# 13 – Dynamic Programming (3)

## Optimal Binary Search Trees

## Subset Sums & Knapsacks

# Average-case analysis

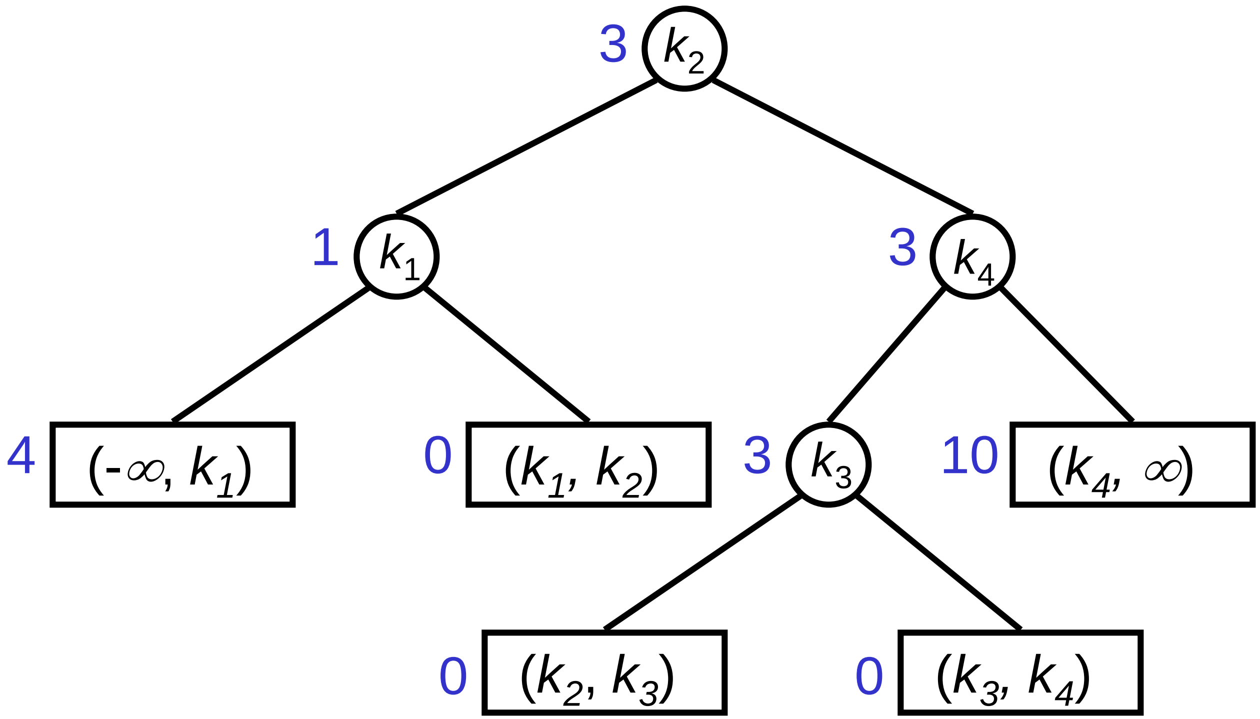
Average-case analysis of algorithms and data structures: Input is generated according to a known probability distribution. This distribution can be learned over time.

Optimal binary search tree: Request probabilities / frequencies for the keys are known in advance. Construct a binary search tree minimizing the expected / average search time.

# Optimal binary search trees

$(-\infty,k_1)\ k_1\ (k_1,k_2)\ k_2\ (k_2,k_3)\ k_3\ (k_3,k_4)\ k_4\ (k_4,\ \infty)$

4 1 0 3 0 3 0 3 10




Weighted path length:

$3 \cdot 1 + 2 \cdot (1 + 3) + 3 \cdot 3 + 2 \cdot (4 + 10)$

# Optimal binary search trees

**Problem:** Set $S$ of keys

$S = \{k_1,\ldots,k_n\}$ $\qquad -\infty = k_0 < k_1 < \ldots < k_n < k_{n+1} = \infty$

