(S')

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



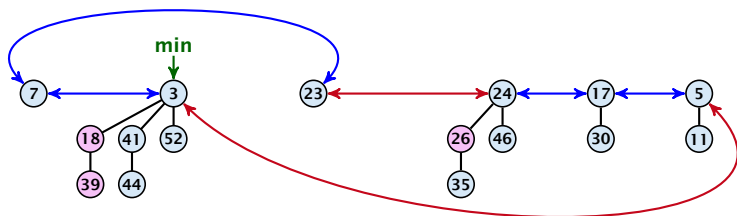
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. merge(S')

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



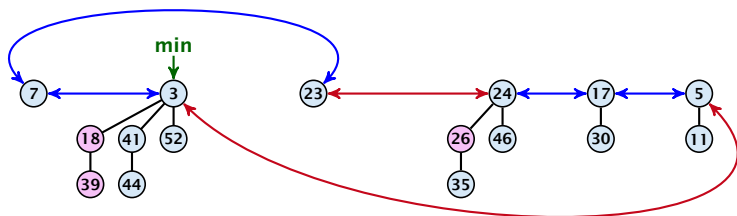
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.

8.3 Fibonacci Heaps

S. merge(S')

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



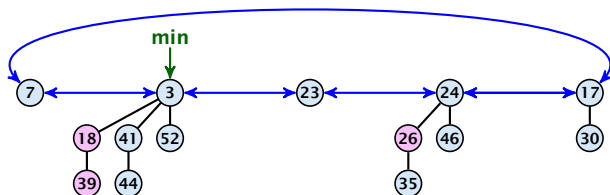
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

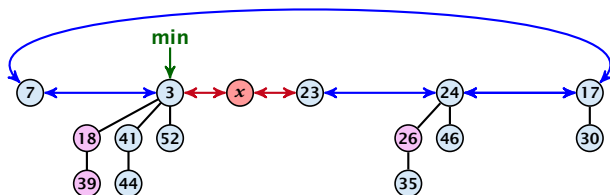
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.



8.3 Fibonacci Heaps

S. insert(x)

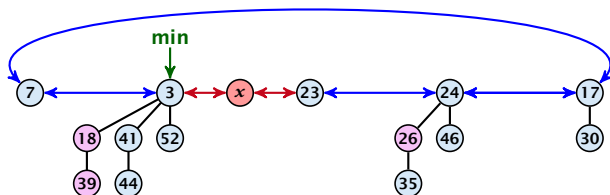
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.



8.3 Fibonacci Heaps

S. insert(x)

- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.

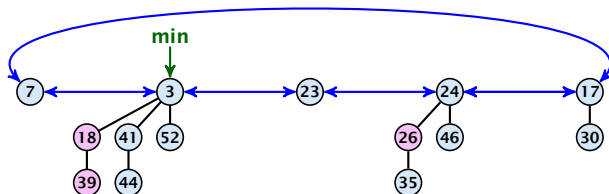


Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

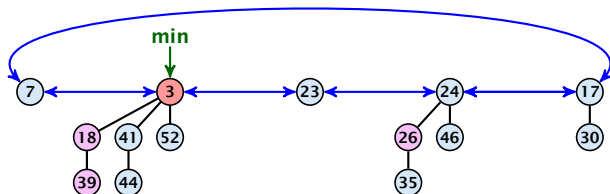
S. delete-min(x)



8.3 Fibonacci Heaps

S. delete-min(x)

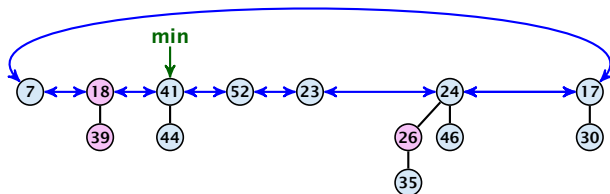
- ▶ Delete minimum; add child-trees to heap;
time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

S. delete-min(x)

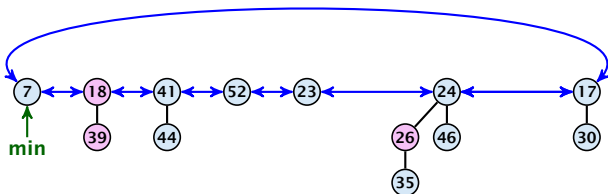
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- ▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

