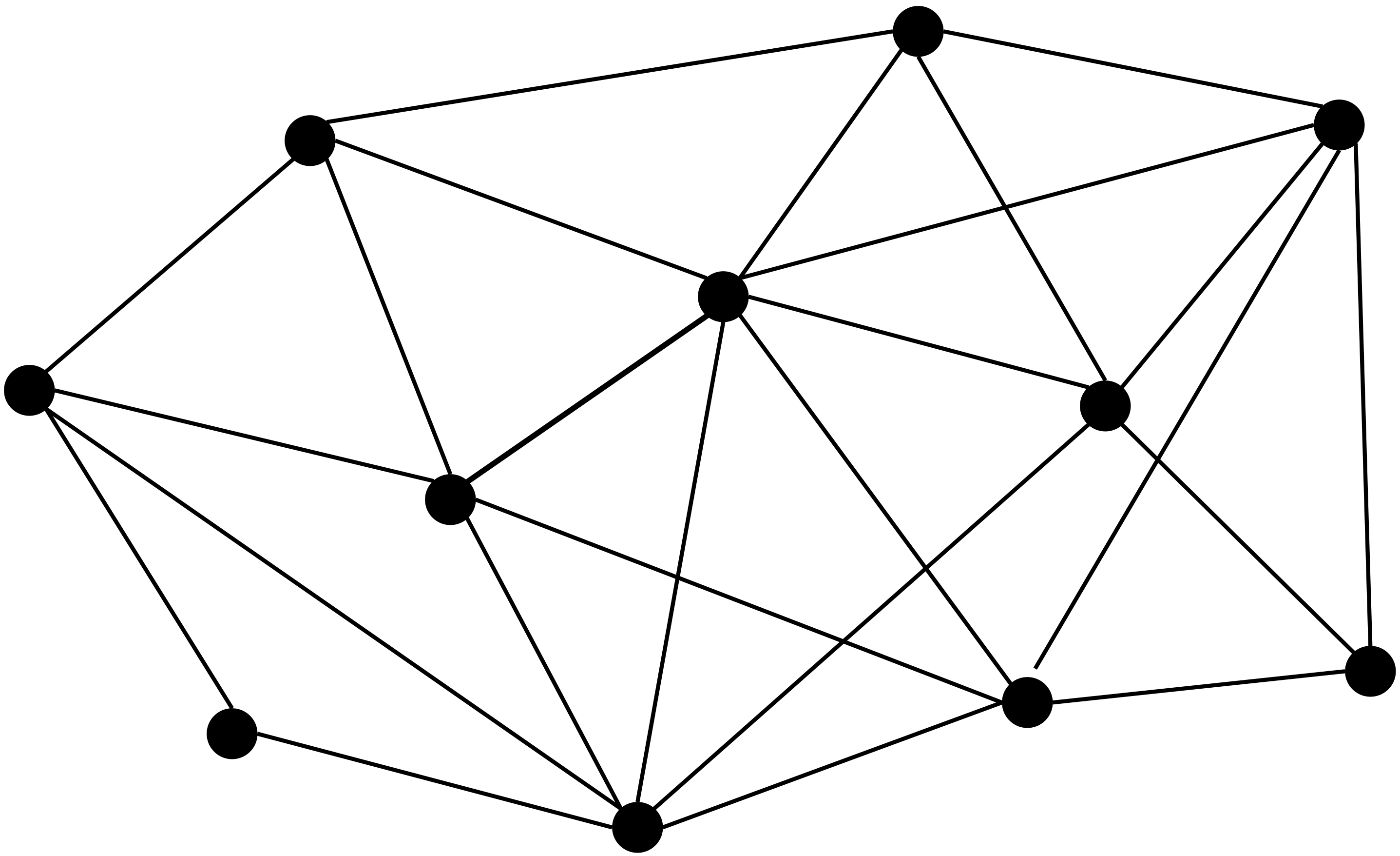
05 – Minimum Cuts

1. Minimum cuts



Minimum cuts

Input: Undirected graph $G = (V,E)$ $\quad n = |V| \quad m = |E|$

Output: $V_1, V_2, \subseteq V$ such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ and the number of edges between V_1 and V_2 is as small as possible.

