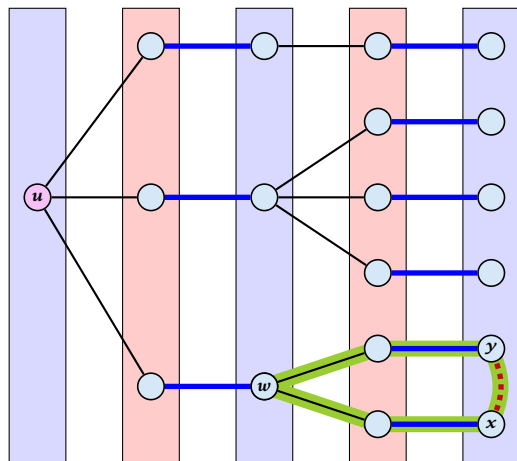# How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w is called the stem of the blossom.

Flowers and Blossoms

Definition 1
A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

▶ A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).

▶ A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

# Flowers and Blossoms
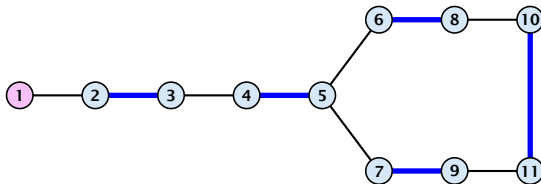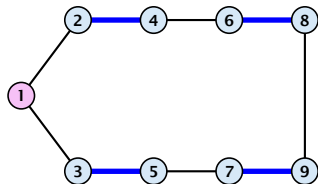
Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.

2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.

3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

5. The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms
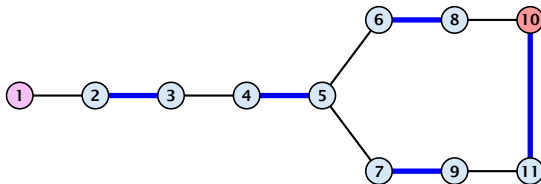
Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- ▶ Delete all vertices in B (and its incident edges) from G.
- ▶ Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

Shrinking Blossoms

▶ Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.

▶ Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.

▶ Nodes that are connected in G to at least one node in B become connected to b in G'.
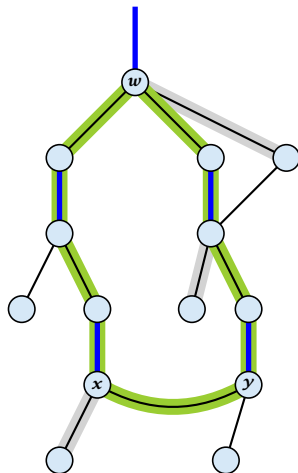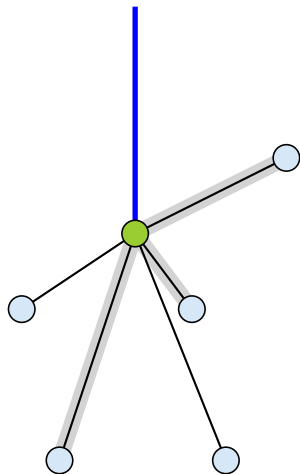
Shrinking Blossoms

▶ Edges of T that connect a node u
  not in B to a node in B become
  tree edges in T' connecting u to
  b.

▶ Matching edges (there is at most
  one) that connect a node u not in
  B to a node in B become
  matching edges in M'.

▶ Nodes that are connected in G to
  at least one node in B become
  connected to b in G'.

Example: Blossom Algorithm

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 2
If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.
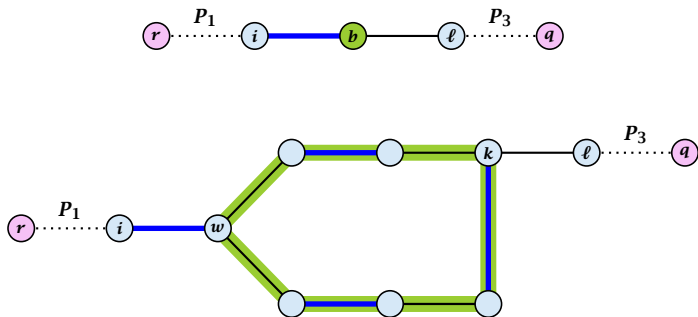
Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

▶ Next suppose that the stem is non-empty.
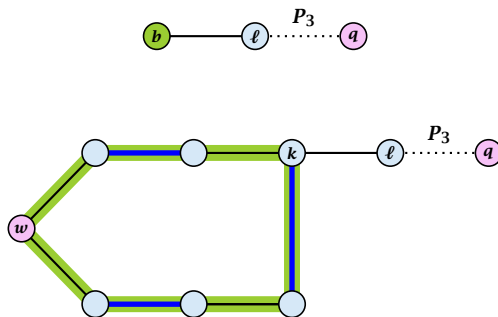
Correctness

▶ After the expansion ℓ must be incident to some node in the blossom. Let this node be k.

▶ If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.

▶ $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

▶ If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom,
$w = r$.

▶ The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 3

If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

- ▶ If P does not contain a node from B there is nothing to prove.
- ▶ We can assume that r and q are the only free nodes in G.
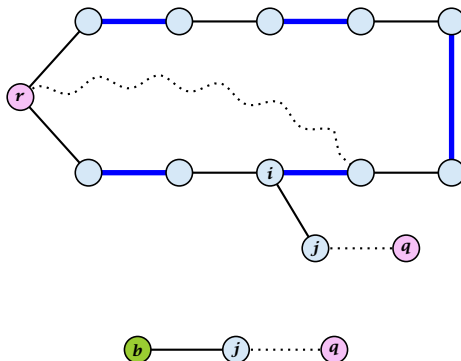
Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

Correctness

Illustration for Case 1:

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

This path must go between r and q.

Algorithm 49 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \emptyset$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

Search for an augmenting path
starting at r.

Algorithm 50 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:     **if** j is even **then** contract(i, j) and **return**
3:     **if** j is unmatched **then**
4:         $q \leftarrow j$;
5:         pred(q) $\leftarrow i$;
6:         *found* $\leftarrow$ true;
7:         **return**
8:     **if** j is matched and unlabeled **then**
9:         pred(j) $\leftarrow i$;
10:         pred(mate(j)) $\leftarrow j$;
11:         add mate(j) to *list*

Examine the neighbours of a node i

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Contract blossom identified by
nodes i and j

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

> b will be an even node, and it has
> unexamined neighbours.

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Every node that was adjacent to a node
in B is now adjacent to b

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only for making a blossom
expansion easier.

Algorithm 51 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only delete links from nodes not in *B* to *B*.

When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

▶ A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

▶ The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

▶ There are at most n contractions as each contraction reduces the number of vertices.

▶ The expansion can trivially be done in the same time as needed for all contractions.

▶ An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.

▶ In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

Example: Blossom Algorithm

20 Maximum Matching in General Graphs   15. Oct. 2018