$a_i$: (absolute) frequency of requests to key $k_i$

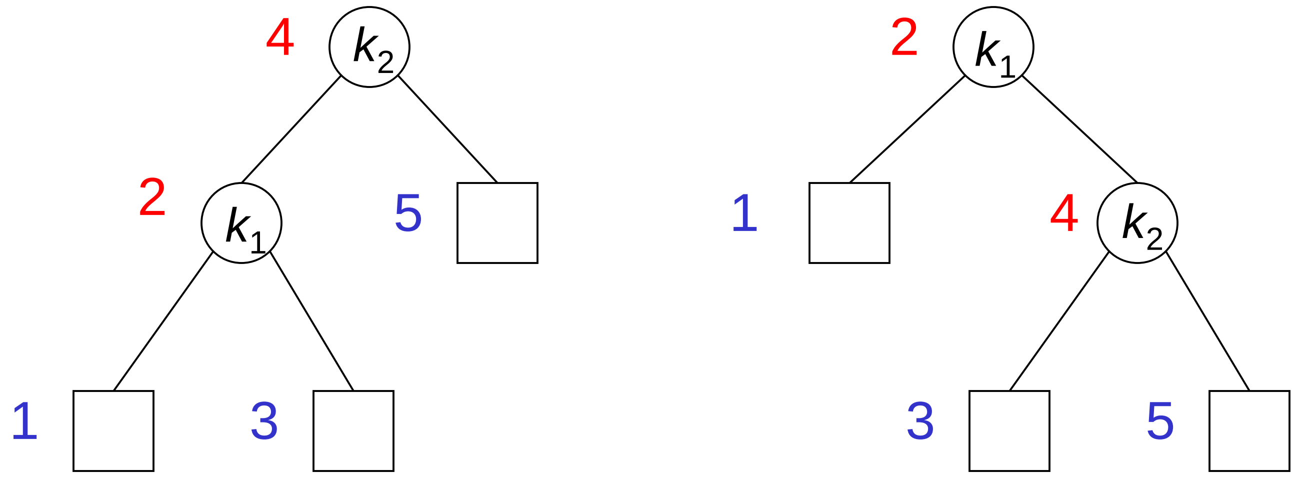
$b_j$: (absolute) frequency of requests to $x \in (k_j, k_{j+1})$

Weighted path length $P(T)$ of a binary search tree $T$ for $S$:

$$P(T) = \sum_{i=1}^{n} (depth(k_i) + 1)a_i + \sum_{j=0}^{n} depth((k_j, k_{j+1}))b_j$$

**Goal:** Binary search tree with minimum weighted path length $P$ for $S$ .

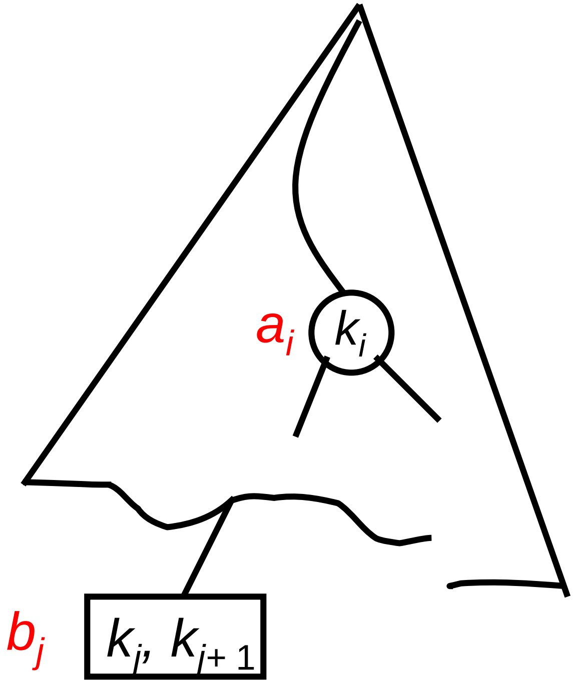
# Example




$P(T_1) = 21$

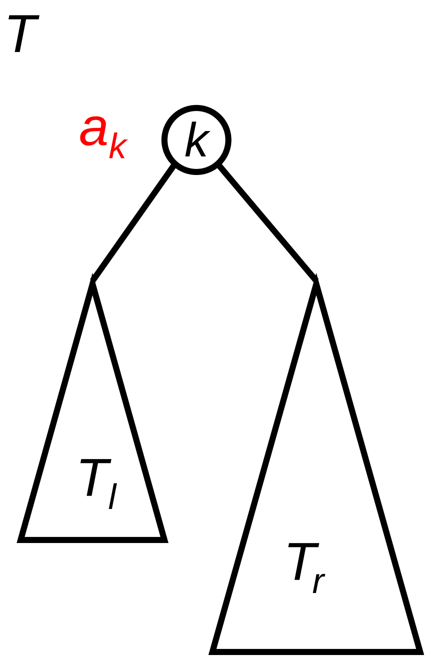
$P(T_2) = 27$

# Construction of optimal binary search trees




An optimal binary search tree is a binary search tree with minimum weighted path length.