S. delete-min(x)

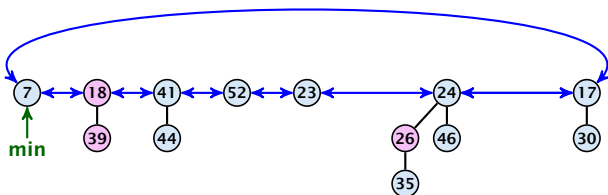
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- ▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

S. delete-min(x)

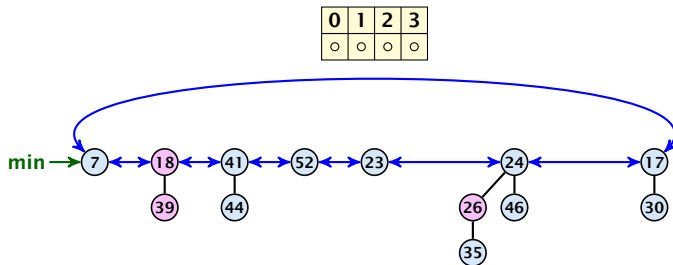
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- ▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



- ▶ Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

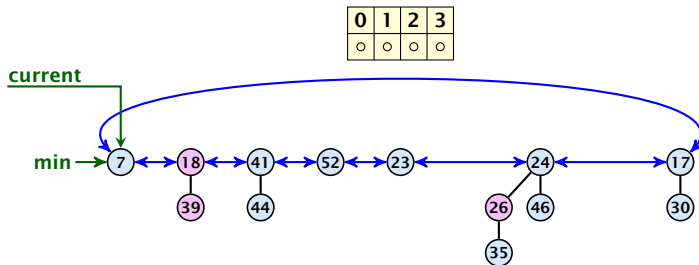
8.3 Fibonacci Heaps

Consolidate:



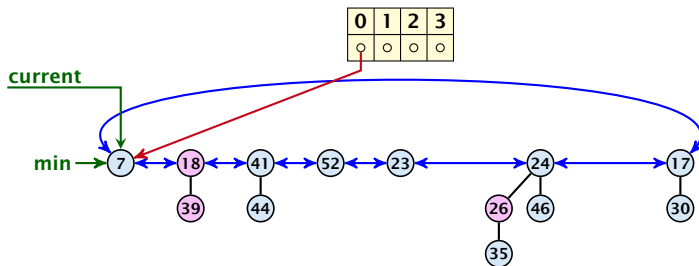
8.3 Fibonacci Heaps

Consolidate:



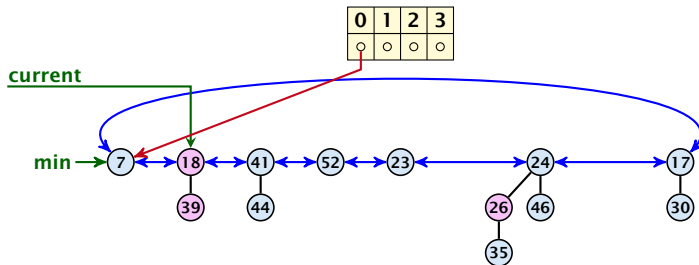
8.3 Fibonacci Heaps

Consolidate:



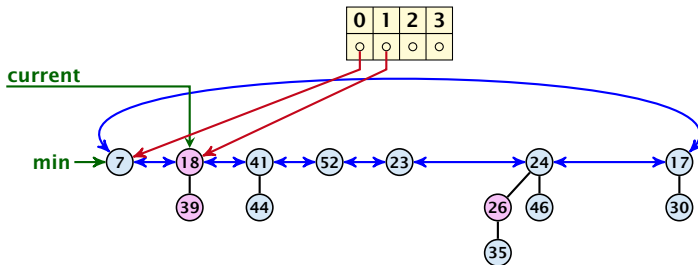
8.3 Fibonacci Heaps

Consolidate:



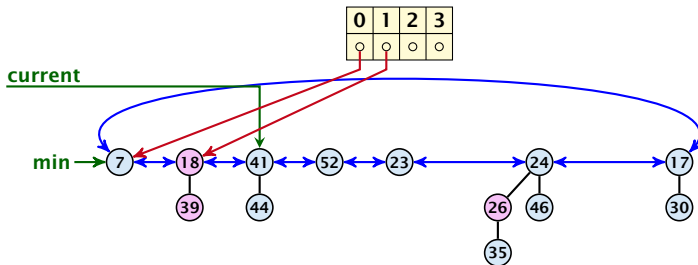
8.3 Fibonacci Heaps

Consolidate:



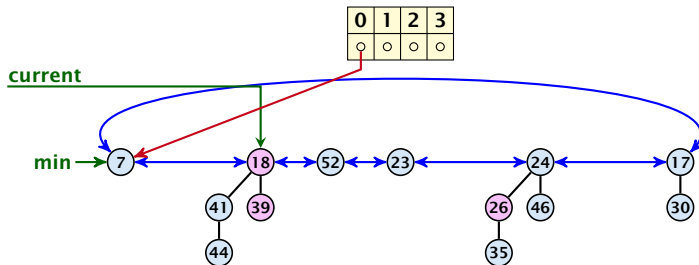
8.3 Fibonacci Heaps

Consolidate:



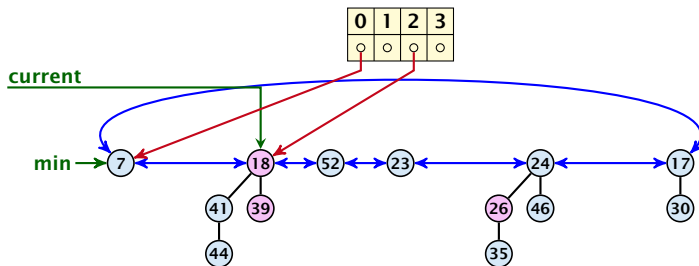
8.3 Fibonacci Heaps

Consolidate:



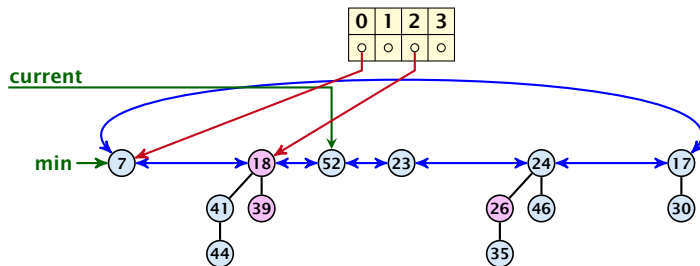
8.3 Fibonacci Heaps

Consolidate:



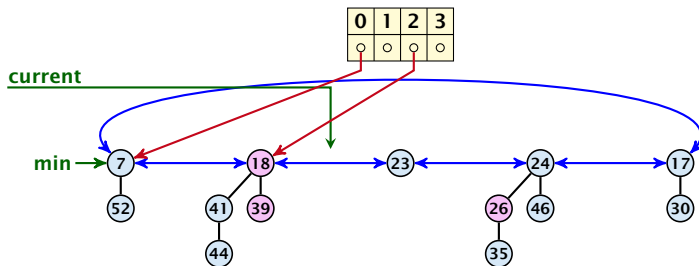
8.3 Fibonacci Heaps

Consolidate:



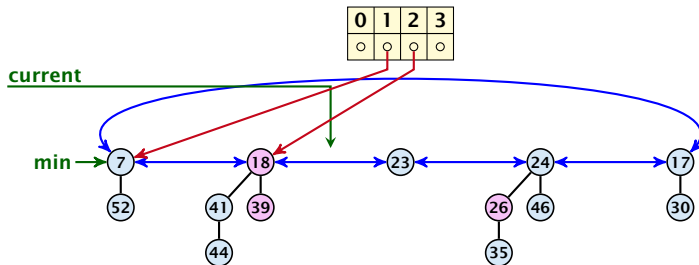
8.3 Fibonacci Heaps

Consolidate:



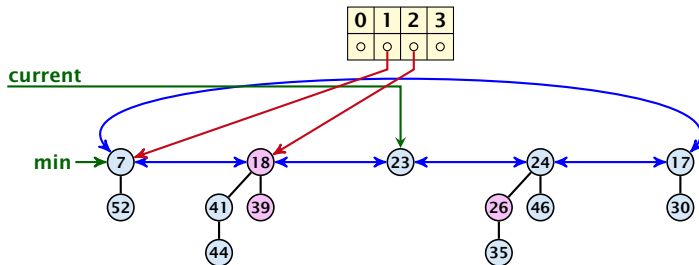
8.3 Fibonacci Heaps

Consolidate:



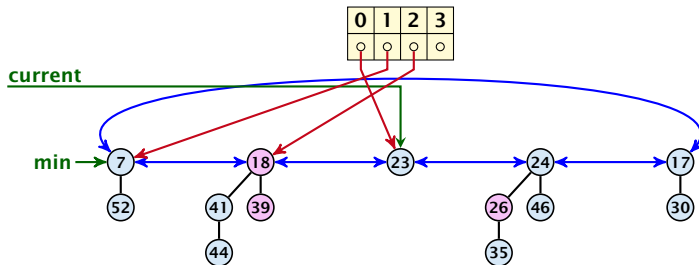
8.3 Fibonacci Heaps

Consolidate:



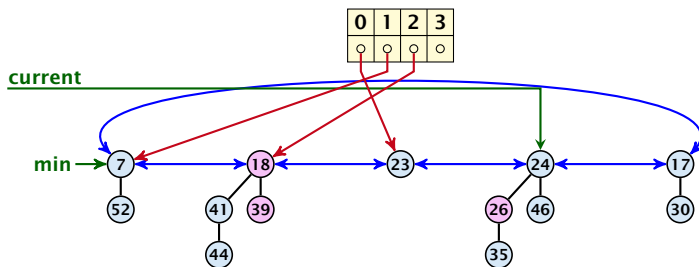
8.3 Fibonacci Heaps

Consolidate:



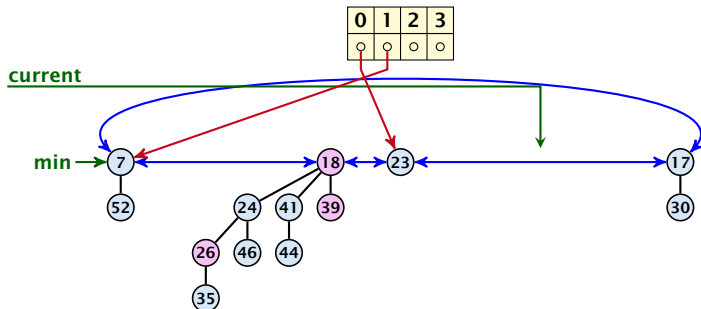
8.3 Fibonacci Heaps

Consolidate:



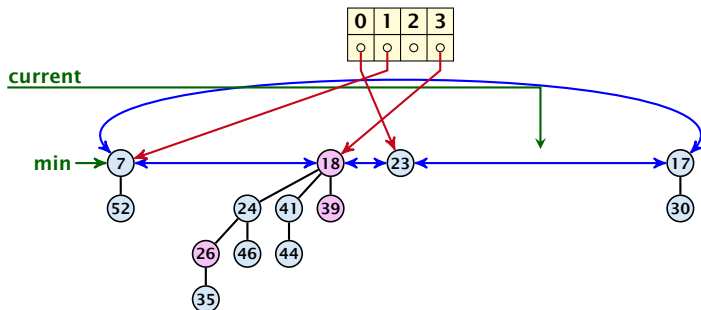
8.3 Fibonacci Heaps

Consolidate:



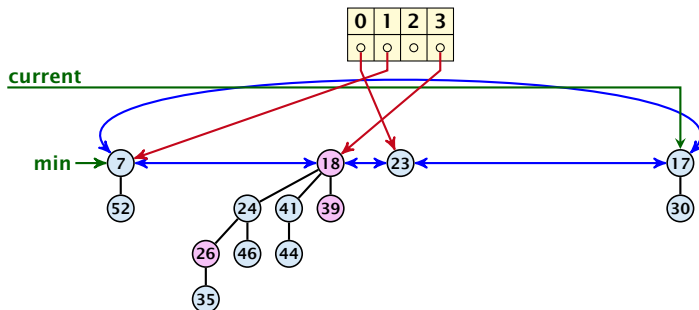
8.3 Fibonacci Heaps

Consolidate:



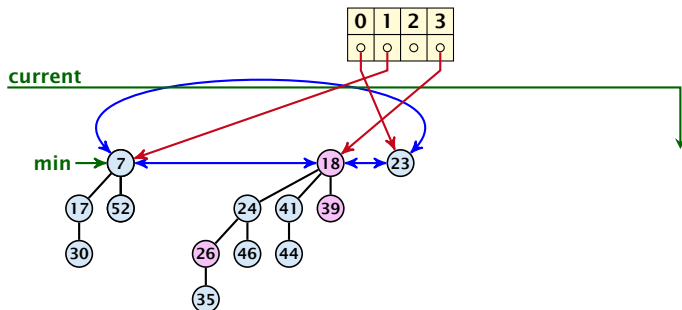
8.3 Fibonacci Heaps

Consolidate:



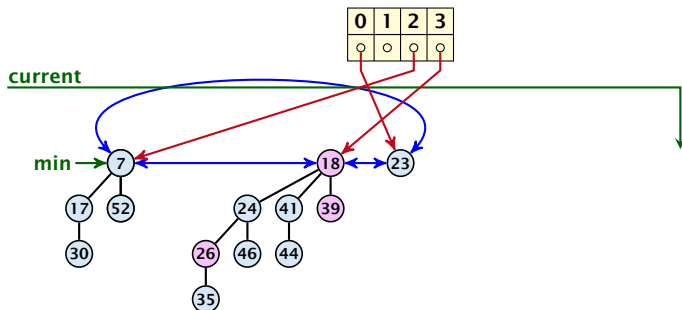
8.3 Fibonacci Heaps

Consolidate:



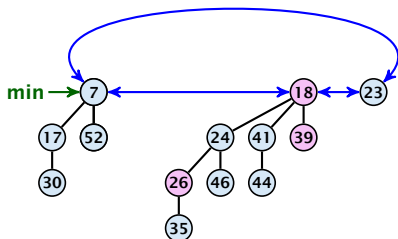
8.3 Fibonacci Heaps

Consolidate:



8.3 Fibonacci Heaps

Consolidate:



8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c\end{aligned}$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1)\end{aligned}$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)\end{aligned}$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)\end{aligned}$$

for $c \geq c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

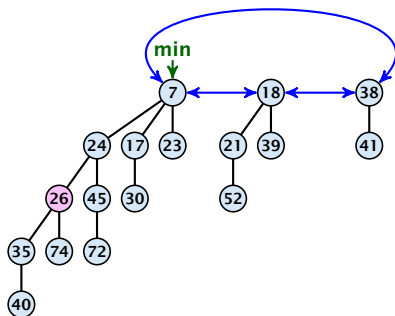
If we do not have delete or decrease-key operations then $D_n \leq \log n$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then
 $D_n \leq \log n$.

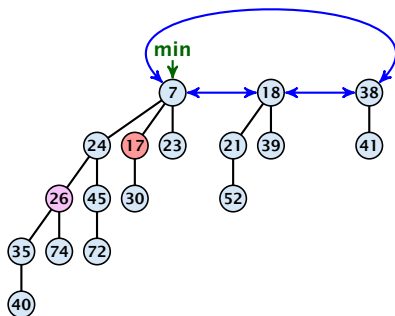
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

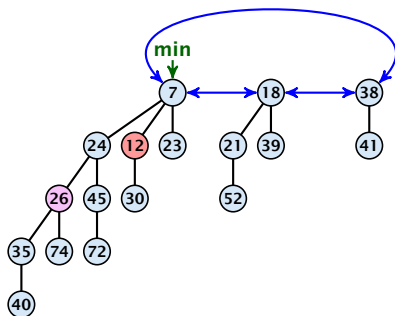
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

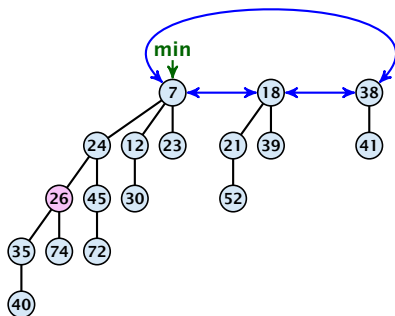
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

