

Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{ll} \min & \sum_{i=1}^k w_i x_i \\ \text{s.t.} & \forall u \in U \quad \sum_{i:u \in S_i} x_i \geq 1 \\ & \forall i \in \{1, \dots, k\} \quad x_i \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{u \in U} y_u \\ \text{s.t.} & \forall i \in \{1, \dots, k\} \quad \sum_{u:u \in S_i} y_u \leq w_i \\ & y_u \geq 0 \end{array}$$

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$$\begin{aligned} \sum_j w_j x_j &= \sum_j \sum_{e \in S_j} y_e = \sum_e |\{j : e \in S_j\}| \cdot y_e \\ &\leq f \cdot \sum_e y_e \leq f \cdot \text{OPT} \end{aligned}$$

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If we would also fulfill **dual slackness conditions**

$$y_e > 0 \Rightarrow \sum_{j: e \in S_j} x_j = 1$$

then the solution would be **optimal!!!!**

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This is sufficient to show that the solution is an f -approximation.

Suppose we have a primal/dual pair

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{s.t.} & \forall i \quad \sum_j a_{ij} x_j \geq b_i \\ & \forall j \quad x_j \geq 0 \end{array}$$

$$\begin{array}{ll} \max & \sum_i b_i y_i \\ \text{s.t.} & \forall j \quad \sum_i a_{ij} y_i \leq c_j \\ & \forall i \quad y_i \geq 0 \end{array}$$

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and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \geq \frac{1}{\alpha} c_j$$

$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \leq \beta b_i$$

Then

$$\sum_j c_j x_j$$

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↑

primal cost

A diagram consisting of two rectangular boxes. The top box contains the mathematical expression $\sum_j c_j x_j$ in a purple font. A blue arrow points upwards from the bottom box to the bottom center of the top box. The bottom box contains the text "primal cost" in a blue font.

Then

right hand side of j -th
dual constraint

$$\sum_j c_j x_j$$

primal cost

Then

$$\boxed{\sum_j c_j x_j} \leq \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j$$

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$$\boxed{\text{primal cost}} = \alpha \sum_i \left(\sum_j a_{ij} x_j \right) y_i$$

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Feedback Vertex Set for Undirected Graphs

- ▶ Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.

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- ▶ Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.
- ▶ Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.

We can encode this as an instance of Set Cover

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- ▶ Each vertex can be viewed as a set that contains some cycles.
- ▶ However, this encoding gives a Set Cover instance of non-polynomial size.
- ▶ The $O(\log n)$ -approximation for Set Cover does not help us to get a good solution.

Let \mathcal{C} denote the set of all cycles (where a cycle is identified by its set of vertices)

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Primal Relaxation:

$$\begin{array}{ll} \min & \sum_v w_v x_v \\ \text{s.t.} & \forall C \in \mathcal{C} \quad \sum_{v \in C} x_v \geq 1 \\ & \forall v \quad x_v \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{C \in \mathcal{C}} y_C \\ \text{s.t.} & \forall v \in V \quad \sum_{C: v \in C} y_C \leq w_v \\ & \forall C \quad y_C \geq 0 \end{array}$$

If we perform the previous dual technique for Set Cover we get the following:

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- ▶ Start with $x = 0$ and $y = 0$
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 - ▶ set $x_v = 1$.

Then

$$\sum_v w_v x_v$$

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$$\begin{aligned}\sum_v w_v x_v &= \sum_v \sum_{C:v \in C} y_C x_v \\ &= \sum_{v \in S} \sum_{C:v \in C} y_C\end{aligned}$$

where S is the set of vertices we choose.

Then

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where S is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.

Algorithm 1 FeedbackVertexSet

- 1: $y \leftarrow 0$
- 2: $x \leftarrow 0$
- 3: **while** exists cycle C in G **do**
- 4: increase y_C until there is $v \in C$ s.t. $\sum_{C:v \in C} y_C = w_v$
- 5: $x_v = 1$
- 6: remove v from G
- 7: repeatedly remove vertices of degree 1 from G

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Always choose a short cycle that is not covered. If we always find a cycle of length at most α we get an α -approximation.

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Observation:

For any path P of vertices of degree 2 in G the algorithm chooses at most one vertex from P .