# Dynamic programming approach




$P(T) \;= \mathrm{P}(T_l) + \mathrm{W}(T_l) + \mathrm{P}(T_r) + \mathrm{W}(T_r) + a_k$

$\quad\;\; = \mathrm{P}(T_l) + P(T_r) + W(T)$

$W(T)$ := total weight of all nodes in $T$

If $T$ is a tree with minimum weighted path length for $S$, then subtrees $T_l$ and $T_r$ are trees with minimum weighted path length for subsets of $S$.

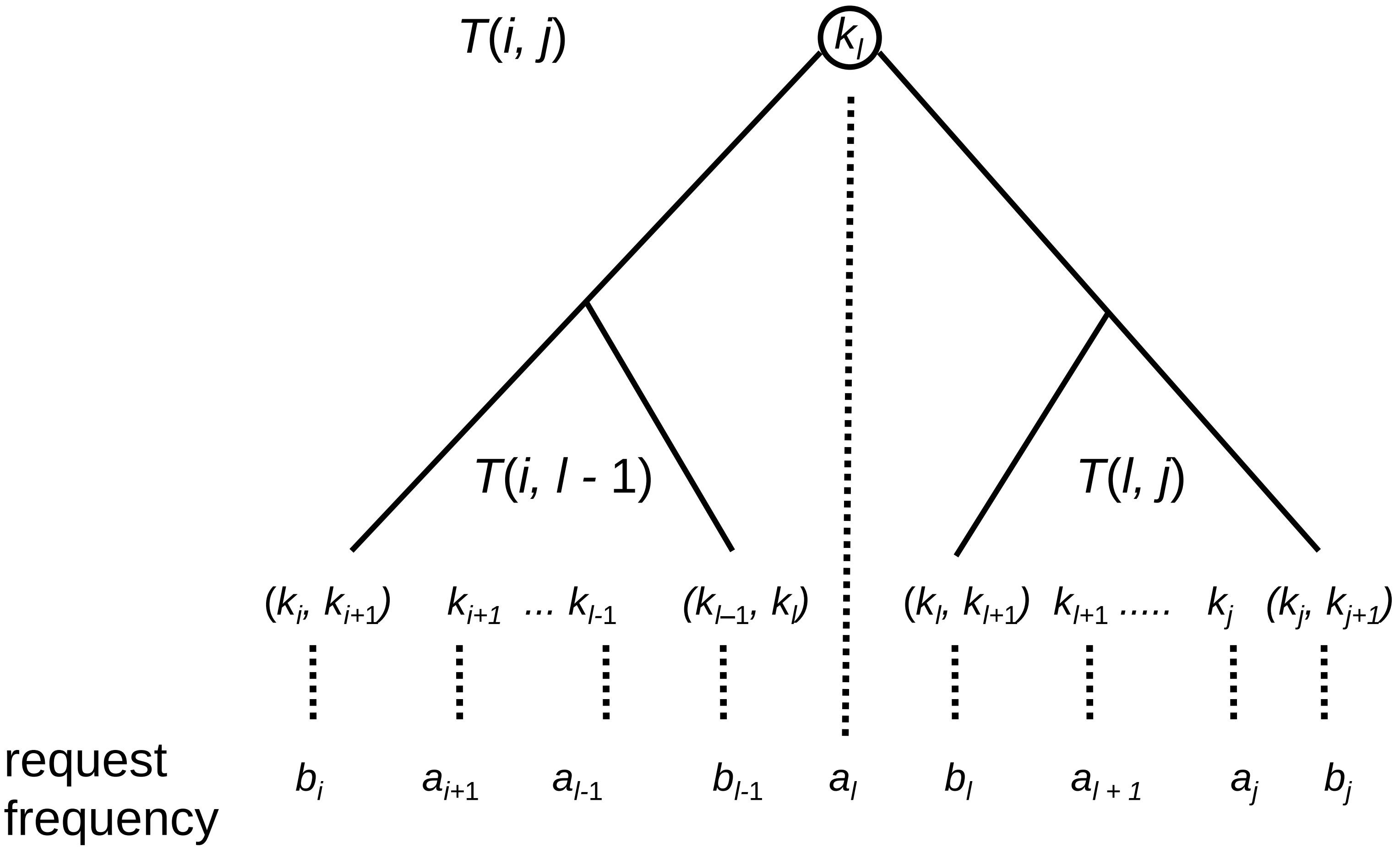
# Definition of subproblems

$T(i, j)$ : optimal binary search tree for $(k_i, k_{i+1})\ k_{i+1} \ldots k_j\ (k_j, k_{j+1})$