$c_{min}(G)$ = # edges of a minimum cut of G

A cut is often represented by the set of edges between V_1, V_2 .

Weighted problem: Edge e has weight $w(e)$.
Find a cut of minimum weight.

Reduction to network flow: For all pairs $s, t \in V$, compute a maximum (s,t)- flow. Time $O(n^5)$

2. Randomized algorithm

Multigraph: Multiple edges may exist between any two vertices.

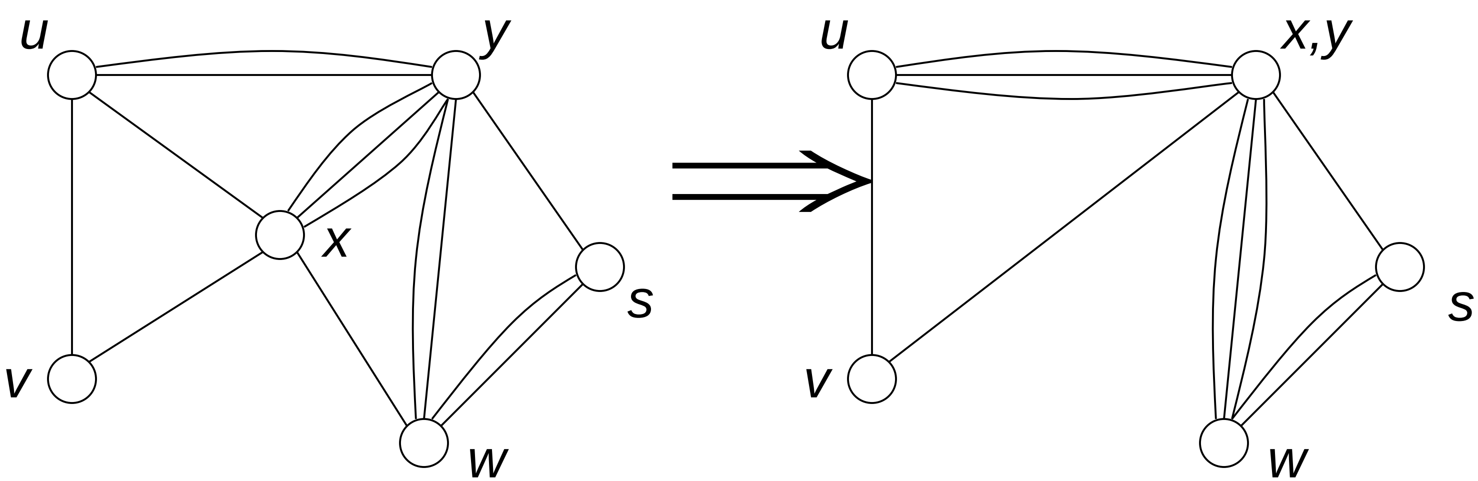
Basic operation: Edge contraction $e = \{x,y\}$.

Replace x,y by a meta-vertex z.