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If we always choose a cycle for which the number of vertices of degree at least 3 is at most α we get a 2α -approximation.

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Theorem 2

In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.

This means we have

$$y_C > 0 \Rightarrow |S \cap C| \leq \mathcal{O}(\log n) .$$

Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \rightarrow \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c .

$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \in \mathcal{S} \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in S , and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

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The Dual:

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Primal Dual for Shortest Path

We can interpret the value γ_S as the width of a moat surrounding the set S .

Each set can have its own moat but all moats must be disjoint.

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Algorithm 1 PrimalDualShortestPath

- 1: $\gamma \leftarrow 0$
- 2: $F \leftarrow \emptyset$
- 3: **while** there is no s - t path in (V, F) **do**
- 4: Let C be the connected component of (V, F) containing s
- 5: Increase γ_C until there is an edge $e' \in \delta(C)$ such that $\sum_{S:e' \in \delta(S)} \gamma_S = c(e')$.
- 6: $F \leftarrow F \cup \{e'\}$
- 7: **Let P be an s - t path in (V, F)**
- 8: **return P**

Lemma 3

At each point in time the set F forms a tree.

Proof:

Assume that at some time t the set F contains a cycle. Then F contains a cycle C and also a path P from C to C . The cycle C and the path P are disjoint. The cycle C is a cycle in F .

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Proof:

- ▶ In each iteration we take the current connected component from (V, F) that contains s (call this component C) and add some edge from $\delta(C)$ to F .
- ▶ Since, at most one end-point of the new edge is in C the edge cannot close a cycle.

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$$\sum_{e \in P} c(e)$$

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$$\sum_{e \in P} c(e) = \sum_S y_S \leq \text{OPT}$$

by weak duality.

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Hence, we find a shortest path.

If S contains two edges from P then there must exist a subpath P' of P that starts and ends with a vertex from S (and all interior vertices are not in S).

When we increased γ_S , S was a connected component of the set of edges F' that we had chosen till this point.

$F' \cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.

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Steiner Forest Problem:

Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i , $i = 1, \dots, k$, and a cost function $c : E \rightarrow \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F .

$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \subseteq V : S \in S_i \text{ for some } i \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

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$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \subseteq V : S \in \mathcal{S}_i \text{ for some } i \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

Here \mathcal{S}_i contains all sets S such that $s_i \in S$ and $t_i \notin S$.

$$\begin{array}{ll}
 \max & \sum_{S: \exists i \text{ s.t. } S \in S_i} \mathcal{Y}_S \\
 \text{s.t.} & \forall e \in E \quad \sum_{S: e \in \delta(S)} \mathcal{Y}_S \leq c(e) \\
 & \mathcal{Y}_S \geq 0
 \end{array}$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

Algorithm 1 FirstTry

- 1: $\gamma \leftarrow 0$
- 2: $F \leftarrow \emptyset$
- 3: **while** not all s_i-t_i pairs connected in F **do**
- 4: Let C be some connected component of (V, F) such that $|C \cap \{s_i, t_i\}| = 1$ for some i .
- 5: Increase γ_C until there is an edge $e' \in \delta(C)$ s.t.
 $\sum_{S \in S_i: e' \in \delta(S)} \gamma_S = c_{e'}$
- 6: $F \leftarrow F \cup \{e'\}$
- 7: **return** $\bigcup_i P_i$

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If we show that $y_S > 0$ implies that $|\delta(S) \cap F| \leq \alpha$ we are in good shape.

However, this is not true:

- ▶ Take a complete graph on $k + 1$ vertices v_0, v_1, \dots, v_k .

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However, this is not true:

- ▶ Take a complete graph on $k + 1$ vertices v_0, v_1, \dots, v_k .
- ▶ The i -th pair is $v_0 - v_i$.

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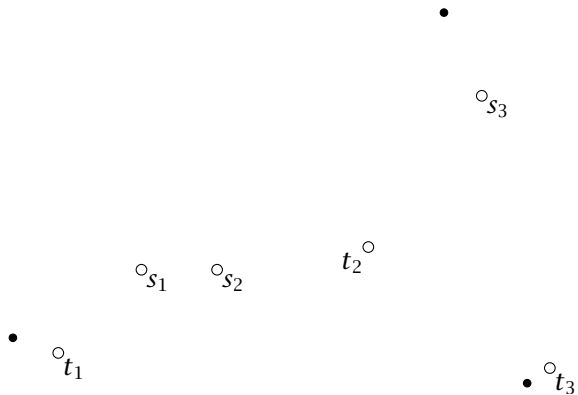
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- ▶ The final set F contains all edges $\{v_0, v_i\}$, $i = 1, \dots, k$.
- ▶ $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\}) \cap F| = k$.

