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Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.

Capacity Scaling

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- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

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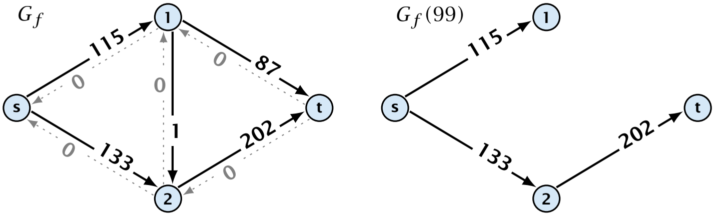

Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path P in $G_f(\Delta)$ **do**
        $f \leftarrow$ augment(f, c, P)
        update$(G_f(\Delta))$
    $\Delta \leftarrow \Delta/2$
return f

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- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

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- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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Theorem 4

We need $\mathcal{O}(m \log C)$ *augmentations. The algorithm can be implemented in time* $\mathcal{O}(m^2 \log C)$.