For $v \notin \{x,y\}$ replace $\{v,x\}$ by $\{v,z\}$

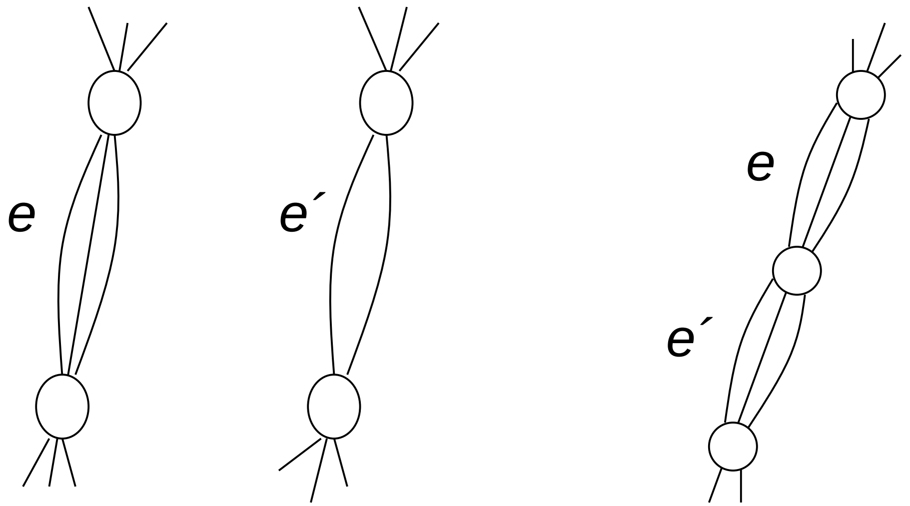
replace $\{v,y\}$ by $\{v,z\}$. No self-loops!

→ $G \setminus \{e\}$



Contractions

The order of the contractions is irrelevant.



Each edge contraction can be implemented in time $O(n)$ using (extended) adjacency lists or matrices.

For each meta-vertex

store the number of edges to other meta-vertices,

store the names of the original vertices it contains.

Algorithm *Contraction*

Algorithm *Contraction*;

1. $H := G$;
2. **while** H consists of more than two vertices **do**
3. Choose an edge e in H uniformly at random;
4. $H := H \setminus \{e\}$;
5. **endwhile**;
6. Let V_1, V_2 be the vertex sets represented by the last two vertices in H.

Running time: $O(n^2)$

Properties

Lemma 1: Partition V_1, V_2 is output by the *Contraction* if and only if no edge between V_1 and V_2 is ever contracted.

Proof: If an edge $\{v_1,v_2\}$ with $v_1 \in V_1$ and $v_2 \in V_2$ is contracted, partition V_1, V_2 cannot be output by the algoritm.

If no edge between V_1 and V_2 is ever contracted, then this partition indeed survives.

Lemma 2: Let G be a multigraph. If $c_{min}(G) = k$, then all vertices have degree $\geq k$ and G has $\geq nk/2$ edges.

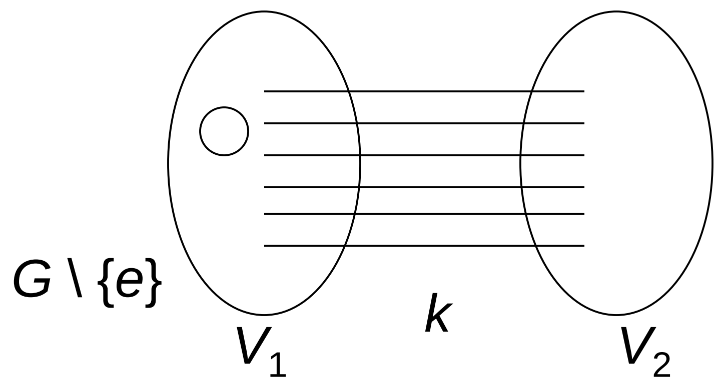
Proof: If there were a vertex v having degree $< k$, then $\{v\}$ and $V\backslash\{v\}$ would be a cut with less than k edges, and $c_{min}(G) < k$.

If all edges have degree $\geq k$, then the total edge degree, summed over all vertices, is at least nk. In this total edge degree, each edge is counted exactly twice.

Properties

Lemma 3: Let G be a multigraph. For each edge e in G there holds $c_{\min}(G) \leq c_{\min}(G \setminus \{e\})$.

Proof: Partition V_1, V_2 of $G \setminus \{e\}$ with k edges is also a partition of G with k edges.



Probability of success

Theorem 1: Let C be a minimum cut in G.
Contraction returns C with probability $\geq 2/n^2$.

