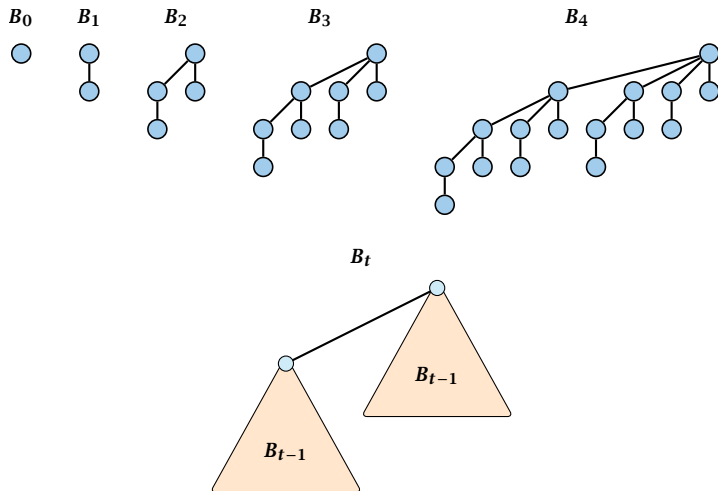


8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees



Properties of Binomial Trees

- ▶ B_k has 2^k nodes.
- ▶ B_k has height k .
- ▶ The root of B_k has degree k .
- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .
- ▶ Deleting the root of B_k gives trees B_0, B_1, \dots, B_{k-1} .

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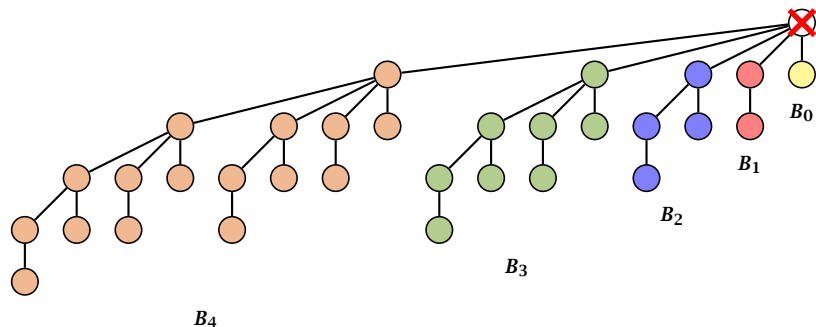
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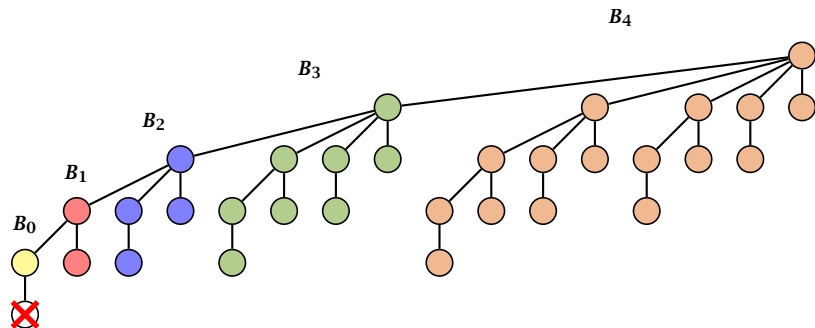
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Binomial Trees



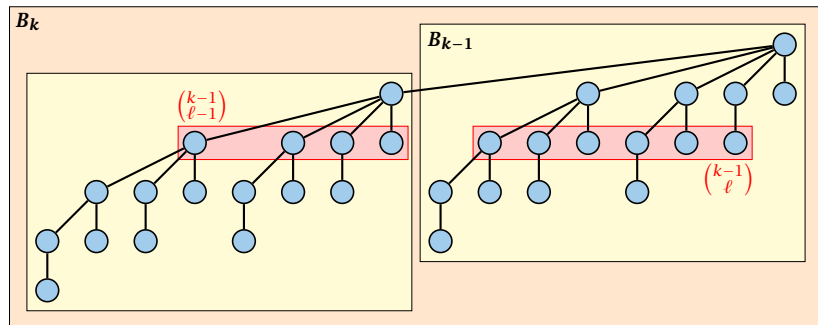
Deleting the root of B_5 leaves sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