Algorithm 1 SecondTry

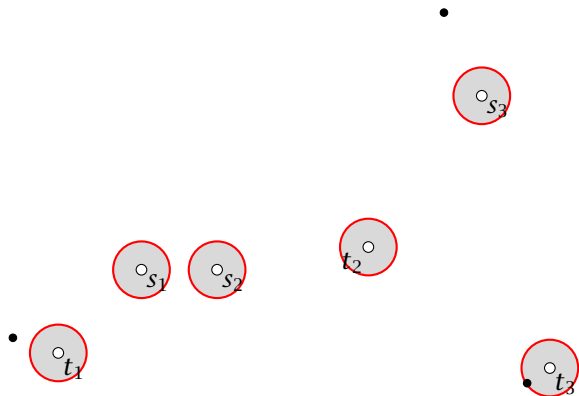
```
1:  $\gamma \leftarrow 0; F \leftarrow \emptyset; \ell \leftarrow 0$ 
2: while not all  $s_i-t_i$  pairs connected in  $F$  do
3:    $\ell \leftarrow \ell + 1$ 
4:   Let  $\mathfrak{C}$  be set of all connected components  $C$  of  $(V, F)$ 
      such that  $|C \cap \{s_i, t_i\}| = 1$  for some  $i$ .
5:   Increase  $\gamma_C$  for all  $C \in \mathfrak{C}$  uniformly until for some edge
       $e_\ell \in \delta(C')$ ,  $C' \in \mathfrak{C}$  s.t.  $\sum_{S: e_\ell \in \delta(S)} \gamma_S = c_{e_\ell}$ 
6:    $F \leftarrow F \cup \{e_\ell\}$ 
7:  $F' \leftarrow F$ 
8: for  $k \leftarrow \ell$  downto 1 do // reverse deletion
9:   if  $F' - e_k$  is feasible solution then
10:     remove  $e_k$  from  $F'$ 
11: return  $F'$ 
```