Proof: Let $c_{\min}(G) = k$.
Consider the i-th iteration of the while-loop.
H has $n_i = n - i + 1$ vertices.
Suppose that the first $i - 1$ iterations do not contract an edge of C.
C is a cut of H and, by Lemma 3, $c_{\min}(H) = k$.
Furthermore, by Lemma 2, H has at least $n_i k/2$ edges.

Prob[i-th iteration contracts an edge of C] $\leq 2/n_i$

Prob[i-th iterationen does not contract an edge of C] $\geq 1 - 2/n_i$

Probability of success

Prob[*C* is output]

$$\geq \prod_{i=1}^{n-2}\left(1-\frac{2}{n_i}\right)=\prod_{i=1}^{n-2}\frac{n-i-1}{n-i+1}=\prod_{j=3}^{n}\frac{j-2}{j}=\frac{1\cdots(n-2)}{3\cdots n}$$

$$=\frac{2}{n(n-1)}\geq\frac{2}{n^2}$$

Increasing the success probability

Repeat *Contraction* $dn^2 \ln n$ times, for some constant d, and select the smallest cut.

$$\text{Prob}\left[C \text{ is not found}\right] \le \left(1 - \frac{2}{n^2}\right)^{n^2 d \ln n} \le e^{-2d \ln n} = n^{-2d}$$

3. Improved running time

Lemma 4: Let *C* be a minimum cut. Stop *Contraction* when exactly *t* vertices are left. There holds

$$\text{Prob}\left[\text{no edge of } C \text{ is contracted}\right] \geq \frac{t(t-1)}{n(n-1)}.$$

Proof:

$$\prod_{i=1}^{n-t}\left(1-\frac{2}{n_i}\right) = \prod_{i=1}^{n-t}\frac{n-i-1}{n-i+1} = \prod_{j=t+1}^{n}\frac{j-2}{j} = \frac{(t-1)\cdots(n-2)}{(t+1)\cdots n}$$

$$= \frac{(t-1)t}{n(n-1)}$$

Algorithm *Fast-Cut*

Input: Multigraph $G = (V, E)$.

Output: Partition $V = V_1 \cup V_2$ or the respective edge set.

1. $n := |V|$;
2. **if** $n \leq 6$ **then**
3. Compute a minimum cut by complete enumeration;
4. **else**
5. $t := \left\lceil 1 + n/\sqrt{2} \right\rceil$ /* $\frac{t(t-1)}{n(n-1)} \geq \frac{1}{2}$
6. Execute *Contraction* twice so that each time exactly t vertices remain. Let H_1 and H_2 be the resulting graphs;
7. Apply *Fast-Cut* recursively to H_1 and H_2;
8. Output the smaller cut;
9. **endif;**

Algorithm *Fast-Cut*

Theorem 2: *Fast-Cut* has a running time of $O(n^2 \log n)$.

Proof: *Contraction* has a running time of $O(n^2)$.

$$T(n) = 2T\left(\left\lceil 1 + n/\sqrt{2} \right\rceil\right) + O\left(n^2\right)$$

Success probability

Theorem 3: *Fast-Cut* finds minimum cut with probability Ω (1/log n).

Proof: Let C be a minimum cut.

Fast-Cut returns *a* minimum cut if

- during the reduction to H_1 or H_2 no edge of C is contracted and
- *Fast-Cut* applied to such an H_i returns C.

$P(n)$ = Prob[*Fast-Cut* finds a minimum cut in graphs with n vertices]

Success probability

$$P(n) \geq 1 - \text{Prob}\left[\mathit{Fast-Cut} \text{ does not find } C \text{ in any of the two trials}\right]$$

$$= 1 - \prod_{i=1,2} \text{Prob}\left[\mathit{Fast-Cut} \text{ does not find } C \text{ in the trial on } H_i\right]$$

$$= 1 - \prod_{i=1,2} \left(1 - \text{Prob}\left[\mathit{Fast-Cut} \text{ finds } C \text{ in the trial on } H_i\right]\right)$$

$$\geq 1 - \left(1 - \frac{1}{2} P(t)\right)^2$$

Success probability

$p(l)$ = lower bound on P if there are l recursive levels

$$p(l+1) = 1 - \left(1 - \frac{1}{2}p(l)\right)^2 = p(l) - \frac{p(l)^2}{4}$$

$$p(0) = 1$$

We prove that $p(l) \geq 1/d$ implies $p(l+1) \geq 1/(d+1)$.