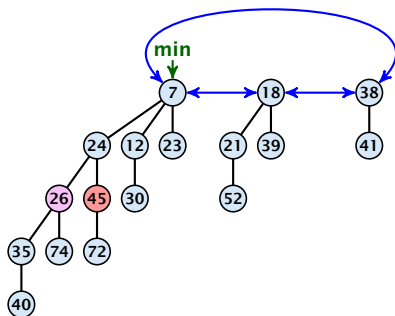
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

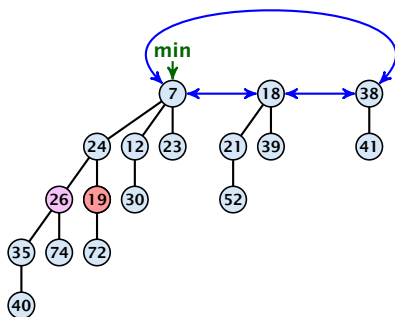
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

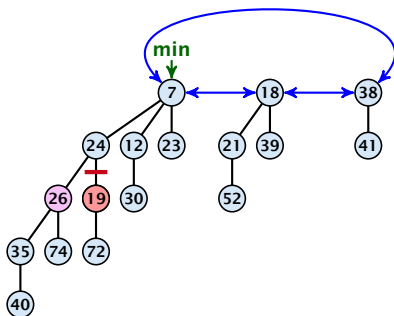
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

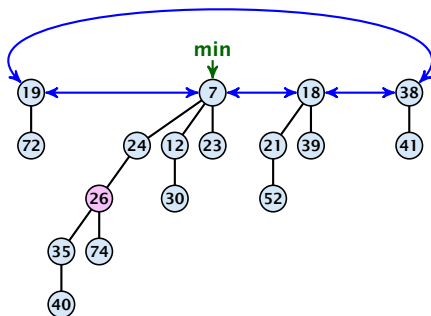
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

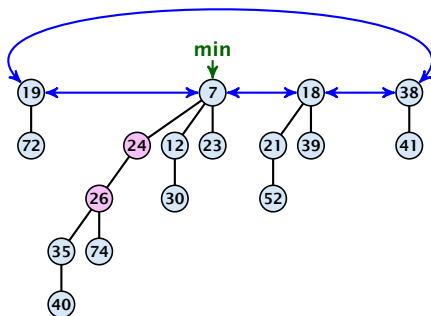
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

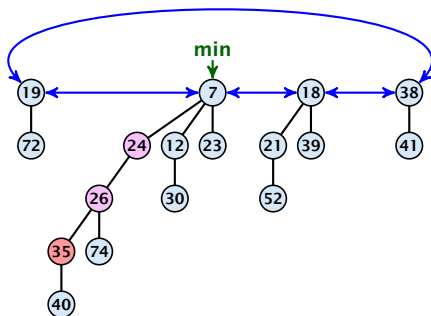
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

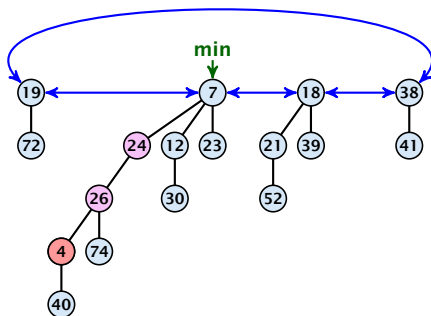
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

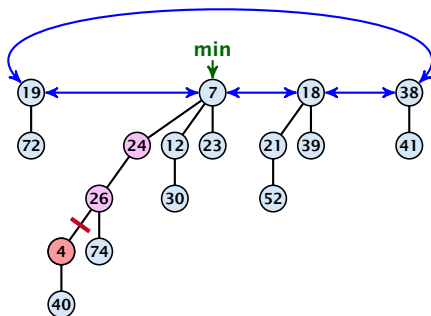
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

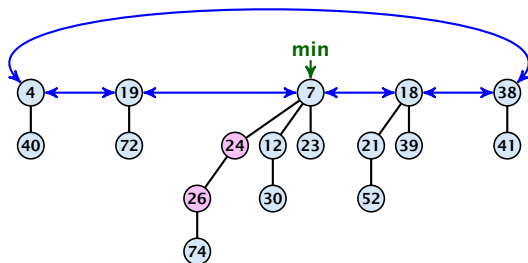
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