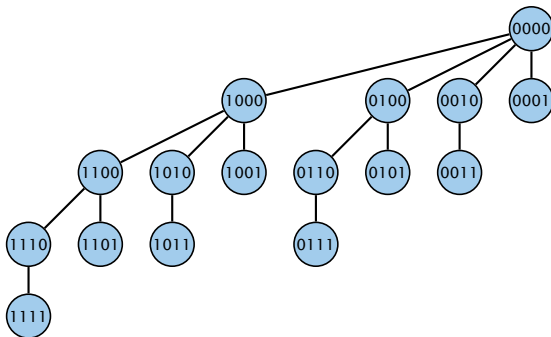
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

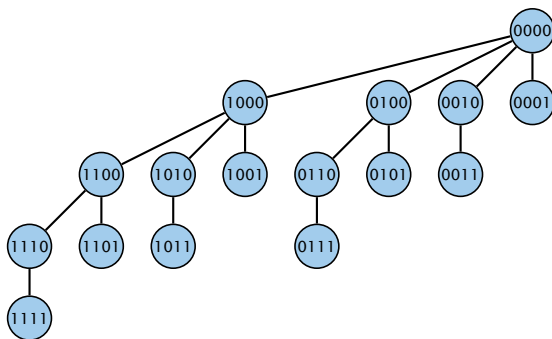


The binomial tree B_k is a sub-graph of the hypercube H_k .

The parent of a node with label b_k, \dots, b_1 is obtained by setting the least significant 1-bit to 0.

The ℓ -th level contains nodes that have ℓ 1's in their label.

Binomial Trees

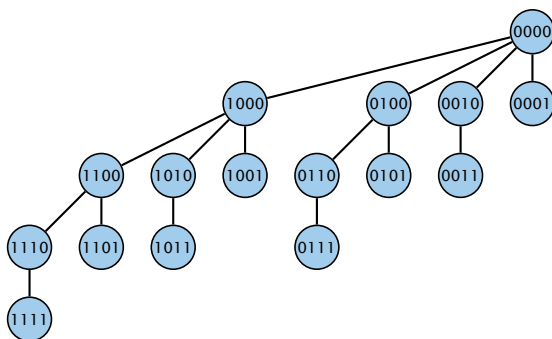


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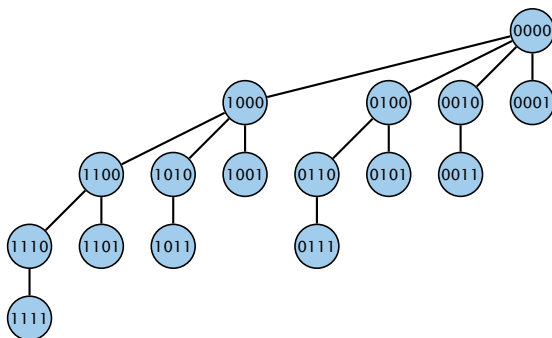


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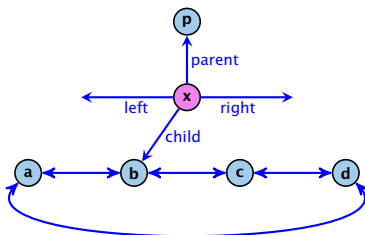
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8.2 Binomial Heaps

How do we implement trees with non-constant degree?

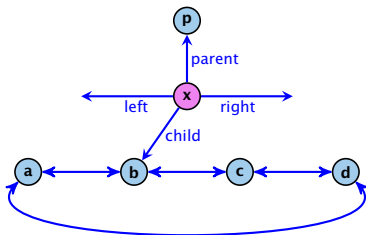
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



