

# Overview: Shortest Augmenting Paths

## Lemma 1

*The length of the shortest augmenting path never decreases.*

## Lemma 2

*After at most  $\mathcal{O}(m)$  augmentations, the length of the shortest augmenting path strictly increases.*

# Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

## Theorem 3

*The shortest augmenting path algorithm performs at most  $\mathcal{O}(mn)$  augmentations. This gives a running time of  $\mathcal{O}(m^2n)$ .*

## Proof.

- ▶ We can find the shortest augmenting paths in time  $\mathcal{O}(m)$  via BFS.
- ▶  $\mathcal{O}(m)$  augmentations for paths of exactly  $k < n$  edges.

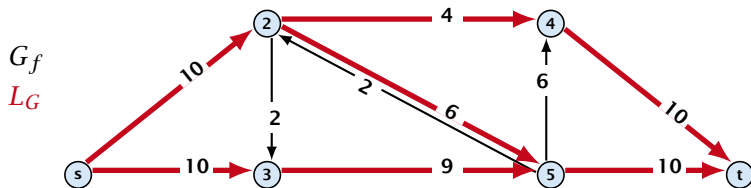


# Shortest Augmenting Paths

Define the level  $\ell(v)$  of a node as the length of the shortest  $s$ - $v$  path in  $G_f$ .

Let  $L_G$  denote the **subgraph** of the residual graph  $G_f$  that contains only those edges  $(u, v)$  with  $\ell(v) = \ell(u) + 1$ .

A path  $P$  is a shortest  $s$ - $t$  path in  $G_f$  if it is an  $s$ - $t$  path in  $L_G$ .



In the following we assume that the residual graph  $G_f$  does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

# Shortest Augmenting Path

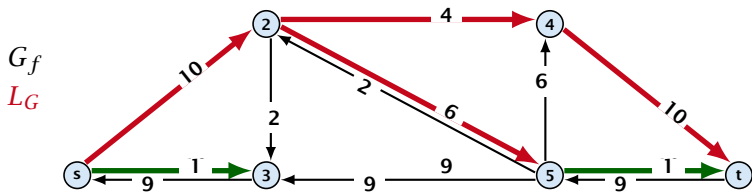
## First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation  $G_f$  changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between  $s$  and  $t$ .



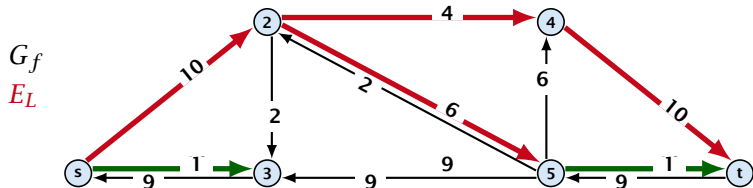
## Shortest Augmenting Path

**Second Lemma:** After at most  $m$  augmentations the length of the shortest augmenting path strictly increases.

Let  $E_L$  denote the set of edges in graph  $L_G$  at the beginning of a round when the distance between  $s$  and  $t$  is  $k$ .

An  $s$ - $t$  path in  $G_f$  that uses edges not in  $E_L$  has length larger than  $k$ , even when considering edges added to  $G_f$  during the round.

In each augmentation one edge is deleted from  $E_L$ .



# Shortest Augmenting Paths

## Theorem 4

*The shortest augmenting path algorithm performs at most  $\mathcal{O}(mn)$  augmentations. Each augmentation can be performed in time  $\mathcal{O}(m)$ .*

## Theorem 5 (without proof)

*There exist networks with  $m = \Theta(n^2)$  that require  $\mathcal{O}(mn)$  augmentations, when we restrict ourselves to only augment along shortest augmenting paths.*

### Note:

There always exists a set of  $m$  augmentations that gives a maximum flow (why?).

# Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to  $\mathcal{O}(mn^2)$  by improving the running time for finding an augmenting path (currently we assume  $\mathcal{O}(m)$  per augmentation for this).



# Shortest Augmenting Paths

We maintain a subset  $E_L$  of the edges of  $G_f$  with the guarantee that a shortest  $s-t$  path using only edges from  $E_L$  is a shortest augmenting path.

With each augmentation some edges are deleted from  $E_L$ .

When  $E_L$  does not contain an  $s-t$  path anymore the distance between  $s$  and  $t$  strictly increases.

Note that  $E_L$  is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between  $s$  and  $t$  in  $G_f$  is  $k$ .

$E_L$  is initialized as the level graph  $L_G$ .

Perform a **DFS search** to find a path from  $s$  to  $t$  using edges from  $E_L$ .

Either you find  $t$  after at most  $n$  steps, or you end at a node  $v$  that does not have any outgoing edges.

You can delete incoming edges of  $v$  from  $E_L$ .

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between  $s$  and  $t$  strictly increases.

Initializing  $E_L$  for the phase takes time  $\mathcal{O}(m)$ .

The total cost for searching for augmenting paths during a phase is at most  $\mathcal{O}(mn)$ , since every search (successful (i.e., reaching  $t$ ) or unsuccessful) decreases the number of edges in  $E_L$  and takes time  $\mathcal{O}(n)$ .

The total cost for performing an augmentation **during** a phase is only  $\mathcal{O}(n)$ . For every edge in the augmenting path one has to update the residual graph  $G_f$  and has to check whether the edge is still in  $E_L$  for the next search.

There are at most  $n$  phases. Hence, total cost is  $\mathcal{O}(mn^2)$ .