The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

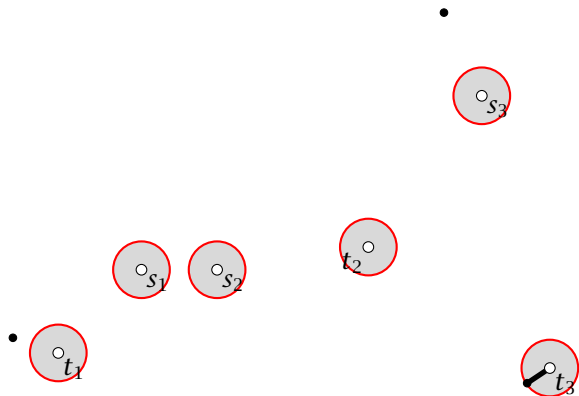
Example



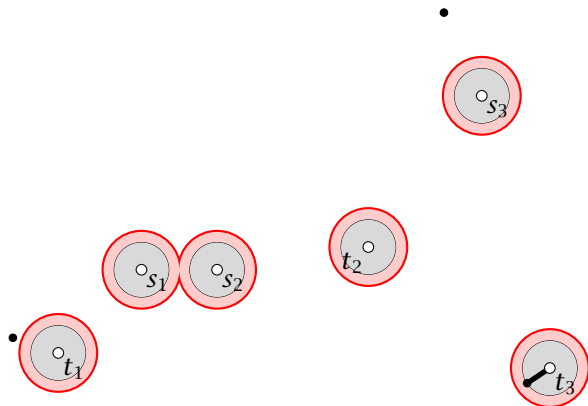
Example



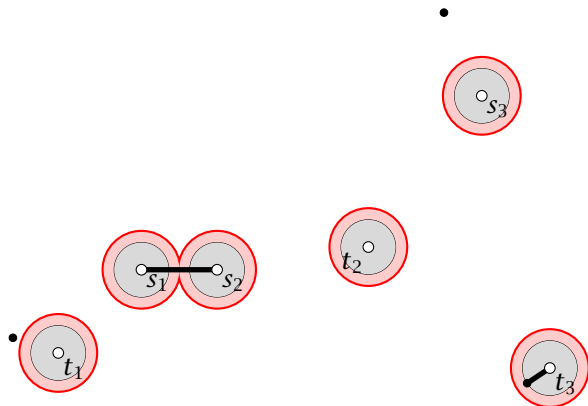
Example



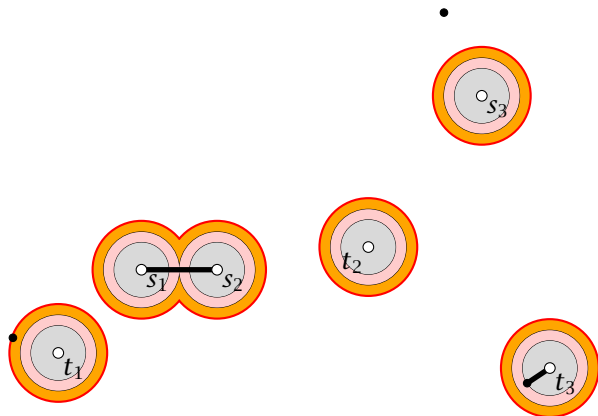
Example



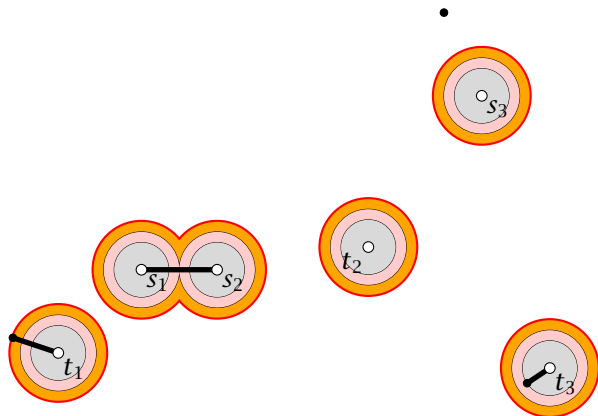
Example



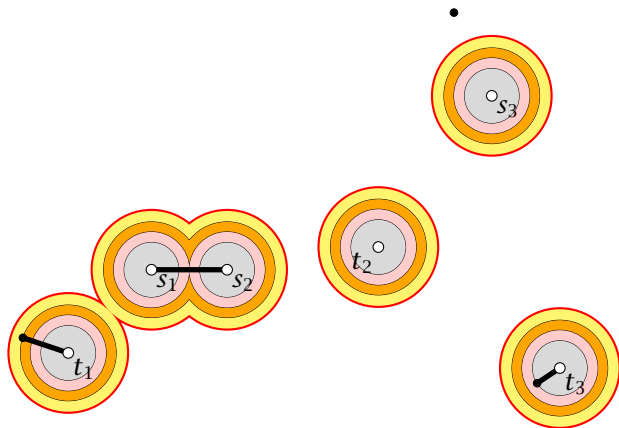
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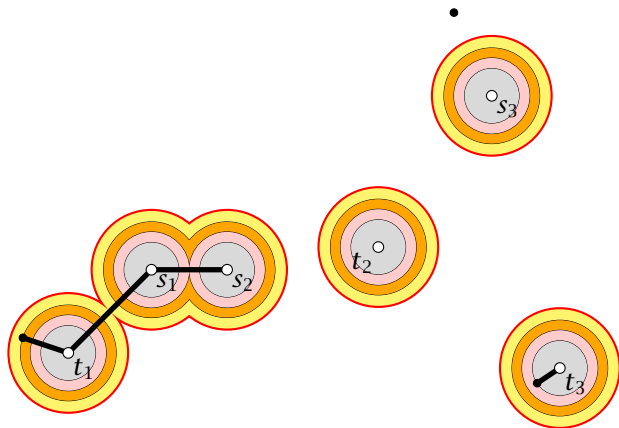
Example