Since $p(0) = 1 \geq 1/1$ it follows $p(l) \geq 1/(l+1) = \Omega(1/l)$.
There are $O(\log n)$ recursive levels so that $P(n) = \Omega(1/\log n)$.

Success probability

$f(x) = x - x^2/4$ is monotonically increasing in [0,1]

Hence $p(l) \geq 1/d$, where $d \geq 1$, and $p(l) \in [0,1]$ imply

$$
\begin{aligned}
p(l+1) &\geq \frac{1}{d} - \frac{1}{4d^2} \\
&= \frac{d+1}{d+1} \cdot \frac{4d-1}{4d^2} \\
&= \frac{1}{d+1} \frac{4d^2+3d-1}{4d^2} \\
&\geq \frac{1}{d+1}.
\end{aligned}
$$

Increasing the success probability

Repeat *Fast-Cut* $d \ln^2 n$ times, for some constant d, and select the smallest cut.

$$\text{Prob}\left[C \text{ not found}\right] \le \left(1 - \frac{c}{\ln n}\right)^{d \ln^2 n} \le e^{-cd \ln n} = n^{-cd}$$

Running time: $O(n^2 \log^3 n)$

4. Minimum weighted cuts

Weighted problem: Undirected graph $G = (V,E)$ $n = |V|$ $m = |E|$

Edge e in $G = (V,E)$ has weight $w(e) \geq 0$.

Output: $V_1, V_2, \subseteq V$ such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ and

$\sum_{e=(u,v)\in V_1 \times V_2} w(e)$ is as small as possible.

Let $c_{\min}(G)$ denote the weight of such a minimum cut.

A minimum (s,t)-cut, where $s, t \in V$, is a cut $V_s, V_t \subseteq V$ with $V_s \cup V_t = V$, $V_s \cap V_t = \emptyset$ and $s \in V_s, t \in V_t$ of minimum weight. This weight is denoted by $c_{\min}(G, s, t)$.

Minimum weighted cuts

$G \setminus \{x,y\}$ = graph if x, y are contracted

The weights of multiple edges add up.

Lemma 5: Let $s, t \in V$ be arbitrary. There holds

$$c_{\min}(G) = \min\{c_{\min}(G,s,t), c_{\min}(G \setminus \{s,t\})\}.$$

Deterministic algorithm

Algorithm *Some-(s,t)-Cut;*

Input: Graph G

Output: s, t (along with a minimum (s,t)-cut)

1. A := {arbitrary vertex of V};
2. **while** $A \neq V$ **do**
3. Add vertex $v \in V - A$ to A for which $w(v,A)$ is maximum;
4. **endwhile;**
5. Let s be the second to last and t be the last vertex added to A;

$w(v,A)$ = total weight of edges between v and vertices in A

Deterministic algorithm

Algorithm *Minimum-Cut*;

1. $Min := \infty$; $n := |V|$;
2. **while** $n \geq 2$ **do**
3. Execute *Some-(s,t)-Cut,* and obtain s, t and a cut C of weight W;
4. **if** $W < Min$ **then** store C; $Min := W$; **endif**;
5. Contract s and t; $n := n - 1$;
6. **endwhile;**
7. Return *Min* and the cut stored last;

Analysis

Theorem 4: Sets $V_t = \{t\}$ and $V_s = V - \{t\}$ computed by *Some-(s,t)-Cut* are a minimum (s,t)-cut.

