7.2 Red Black Trees

Definition 1

A red black tree is a balanced binary search tree in which each
internal node has two children. Each internal node has a color,
such that

1. The root is black.

2. All leaf nodes are black.

3. For each node, all paths to descendant leaves contain the
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The null-pointers in a binary search tree are replaced by pointers
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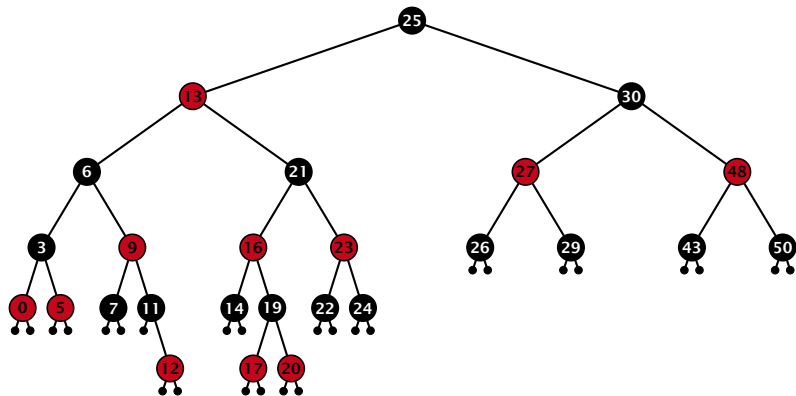
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Red Black Trees: Example

7.2 Red Black Trees

Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 3

The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 4

A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

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Induction on the height of v.

base case (height(v) = 0)

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▶ If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

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Proof (cont.)

7.2 Red Black Trees

Proof (cont.)

induction step

▶ Supose v is a node with $\text{height}(v) > 0$.

▶ v has two children with strictly smaller height.

▶ These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.

▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.

▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

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Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

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We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:

Red Black Trees: Insert

Insert:

▶ first make a normal insert into a binary search tree
▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

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Invariant of the fix-up algorithm:

▶ z is a red node

▶ the black-height property is fulfilled at every node

▶ the only violation of red-black properties occurs at z and parent[z]

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

 1: **while** parent[z] ≠ null **and** col[parent[z]] = red **do**
 2:     **if** parent[z] = left[gp[z]] **then**
 3:         *uncle* ← right[grandparent[z]]
 4:         **if** col[*uncle*] = red **then**
 5:             col[p[z]] ← black; col[u] ← black;
 6:             col[gp[z]] ← red; z ← grandparent[z];
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 8:             **if** z = right[parent[z]] **then**
 9:                 z ← p[z]; LeftRotate(z);
10:             col[p[z]] ← black; col[gp[z]] ← red;
11:             RightRotate(gp[z]);
12:     **else** same as then-clause but right and left exchanged
13: col(root[T]) ← black;

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3:          *uncle* ← right[grandparent[z]]
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8:             **if** z = right[parent[z]] **then**   2a: z right child
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13: col(root[T]) ← black;

