13.3 Highest Label

Algorithm 6 highest-label(G, s, t)

1: initialize preflow
2: **foreach** $u \in V \setminus \{s, t\}$ **do**
3: $\quad u.\textit{current-neighbour} \leftarrow u.\textit{neighbour-list-head}$
4: **while** $\exists$ active node u **do**
5: $\quad$ select active node u with highest label
6: $\quad$ discharge(u)

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Lemma 1
When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a non-saturating push from u a relabel is required to make u active again.

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#\textit{relabels} + 1) = \mathcal{O}(n^3)$.

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Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?

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Maintain lists L_i, $i \in \{0, \dots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \dots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

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Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#non\text{-}saturating\text{-}pushes\text{-}to\text{-}s\text{-}or\text{-}t)$$

Lemma 2
The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 3
The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

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Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.