**How to choose augmenting paths?**

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

**Several possibilities:**

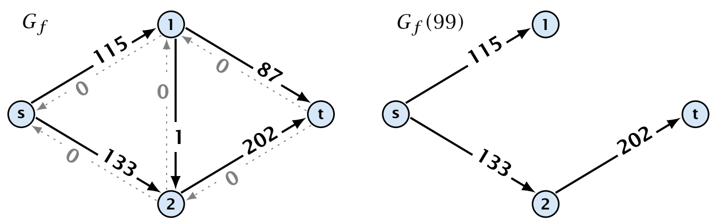
- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.



## Capacity Scaling

**Intuition:**

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter $\Delta$.
- $G_f(\Delta)$ is a sub-graph of the residual graph $G_f$ that contains only edges with capacity at least $\Delta$.






## Capacity Scaling

**Algorithm 2** maxflow$(G, s, t, c)$

**foreach** $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
**while** $\Delta \geq 1$ **do**
  $G_f(\Delta) \leftarrow \Delta$-residual graph
  **while** there is augmenting path $P$ in $G_f(\Delta)$ **do**
    $f \leftarrow$ augment$(f, c, P)$
    update$(G_f(\Delta))$
  $\Delta \leftarrow \Delta/2$
**return** $f$



## Capacity Scaling

**Assumption:**
All capacities are integers between $1$ and $C$.

**Invariant:**
All flows and capacities are/remain integral throughout the algorithm.

**Correctness:**
The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

## Capacity Scaling

**Lemma 1**

*There are* $\lceil \log C \rceil + 1$ *iterations over* $\Delta$.

**Proof:** obvious.

**Lemma 2**

*Let* $f$ *be the flow at the end of a* $\Delta$*-phase. Then the maximum flow is smaller than* $\text{val}(f) + m\Delta$.

**Proof:** less obvious, but simple:

- There must exist an $s$-$t$ cut in $G_f(\Delta)$ of zero capacity.
- In $G_f$ this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.



## Capacity Scaling

**Lemma 3**

*There are at most* $2m$ *augmentations per scaling-phase.*

**Proof:**

- Let $f$ be the flow at the end of the previous phase.
- $\text{val}(f^*) \le \text{val}(f) + 2m\Delta$
- Each augmentation increases flow by $\Delta$.

**Theorem 4**

*We need* $\mathcal{O}(m \log C)$ *augmentations. The algorithm can be implemented in time* $\mathcal{O}(m^2 \log C)$.