Case 1: Red Uncle
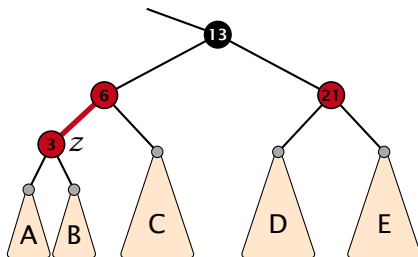
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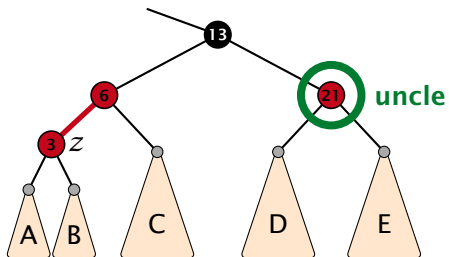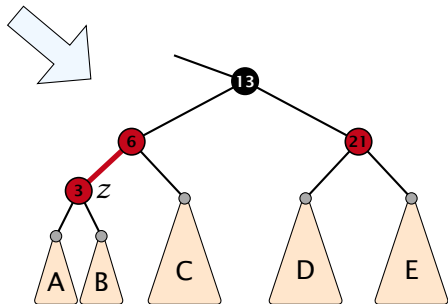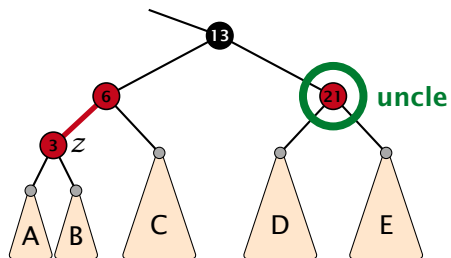
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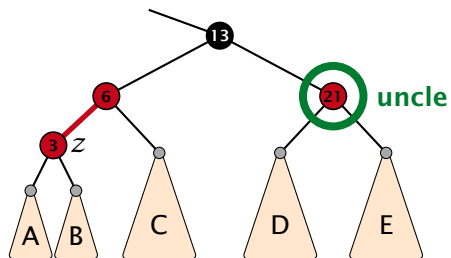
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1. recolour
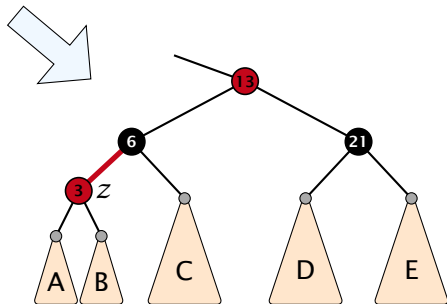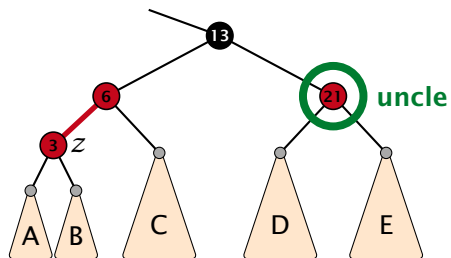
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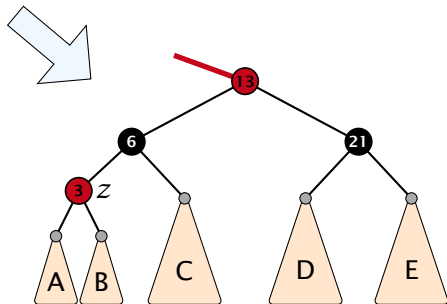
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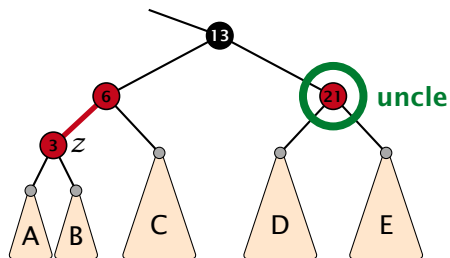
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1. recolour
2. move *z* to grand-parent

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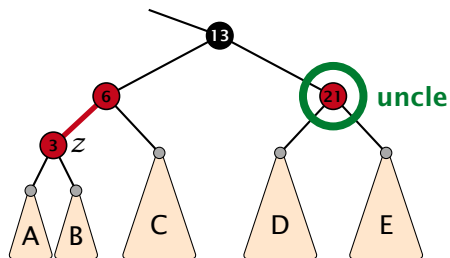
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2. move z to grand-parent
3. invariant is fulfilled for new z