Proof:

Number the vertices from 1 to n so that vertex i is added to A in the i-th iteration.

$s = n - 1$ and $t = n$

Let C be an arbitrary (s,t)-cut.

C_i = edges in C having both endpoints in $\{1,\dots,i\}$

$w(C_i)$ = total weight of edges in C_i

Analysis

Vertex i is active (with respect to C) if i and $i - 1$ belong to different parts of C.

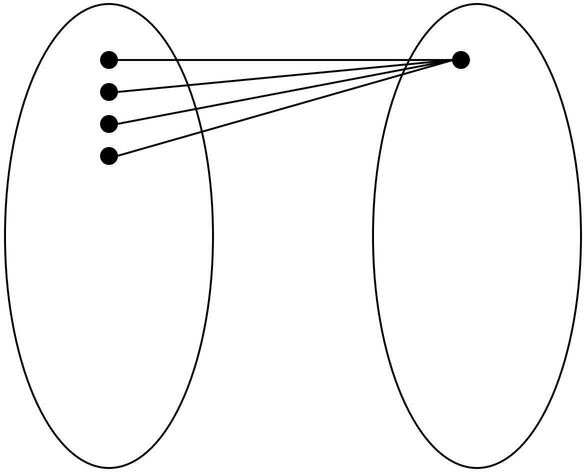
Claim: For each active vertex i there holds $w(i, \{1, ..., i - 1\}) \leq w(C_i)$.

n is active and hence $w(n, \{1, ... , n - 1\}) \leq w(C)$.

Analysis

Claim: For each active vertex *i* there holds $w(i,\{1,..,i-1\}) \leq w(C_i)$.

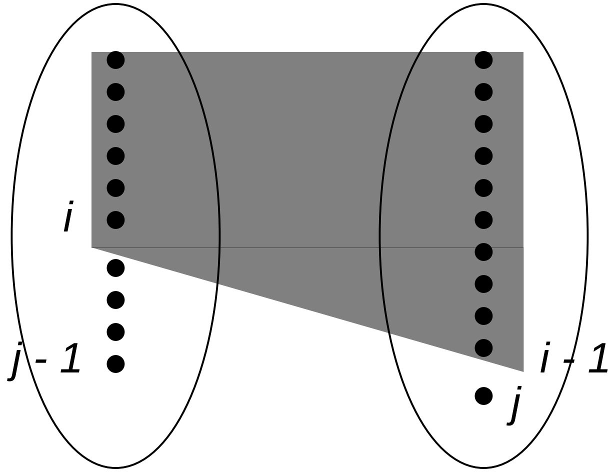
Proof: The claim holds for the first active vertex *i*.



Analysis

Suppose the claim holds for active vertex *i* and the next active vertex is *j*.

$$
\begin{aligned}
w\left(j,\{1,\ldots,j-1\}\right) &= w\left(j,\{1,\ldots,i-1\}\right)+w\left(j,\{i,\ldots,j-1\}\right)\\
&\le w\left(i,\{1,\ldots,i-1\}\right)+w\left(j,\{i,\ldots,j-1\}\right)\\
&\le w\left(C_i\right)+w\left(j,\{i,\ldots,j-1\}\right)\\
&\le w\left(C_j\right)
\end{aligned}
$$



Analysis

Theorem 5: *Minimum-Cut* computes a minimum cut in time $O(mn + n^2 \log n)$.

Proof: Correctness: Induction on the number of vertices.

Algorithm works correctly for multigraphs with $n = 2$ vertices.

n-1 → n: *Some-(s,t)-Cut* computes s,t and $c_{min}(G,s,t)$ correctly.
MinimumCut computes $c_{min}(G \setminus \{s,t\})$ correctly.

Some-(s,t)-Cut has a running time of $O(m + n \log n)$.

Maintain a priority queue for $v \in V - A$ with $key(v) = w(v,A)$.

n *DeleteMax* and m *IncreaseKey* operations (Fibonacci Heaps).