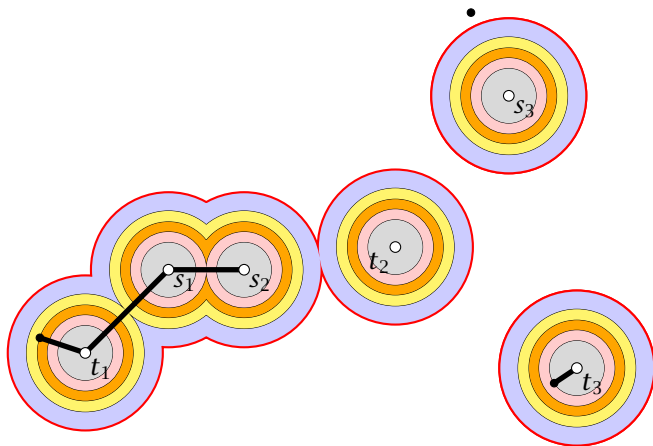
Example



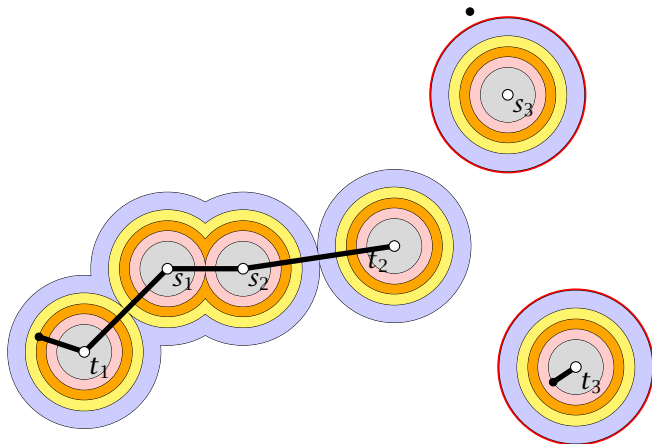
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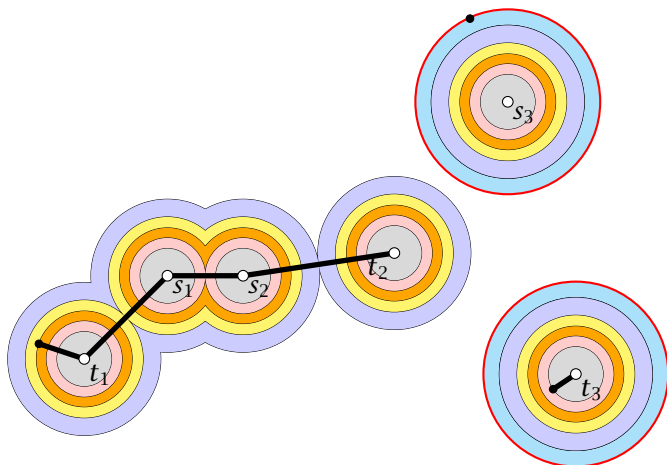
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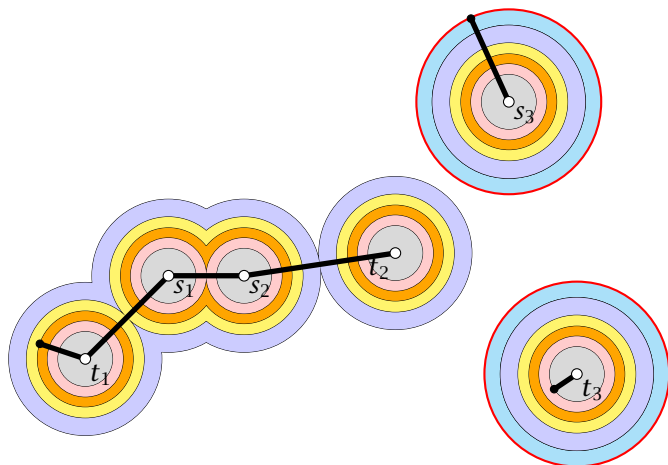
Example