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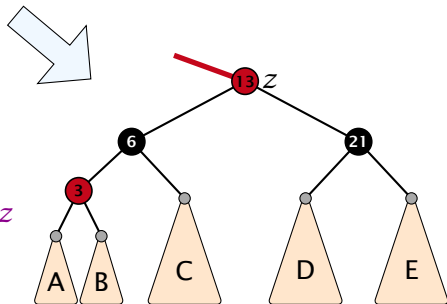
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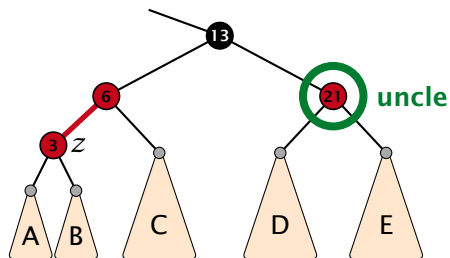
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that
   black height property holds
3. you have a red black tree

Case 2b: Black uncle and *z* is left child

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Case 2a: Black uncle and *z* is right child

1. rotate around parent
2. move *z* downwards
3. you have Case 2b.

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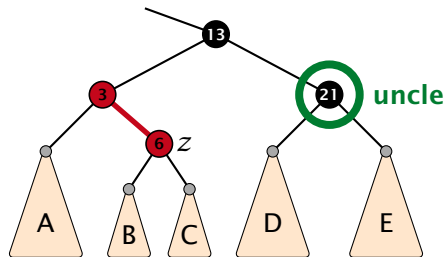
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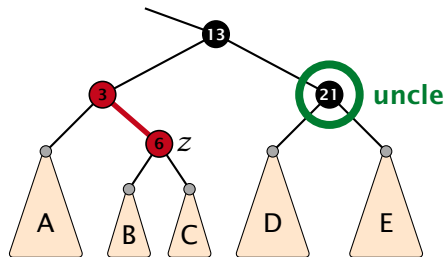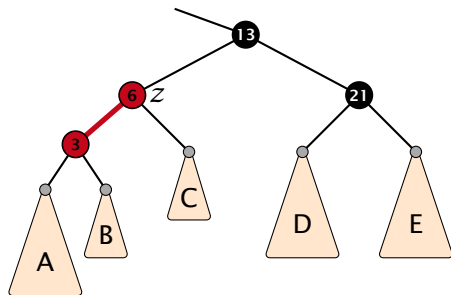
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Red Black Trees: Insert

Running time:

▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

▶ Case 2a → Case 2b → red-black tree

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Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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Red Black Trees: Delete
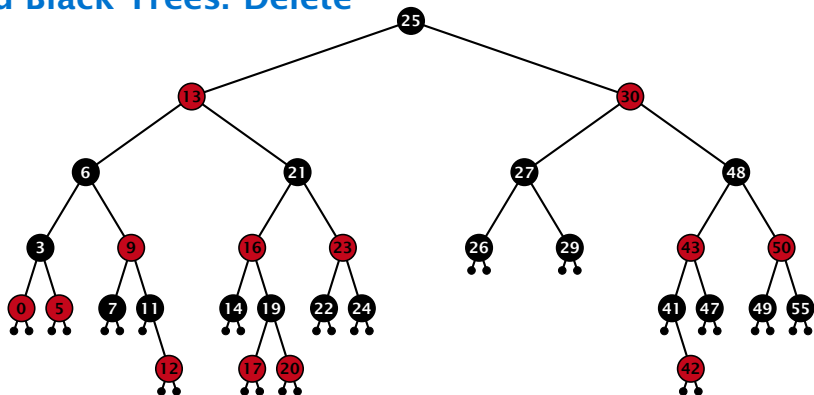
Red Black Trees: Delete

First do a standard delete.

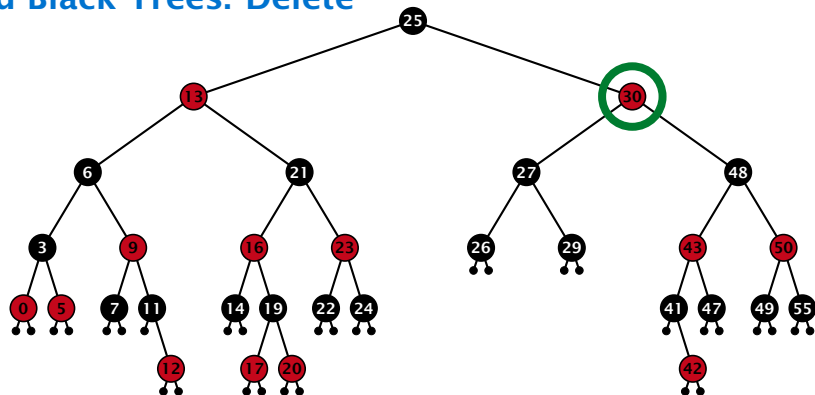If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- The parent and child of x were red; now adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a (deleted) leaf descendant of x changes the number of black nodes. Black height property might be violated.

