How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

## Several possibilities:

▶ Choose path with maximum bottleneck capacity.

▶ Choose path with sufficiently large bottleneck capacity.

▶ Choose the shortest augmenting path.

Capacity Scaling

Intuition:

▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

▶ Don't worry about finding the exact bottleneck.

▶ Maintain scaling parameter Δ.

▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.
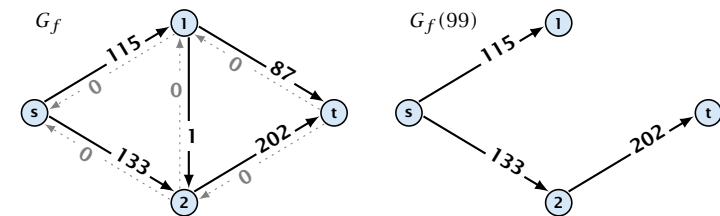
Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

1: **foreach** $e \in E$ **do** $f_e \leftarrow 0$;
2: $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
3: **while** $\Delta \geq 1$ **do**
4: $\quad$ $G_f(\Delta) \leftarrow \Delta$-residual graph
5: $\quad$ **while** there is augmenting path P in $G_f(\Delta)$ **do**
6: $\quad\quad$ $f \leftarrow$ augment(f, c, P)
7: $\quad\quad$ update($G_f(\Delta)$)
8: $\quad$ $\Delta \leftarrow \Delta/2$
9: **return** f

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

▶ because of integrality we have $G_f(1) = G_f$

▶ therefore after the last phase there are no augmenting paths anymore

▶ this means we have a maximum flow.

Capacity Scaling

Lemma 1
There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

Lemma 2
Let f be the flow at the end of a Δ-phase. Then the maximum flow is smaller than $\mathrm{val}(f) + m\Delta$.

Proof: less obvious, but simple:

▶ There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.

▶ In G_f this cut can have capacity at most $m\Delta$.

▶ This gives me an upper bound on the flow that I can still add.

Capacity Scaling

Lemma 3
There are at most $2m$ augmentations per scaling-phase.

Proof:

▶ Let f be the flow at the end of the previous phase.

▶ $\mathrm{val}(f^*) \le \mathrm{val}(f) + 2m\Delta$

▶ Each augmentation increases flow by Δ.

Theorem 4
We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.