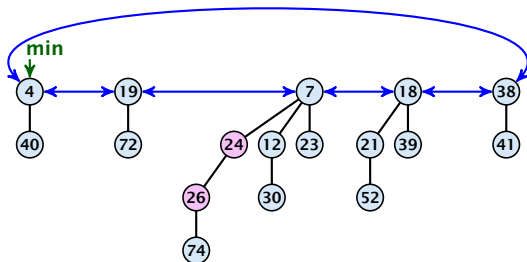
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

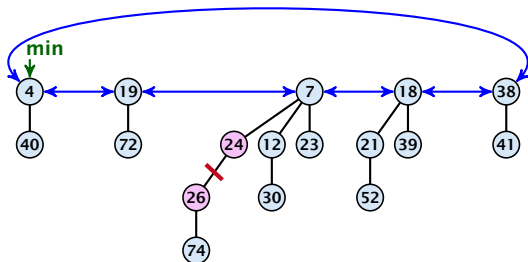
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

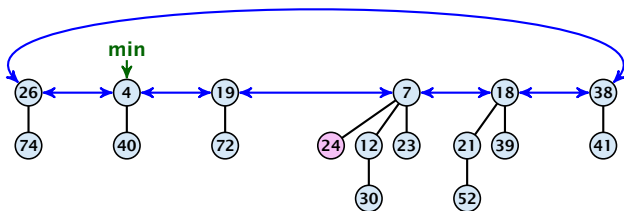
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

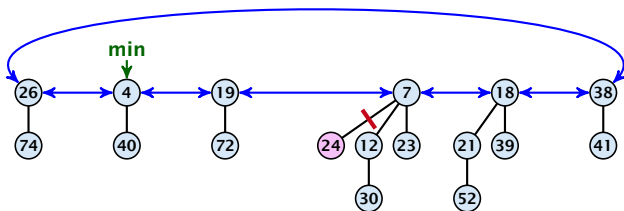
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

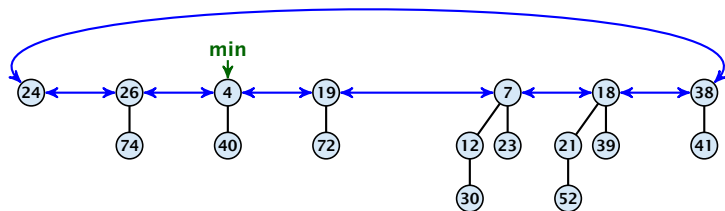
Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)



Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ Each time we cut, we create one new root.
- ▶ Each time we cut, we mark a node, and all but the first cut marks a node, the last cut may mark a node.
- ▶ Amortized cost is at most c_2 .

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ Every cut creates one new root.
- ▶ Every root has at most ℓ children.
- ▶ Every root has at most ℓ children, and all but the first cut marks a node, the last cut may mark a node.
- ▶ Amortized cost is $O(\ell)$.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ Every cut creates one new root.
- ▶ Every root has at least one child.
- ▶ Hence, every root has at least one child that is not a root.
- ▶ Hence, every root has at least one child that is not a root.
- ▶ Hence, every root has at least one child that is not a root.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

$$c_2(\ell+1) + c(4-\ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

if $c \geq c_2$.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

if $c \geq c_2$.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

if $c \geq c_2$.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most
$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if $c \geq c_2$.

Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 1

Let x be a node with degree k and let y_1, \dots, y_k denote the children of x in the order that they were linked to x . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- ▶ When y_i was linked to x , at least y_1, \dots, y_{i-1} were already linked to x .
- ▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

Proof

- ▶ When y_i was linked to x , at least y_1, \dots, y_{i-1} were already linked to x .
- ▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

Proof

- ▶ When y_i was linked to x , at least y_1, \dots, y_{i-1} were already linked to x .
- ▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

Proof

- ▶ When y_i was linked to x , at least y_1, \dots, y_{i-1} were already linked to x .
- ▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let y_1, \dots, y_k be its children.

$$s_k = 2 + \sum_{i=2}^k \text{size}(y_i)$$

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let y_1, \dots, y_k be its children.

$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \end{aligned}$$

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let y_1, \dots, y_k be its children.

$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

$\phi = \frac{1}{2}(1 + \sqrt{5})$ denotes the *golden ratio*.
Note that $\phi^2 = 1 + \phi$.

Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$