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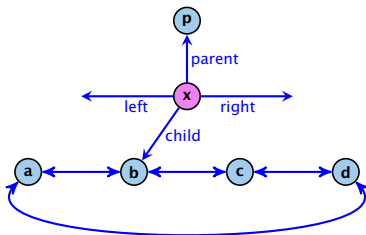
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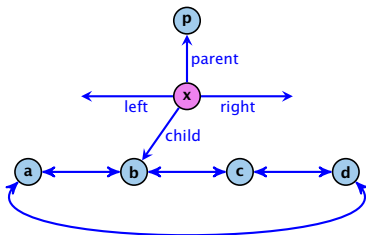
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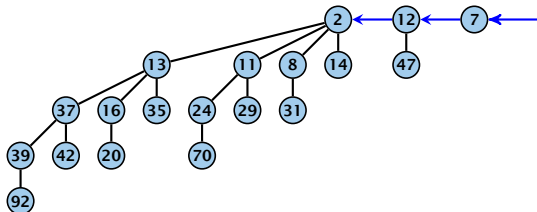
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8.2 Binomial Heaps

- ▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- ▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T .

Binomial Heap

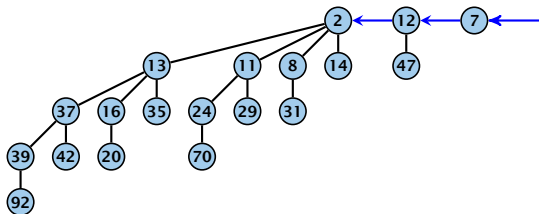


In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0 , B_1 , and B_4 .

Binomial Heap

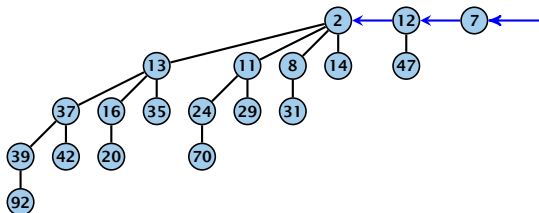


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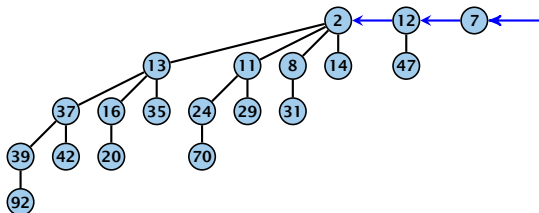


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Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n .

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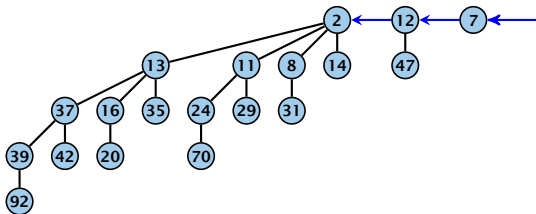
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Properties of a heap with n keys:

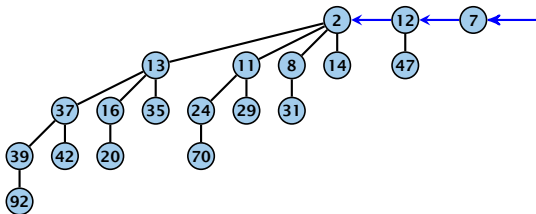
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- ▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.
- ▶ The minimum must be contained in one of the roots.
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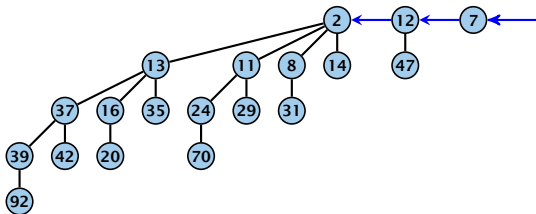
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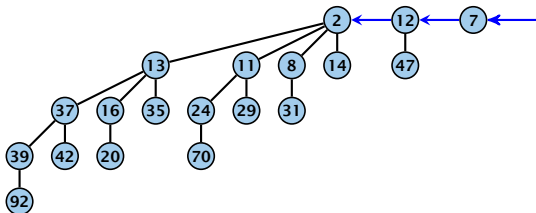
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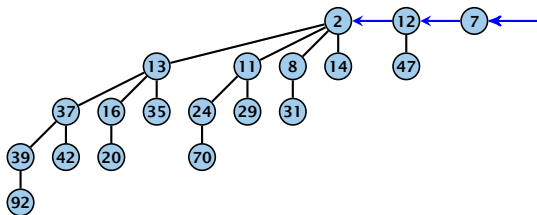
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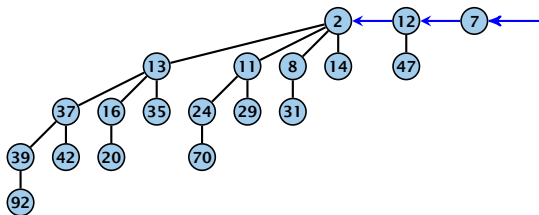
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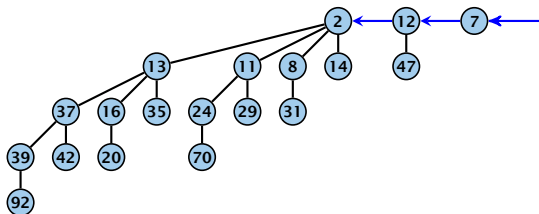
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Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For many trees this technique is analogous to binary addition.