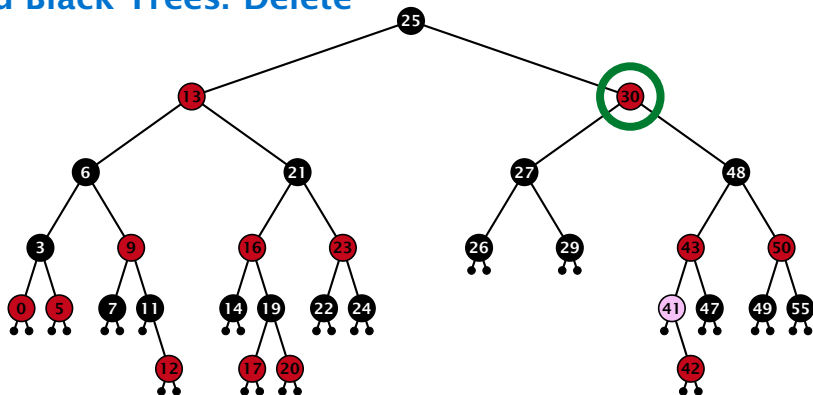# Red Black Trees: Delete

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Case 3:

Element has two children

  ▶ do normal delete

  ▶ when replacing content by content of successor, don't change color of node

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Delete:
- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

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▶ the node z is black

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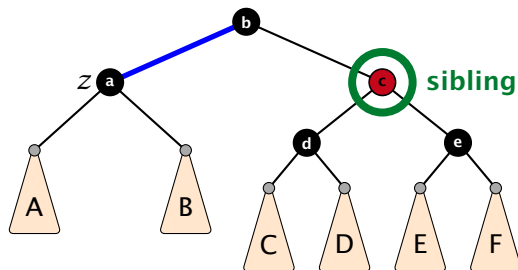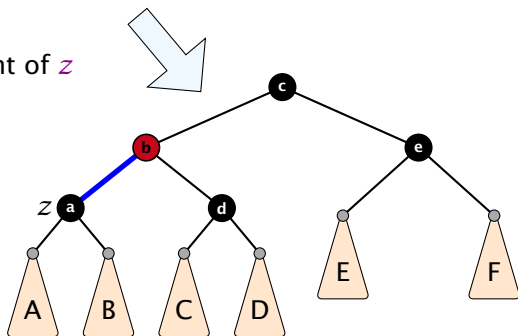
Case 1: Sibling of *z* is red

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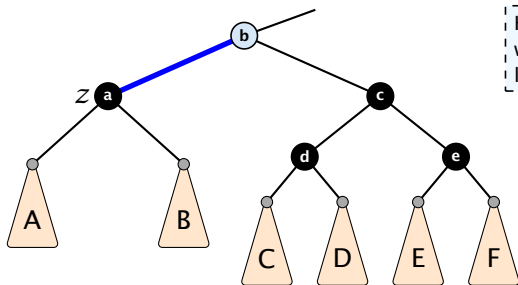
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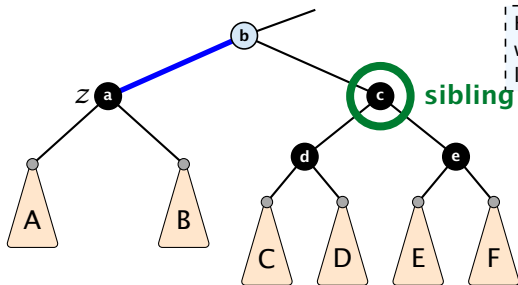   or Case 3, or Case 4