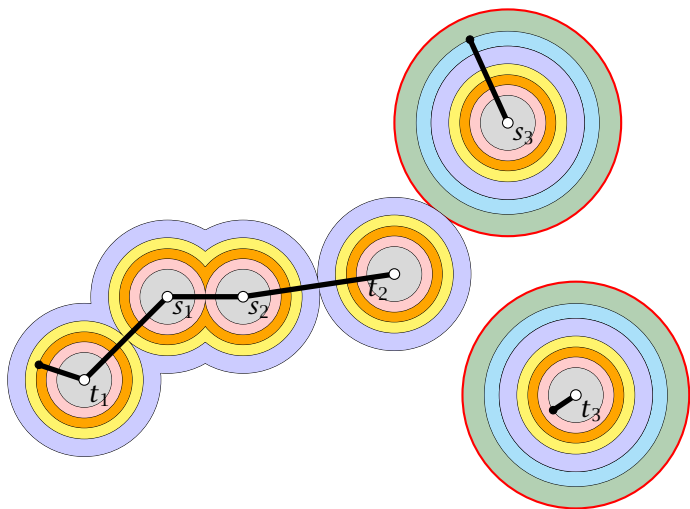
Example



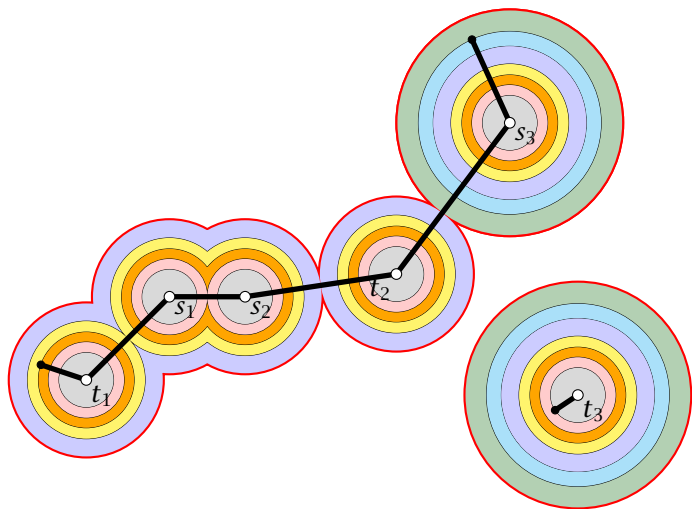
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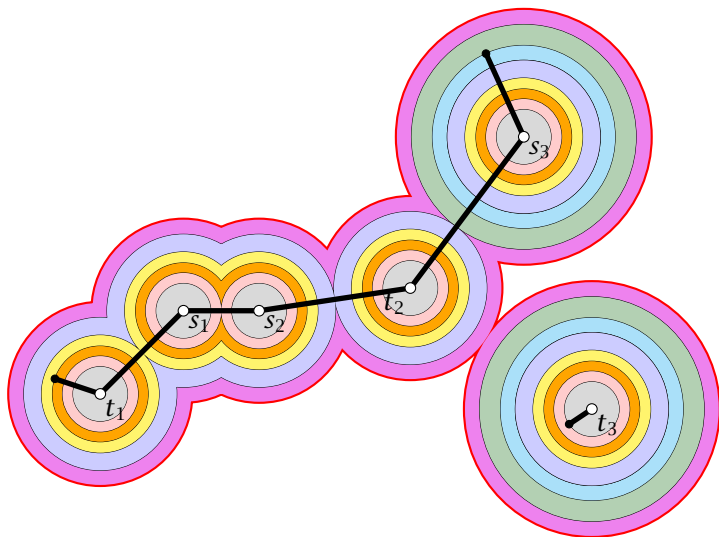
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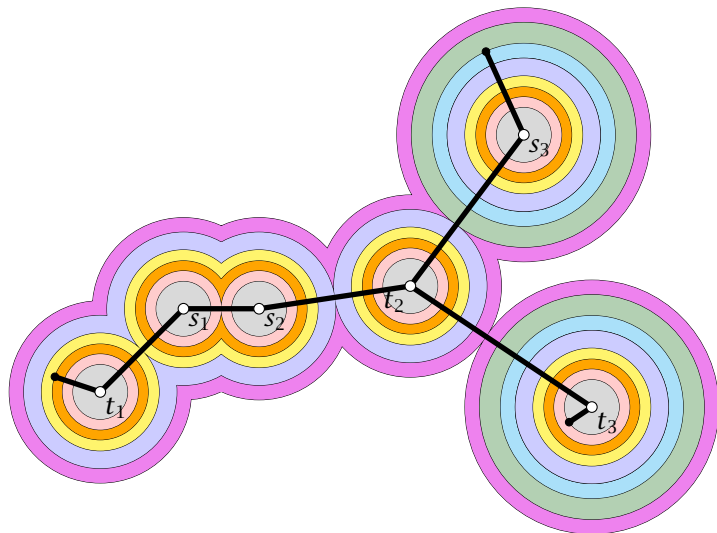
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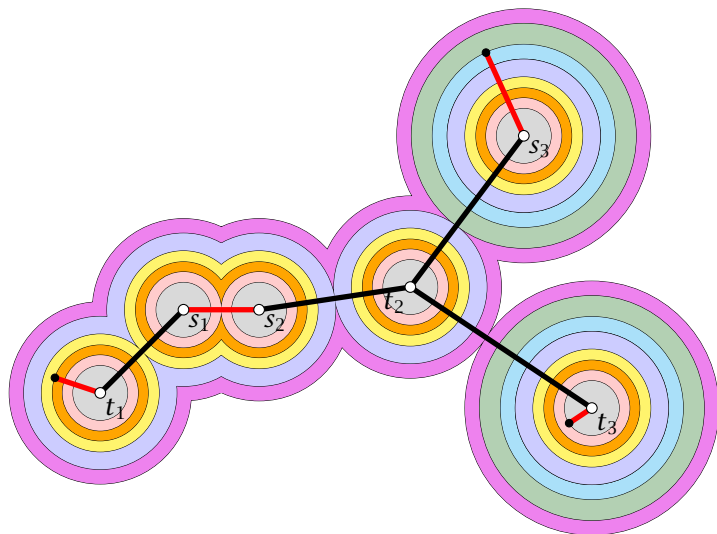
Example