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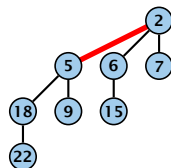
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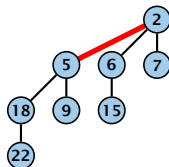
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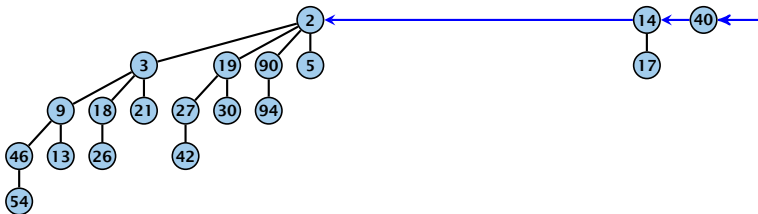
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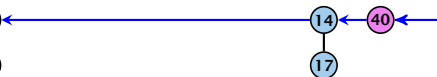
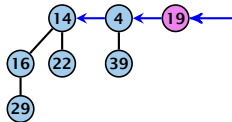
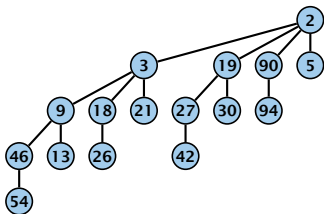
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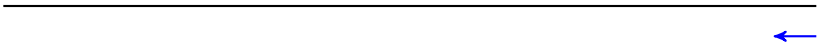
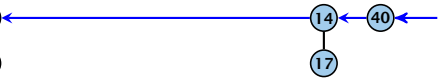
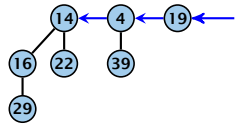
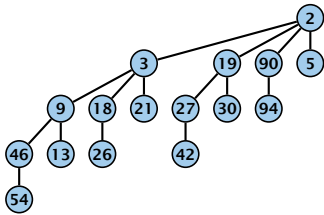
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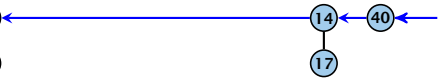
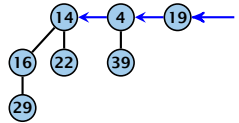
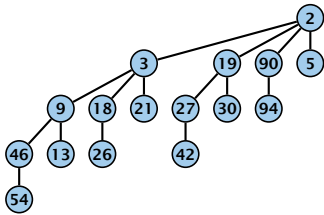
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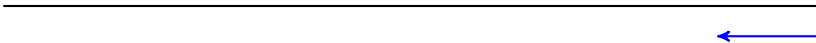
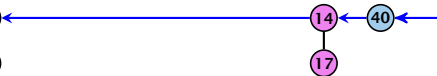
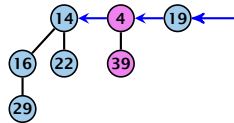
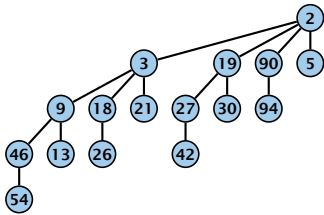