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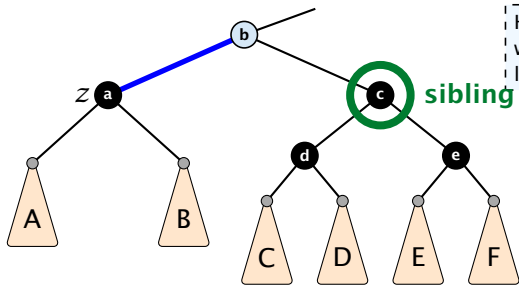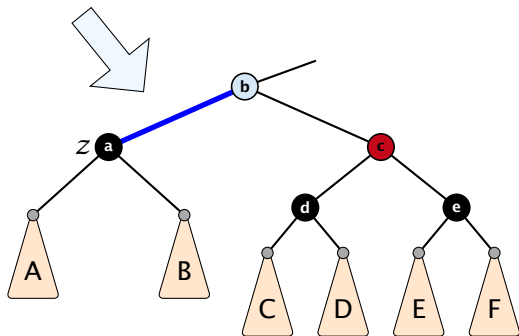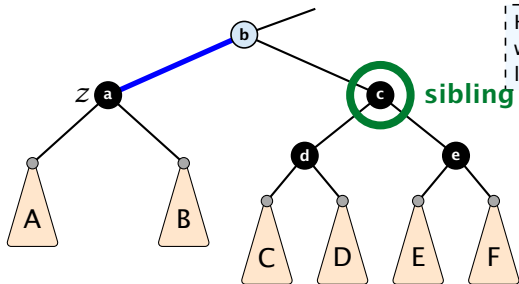# Case 2: Sibling is black with two black children

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
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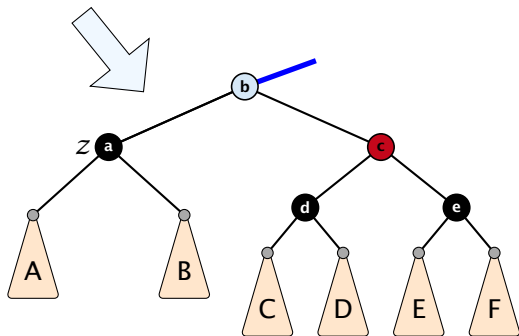5. if b is red we color it black and are done

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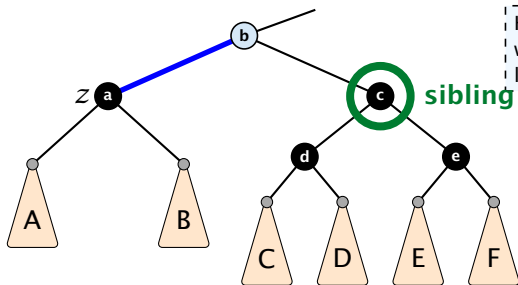# Case 2: Sibling is black with two black children

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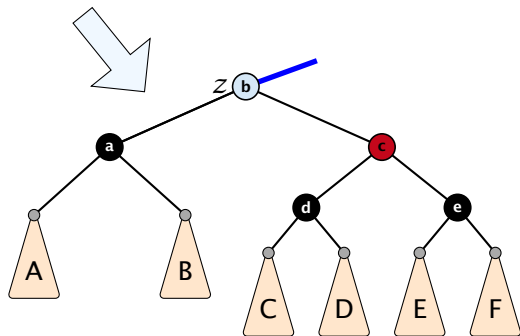5. if b is red we color it black and are done