$W(i, j)$: weight of $T(i, j)$, i.e. $W(i, j) = b_i + a_{i+1} + \ldots + a_j + b_j$

$P(i, j)$ : weighted path length of $T(i, j)$

# Subproblems



# Recurrences

$W(i, i) = b_i$ for $0 \leq i \leq n$

$W(i, j) = W(i, j{-}1) + a_j + b_j$ for $0 \leq i < j \leq n$

$P(i, i) = 0$ for $0 \leq i \leq n$

$P(i, j) = W(i, j) + \min_{i < l \leq j} \{ P(i, l-1) + P(l, j) \}$ for $0 \leq i < j \leq n$ (*)

$r(i, j)$ = the index $l$ for which the minimum is achieved in (*)
(index of key in the root)

# Bottom-up approach

## Base cases

Case 1: $s = j - i = 0$

$T(i, i) = (k_i, k_{i+1})$

$W(i, i) = b_i$

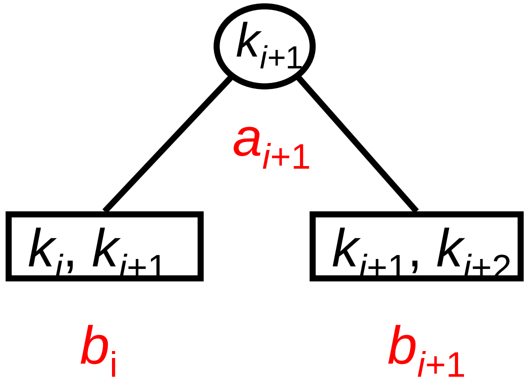
$P(i, i) = 0$

$r(i, i)$ not defined

# Bottom-up approach

Case 2: $s = j - i = 1$

$T(i, i+1)$




$W(i, i+1) = b_i + a_{i+1} + b_{i+1} = W(i, i) + a_{i+1} + W(i+1, i+1)$

$P(i, i+1) = W(i, i+1)$

$r(i, i+1) = i + 1$

# Bottom-up approach

Case 3: $s = j - i > 1$

**for** $s := 2$ **to** $n$ **do**<br>
  **for** $i := 0$ **to** $n - s$ **do**<br>
      $j := i + s$;<br>
      Determine (greatest) $l$, $i < l \leq j$, s.t. $P(i, l - 1) + P(l, j)$ is minimal;<br>
      $W(i, j) := W(i, j-1) + a_j + b_j$;<br>
      $P(i, j) := P(i, l-1) + P(l, j) + W(i, j)$;<br>
      $r(i, j) := l$;<br>
  **endfor;**<br>
**endfor;**

Computing solution $P(0,n)$ takes $O(n^3)$ time and requires $O(n^2)$ space.

# Improvement

**Lemma:** For all $i$, $j$ such that $0 \le i < j \le n$, $r(i, j-1) \le r(i, j) \le r(i+1, j)$.

Given this lemma, for fixed $s$, the total time of the inner for-loop is:

$$O(n - s + \sum_{i=0}^{n-s} (r(i+1, i+s) - r(i, i+s-1) + 1) )$$

$$\begin{aligned} &= O(n - s + r(1, s) - r(0, s-1) + 1 \\ &\qquad + r(2, s+1) - r(1, s) + 1 \\ &\qquad + r(3, s+2) - r(2, s+1) + 1 \\ &\qquad \ldots \\ &\qquad + r(n-s+1, n) - r(n-s, n-1) + 1) \\ &= O(n - s + r(n-s+1, n) - r(0, s-1)) \\ &= O(n) \end{aligned}$$



# Improvement

**Proof of the Lemma:** Induction on $s = j-i$.

For $s = 1$, the statement is vacuous. So consider $s = 2$.

In $T(i, i+1)$ key $k_{i+1}$ is in the root, so that $r(i, i+1) = i+1$. In $T(i+1, i+2)$ key $k_{i+2}$ is in the root, so that $r(i+1, i+2) = i+2$. In $T(i, i+2)$ key $k_{i+1}$ or $k_{i+2}$ can reside in the root, which implies $r(i, i+2) \in \{i+1, i+2\}$ and the desired inequality holds.