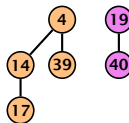
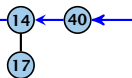
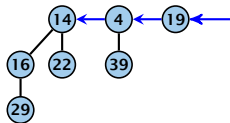
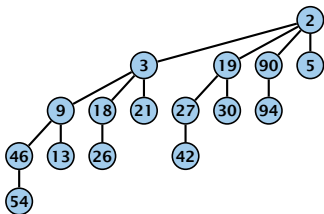


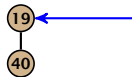
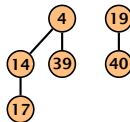
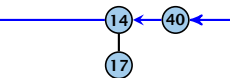
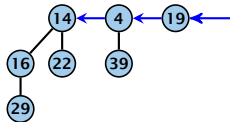
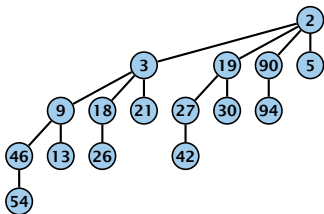


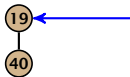
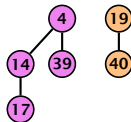
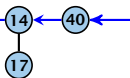
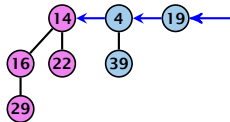
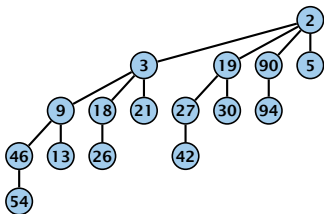


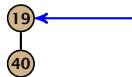
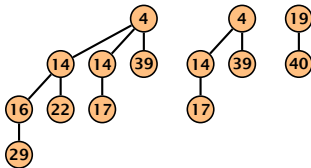
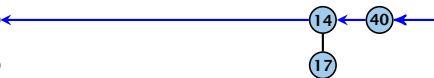
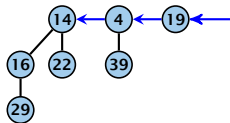
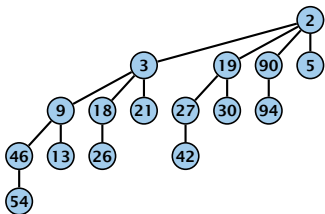


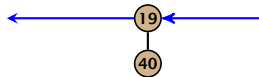
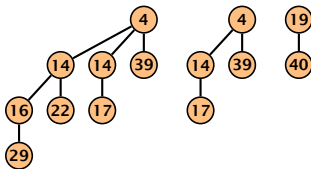
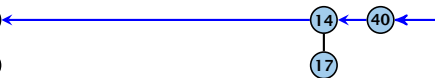
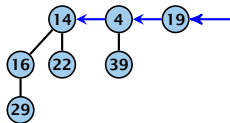
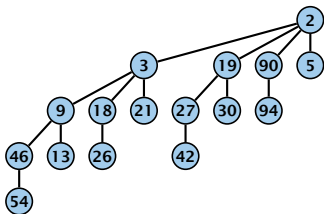


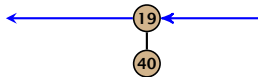
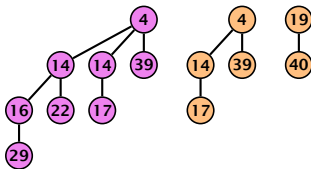
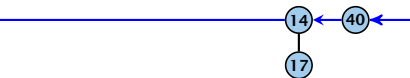
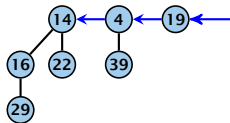
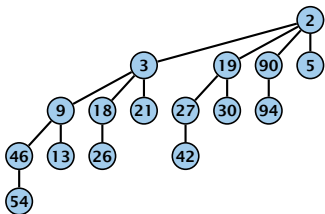




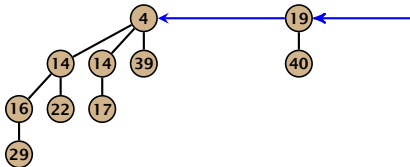
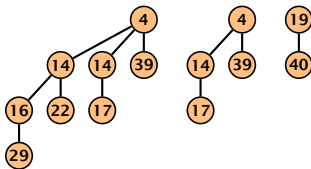
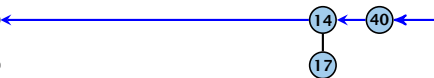
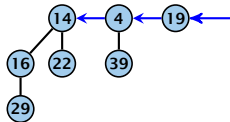
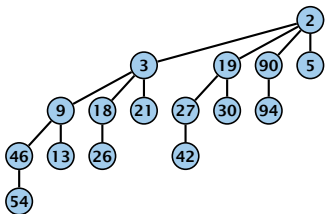