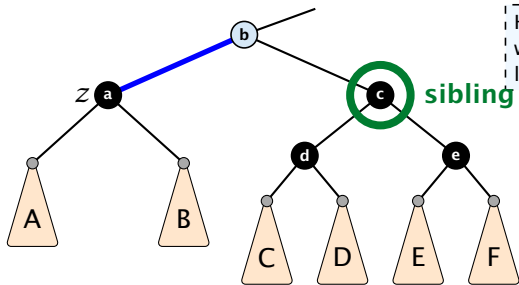# Case 2: Sibling is black with two black children

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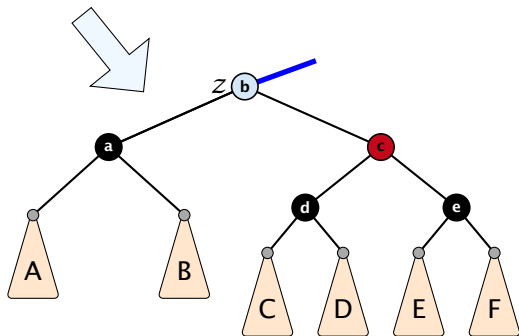5. if b is red we color it black and are done

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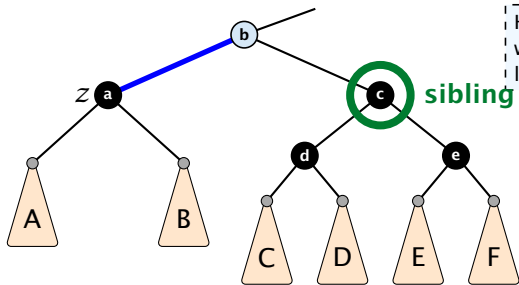# Case 2: Sibling is black with two black children

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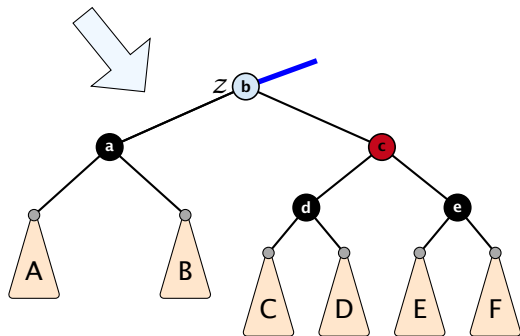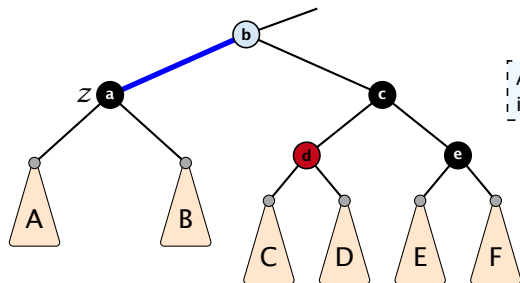5. if b is red we color it black and are done

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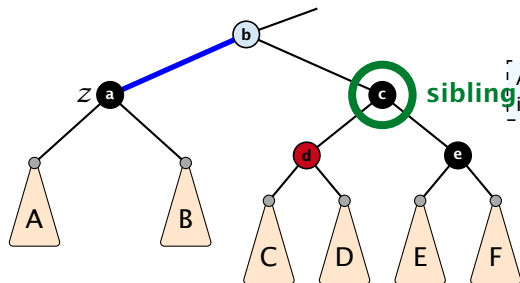# Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
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Again the blue color of b indicates that it can either be black or red.