We study $s > 2$ and prove $r(i, j-1) \leq r(i, j)$. The second inequality $r(i, j) \leq r(i+1, j)$ can be shown analogously. Consider an optimal tree $T(i, j-1)$. Replace leaf $(k_{j-1},k_j)$ by a node containing $k_j$ along with the leaves $(k_{j-1},k_j)$ and $(k_j,k_{j+1})$. Let $T$ be the resulting tree. There holds

$$P(T) = P(i, j-1) + b_{j-1} + (d+1)(a_j + b_j),$$

where $d$ denotes the depth of $k_j$ in $T$.

# Proof of the lemma

Suppose that $P(T) > P(i,j)$ since otherwise we are done.

Consider an optimal tree $T(i, j)$ and let $d$‘ be the depth of $k_j$.

**Claim 1:** There holds $d > d$‘.

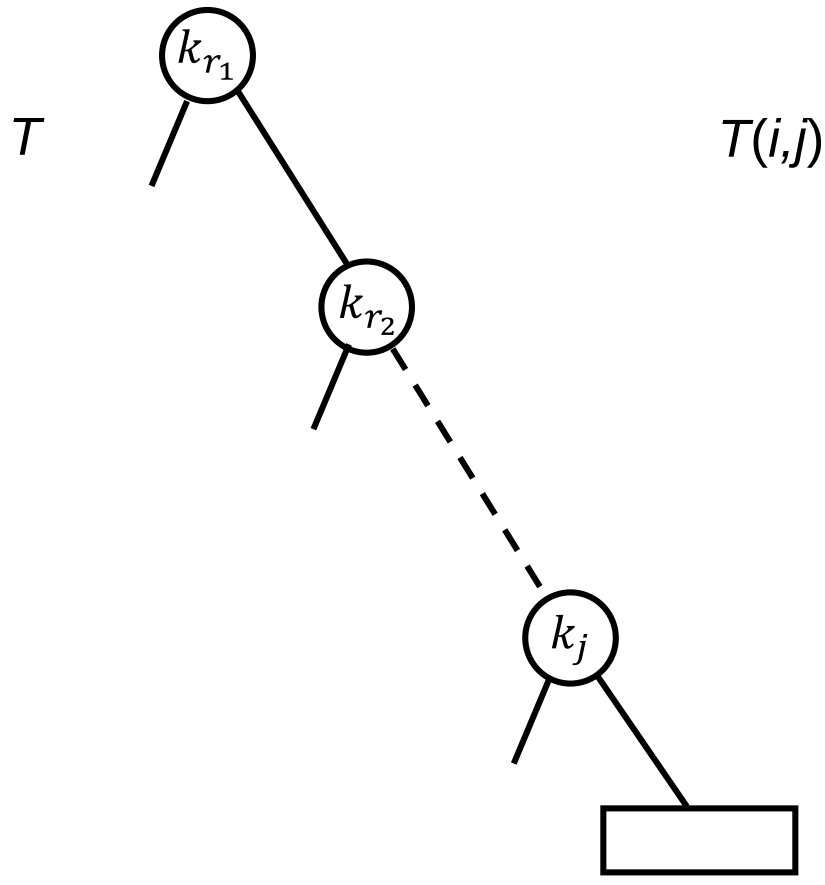
For the proof of the claim take $T(i, j)$ and replace key $k_j$ along with leaves $(k_{j-1},k_j)$ and $(k_j,k_{j+1})$ by leaf $(k_{j-1},k_j)$. The resulting tree has a weighted path length of

$$P(i, j) - b_{j-1} - (d'+1)(a_j + b_j) < P(T) - b_{j-1} - (d'+1)(a_j + b_j)$$
$$= P(i, j-1) + (d-d')(a_j + b_j).$$

Hence, if $d \leq d$‘, the resulting tree has a weighted path length strictly smaller than $P(i, j-1)$, contradicting the optimality of $T(i, j-1)$.

In $T$ and $T(i, j)$ consider the paths from the root to $k_j$; cf. the figure on the next page.

# Proof of the lemma



# Proof of the lemma

Suppose that $r_1 > r_1'$; otherwise there is nothing to show.

**Claim 2:** There exists an *l* such that $r_l = r_l'$.

By assumption $r_1' < r_1$. The induction hypothesis implies $r_2' \leq r_2$. If $r_2' < r_2$, then $r_3' \leq r_3$. In general if $r_{l-1}' < r_{l-1}$, the induction hypothesis implies $r_l' \leq r_l$. Since *d*' < *d* and $k_j$ is the largest key in the trees, there must exist an *l* such that $r_l = r_l'$.