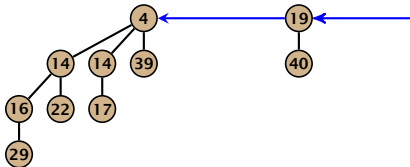
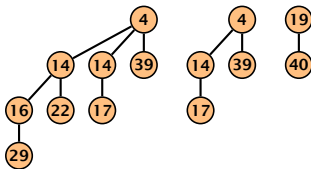
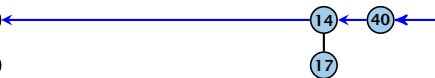
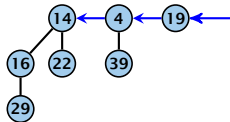
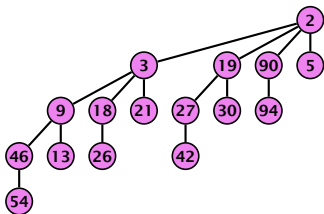




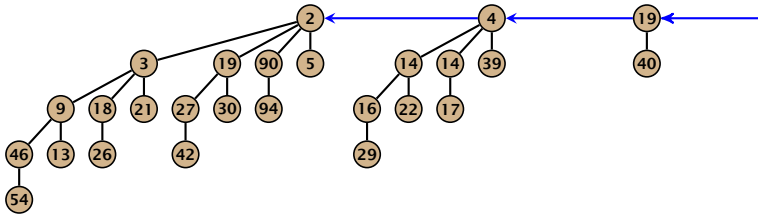
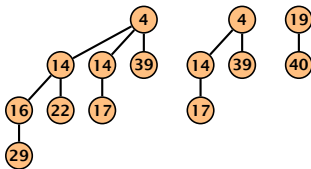
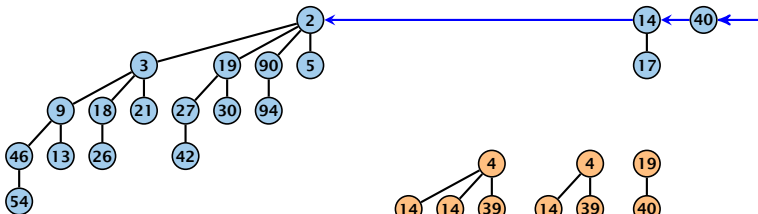
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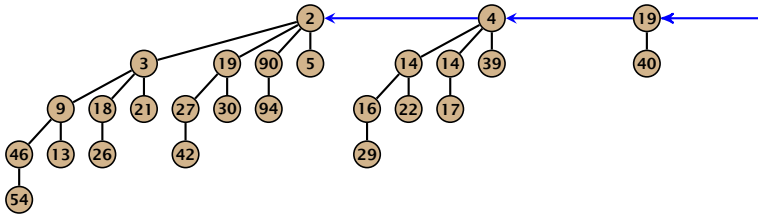
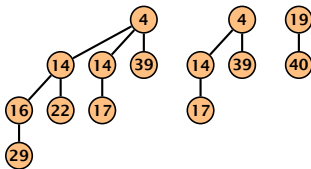
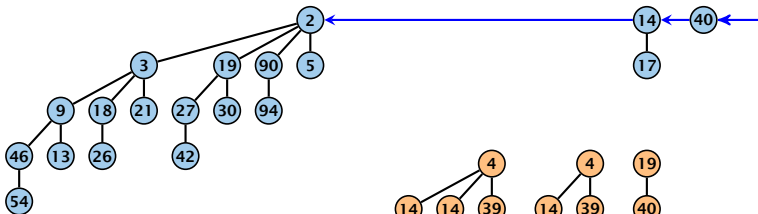
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S_1 . merge(S_2):

- ▶ Analogous to binary addition.
- ▶ Time is proportional to the number of trees in both heaps.
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`S.insert(x)`:

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