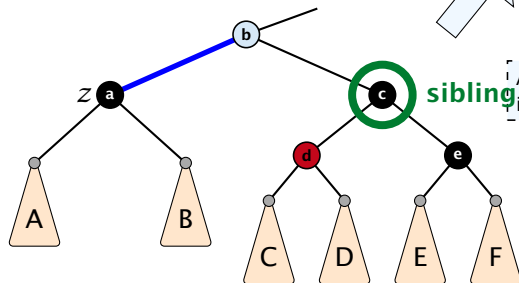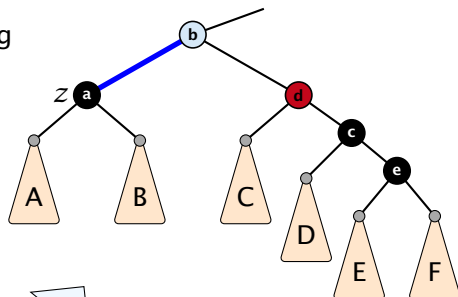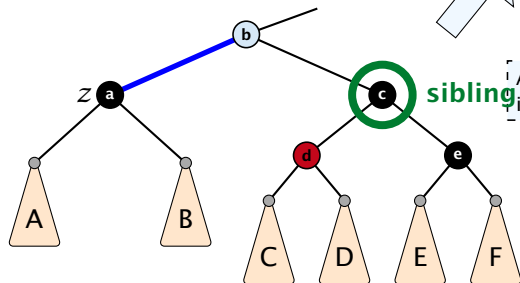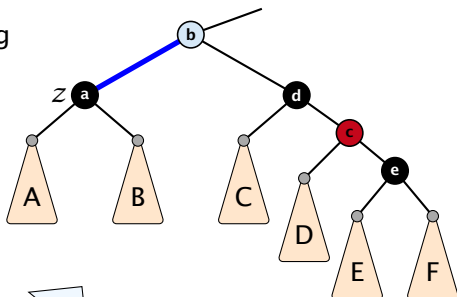# Case 3: Sibling black with one black child to the right

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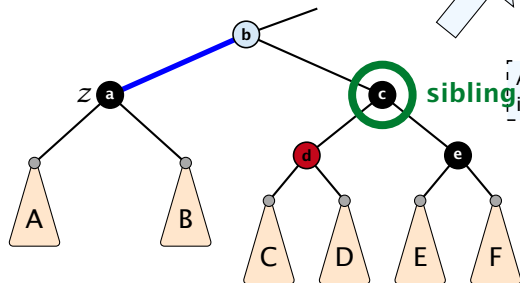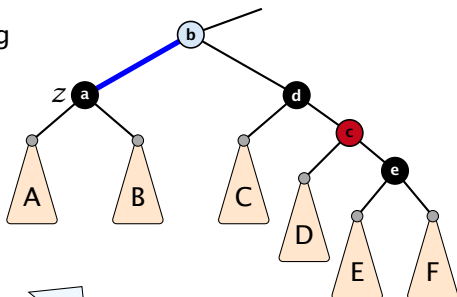# Case 3: Sibling black with one black child to the right

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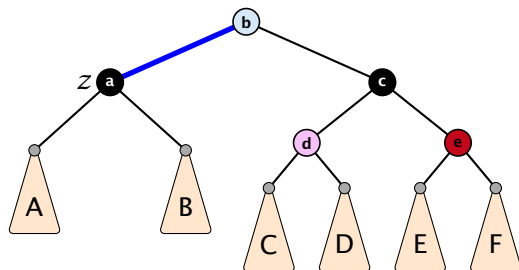
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Case 4: Sibling is black with red right child

- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree
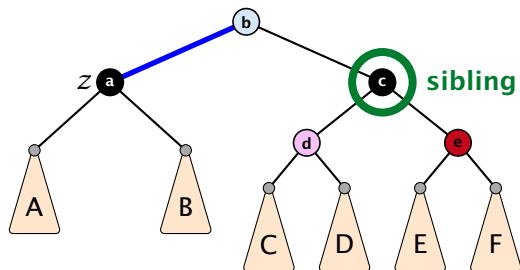
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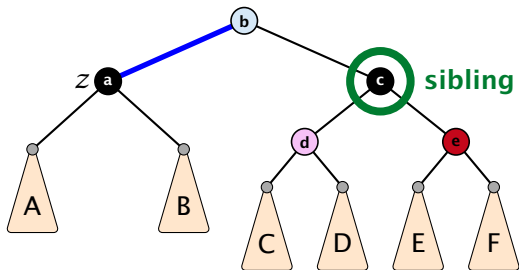
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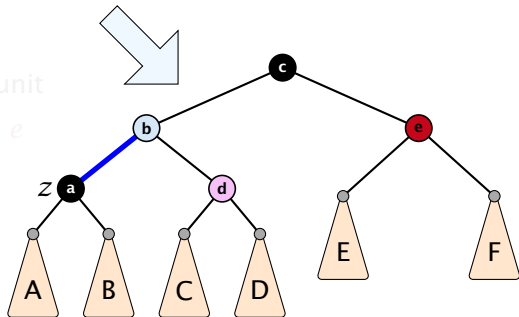3. recolor nodes *b*, *c*, and *e*
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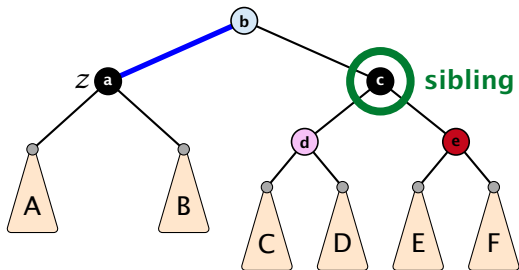- Here b and d are either red or black but have possibly different colors.
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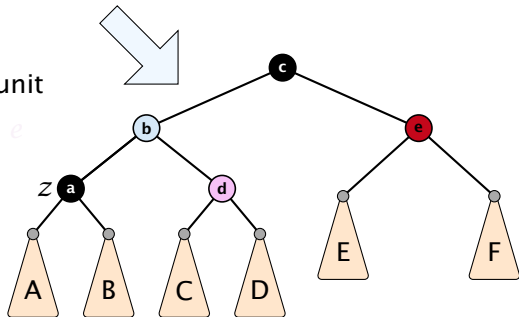3. recolor nodes b, c, and e
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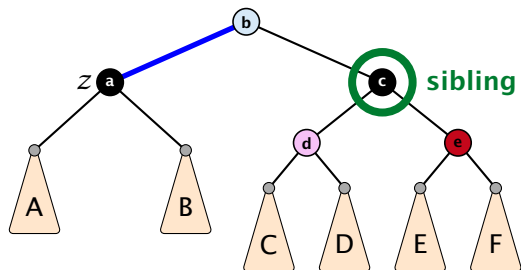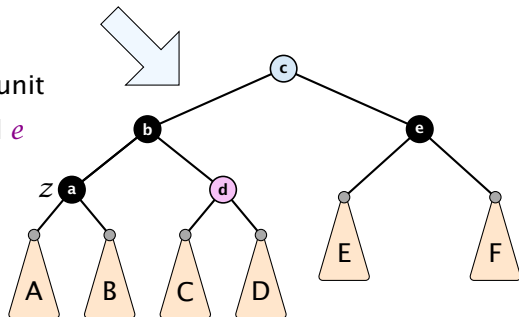
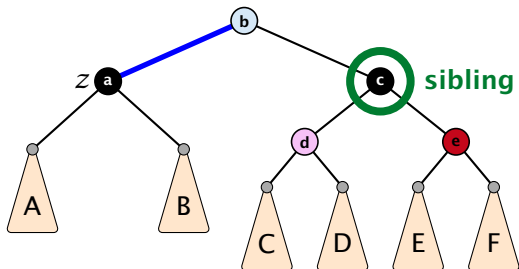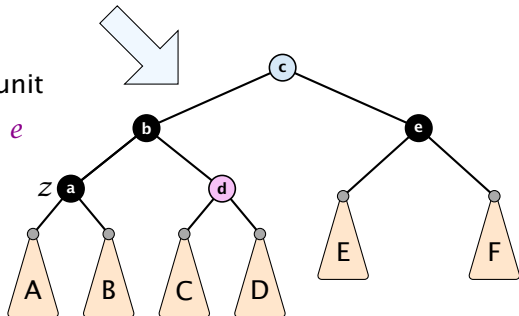

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Running time:

▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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