In *T* we now replace the right subtree of $k_{r_l}$ by the corresponding subtree in *T*(*i*,*j*). The new tree has a minimum weighted path length: Otherwise in *T*(*i*, *j*-1) the nodes containing $k_{r_1}, \ldots, k_{r_l}$ and their left subtrees could be replaced by the corresponding structure in *T(i,j)*, contradicting the optimality of *T*(*i*, *j*-1).

In the new, optimal tree key $k_{r_1}$ is in the root.

# Main result

**Theorem:**

An optimal binary search tree for $n$ keys and $n + 1$ intervals with known request frequencies can be constructed in $O(n^2)$ time.

# Subset Sums and Knapsacks

**Subset Sum Problem:** $n$ items $\{1,\ldots,n\}$

Item $i$ has a non-negative weight $w_i$. Bound $W \geq 0$

**Goal:** Find $S \subseteq \{1,\ldots,n\}$ that maximizes $\Sigma_{i \in S}\, w_i$ subject to the restriction $\Sigma_{i \in S}\, w_i \leq W$.

**Knapsack Problem:** $n$ items $\{1,\ldots,n\}$

Item $i$ has a non-negative weight $w_i$ and a value $v_i$. Bound $W \geq 0$

**Goal:** Find $S \subseteq \{1,\ldots,n\}$ that maximizes $\Sigma_{i \in S}\, v_i$ subject to the restriction $\Sigma_{i \in S}\, w_i \leq W$.

# Subset Sums: Dynamic programming

Assume that $W \in \mathbb{N}$ and $w_i \in \mathbb{N}$ for $i$=1,…$n$.

For $1 \le i \le n$ and each integer $w$ with $0 \le w \le W$

$\text{OPT}(i,w) = \max_S \Sigma_{j \in S}\, w_j$ where $S \subseteq \{1,\ldots,i\}$ and $\Sigma_{j \in S}\, w_j \le w$.

$O$ = optimal solution

- $n \notin O$: $\text{OPT}(n,W) = \text{OPT}(n-1,W)$
- $n \in O$: $\text{OPT}(n,W) = w_n + \text{OPT}(n-1, W - w_n)$

If $w < w_i$, then $\text{OPT}(i,w) = \text{OPT}(i-1,w)$. Otherwise

$\text{OPT}(i,w) = \max\{\, \text{OPT}(i-1,w),\; w_i + \text{OPT}(i-1, w - w_i)\,\}$.

# Dynamic programming algorithm

Array M[0..$n$,0..$W$] optimal solutions OPT($i$,$w$),   $0 \le i \le n$   $0 \le w \le W$.

**Algorithm** SubsetSum($n$,$W$)

```
for w := 0 to W do
      M[0,w] := 0;
endfor;
for i := 1 to n do
   for w := 0 to W do
         if  w < w_i
            then OPT(i,w) = OPT(i-1,w);
            else OPT(i,w) := max{ M[i-1,w],  w_i + M[i-1, w - w_i] };
         endif;
   endfor;
endfor;
```

Pseudopolynomial running time: $\Theta(nW)$

# Knapsack: Dynamic programming

Assume that $W \in \mathbb{N}$ and $w_i \in \mathbb{N}$ for $i=1,\ldots n$.

For $1 \leq i \leq n$ and each integer $w$ with $0 \leq w \leq W$

$$\mathrm{OPT}(i,w) = \max_S \Sigma_{j \in S} v_j \quad \text{where } S \subseteq \{1,\ldots,i\} \text{ and } \Sigma_{j \in S} w_j \leq w.$$

$O$ = optimal solution

- $n \notin O$: $\mathrm{OPT}(n,W) = \mathrm{OPT}(n\text{-}1,W)$
- $n \in O$: $\mathrm{OPT}(n,W) = v_n + \mathrm{OPT}(n\text{-}1,W - w_n)$

If $w < w_i$, then $\mathrm{OPT}(i,w) = \mathrm{OPT}(i\text{-}1,w)$. Otherwise

$$\mathrm{OPT}(i,w) = \max\{ \mathrm{OPT}(i\text{-}1,w),\ v_i + \mathrm{OPT}(i\text{-}1, w - w_i) \}.$$

Pseudopolynomial running time: $\Theta(nW)$