Example



Example



Lemma 4

For any \mathcal{C} in any iteration of the algorithm

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|$$

This means that the number of times a moat from \mathcal{C} is crossed in the final solution is at most twice the number of moats.

Proof: later...

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} \gamma_S = \sum_S |F' \cap \delta(S)| \cdot \gamma_S .$$

We want to show that

$$\sum_S |F' \cap \delta(S)| \cdot \gamma_S \leq 2 \sum_S \gamma_S$$

At the i -th iteration the increase of the left-hand side is

$\sum_{S \in \mathcal{S}_i} |F' \cap \delta(S)| \cdot \gamma_S$ and the increase of the right hand side is $2 \sum_{S \in \mathcal{S}_i} \gamma_S$.

By the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

Since, by the previous lemma, the inequality holds after the iteration if it holds in the beginning of the iteration, it holds in the beginning of the iteration.

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by induction over the increase of the left-hand side. In

the base case, $F' = \emptyset$, the inequality holds trivially.

Assume that the inequality holds for a certain F' . If we

add $\{e\}$ to F' , the increase of the right hand side is $2\gamma_S$.

Since, by the previous lemma, the inequality holds after the
removal of e , it holds in the beginning of the induction.

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Since the left hand side is the increase of the left hand side of

$\sum_S |F \cap \delta(S)| \cdot \gamma_S$ when we add F' to F ,

and since the increase of the right hand side is $2 \sum_S \gamma_S$,

we conclude that the inequality holds if it holds for F .

Since, by the previous lemma, the inequality holds also for $F \cup F'$,

we conclude that it holds in the beginning of the proof. \square

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- ▶ In the i -th iteration the increase of the left-hand side is

$$\epsilon \sum_{C \in \mathcal{C}} |F' \cap \delta(C)|$$

and the increase of the right hand side is $2\epsilon|\mathcal{C}|$.

- ▶ Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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- ▶ Fix iteration i . Let F_i be the set of edges in F at the beginning of the iteration.
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- ▶ All edges in H are necessary for the solution.

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- ▶ Contract all edges in F_i into single vertices V' .
- ▶ We can consider the forest H on the set of vertices V' .
- ▶ Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- ▶ Color a vertex $v \in V'$ **red** if it corresponds to a component from \mathbb{C} (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
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 - ▶ But this means that the cluster corresponding to b must separate a source-target pair.

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 - ▶ Suppose not. The single edge connecting $b \in B$ comes from H , and, hence, is necessary.
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 - ▶ But then it must be a red node.