9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- $\mathcal{P}.\mathbf{makeset}(x)$**:** Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.
- $\mathcal{P}.\mathbf{find}(x)$**:** Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
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Applications:

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9 Union Find

Algorithm 16 Kruskal-MST($G = (V, E), w$)

```
1: A ← ∅;
2: for all v ∈ V do
3:     v.set ← P.makeset(v.label)
4: sort edges in non-decreasing order of weight w
5: for all (u, v) ∈ E in non-decreasing order do
6:     if P.find(u.set) ≠ P.find(v.set) then
7:         A ← A ∪ {(u, v)}
8:         P.union(u.set, v.set)
```

List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.

- $\text{makeset}(x)$ can be performed in constant time.
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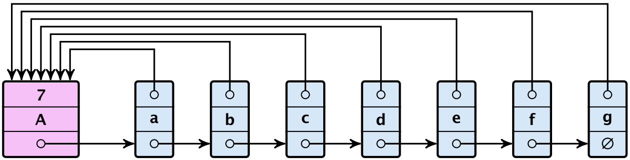

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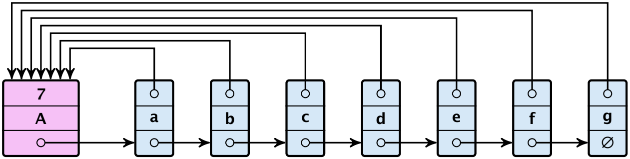

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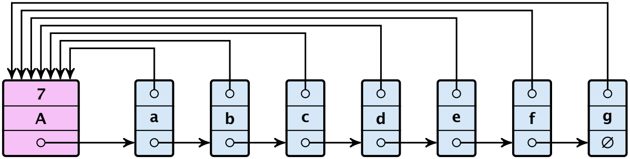

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union(x, y)

- Determine sets S_x and S_y.
- Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- Insert list S_y at the head of S_x.
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- Time: $\min\{|S_x|, |S_y|\}$.

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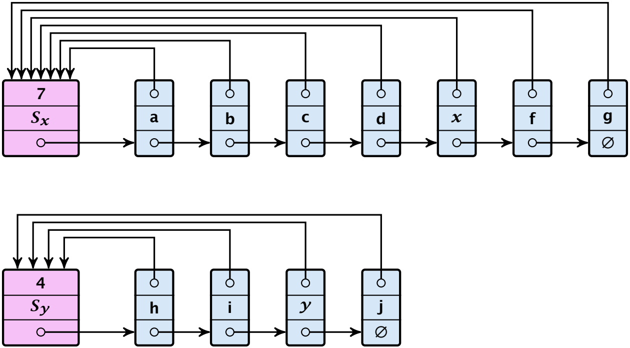

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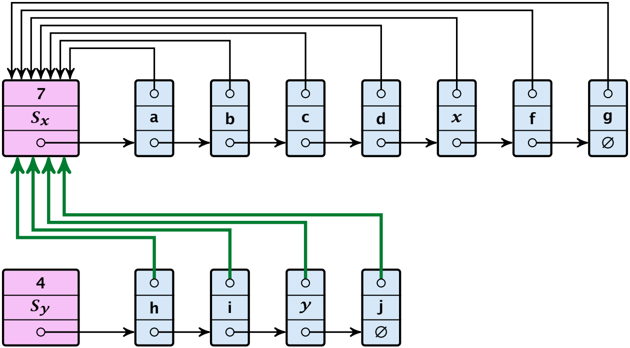

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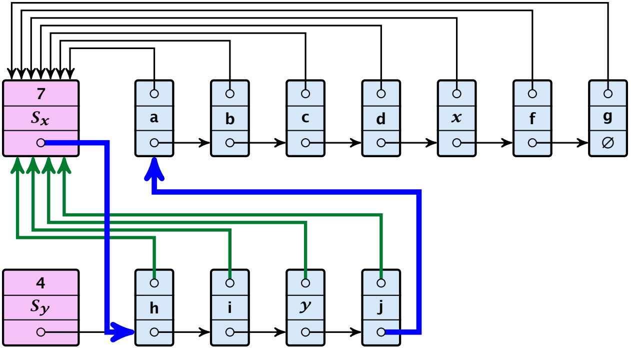

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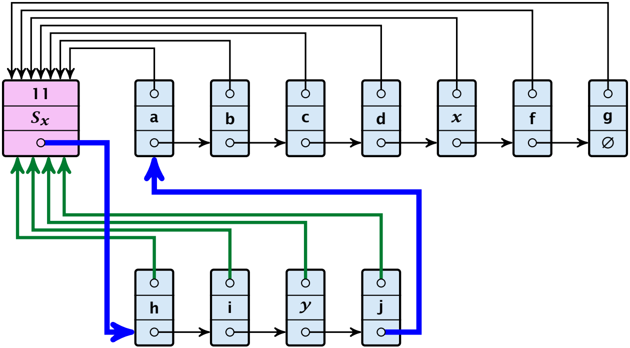

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 1

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x, y)$: $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
- Initially the balance on all accounts is zero.
- Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
- Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.

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makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

- If $S_x = S_y$ the cost is constant; no bank accounts change.
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Lemma 2

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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- Maintain nodes of a set in a tree.
- The root of the tree is the label of the set.
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- Example:

Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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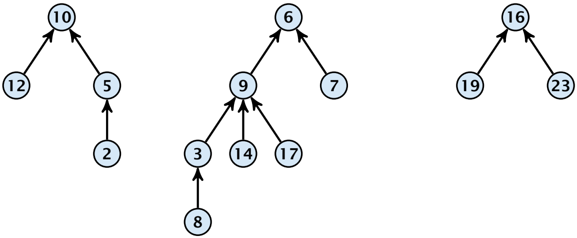

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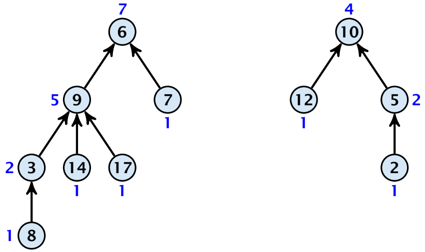

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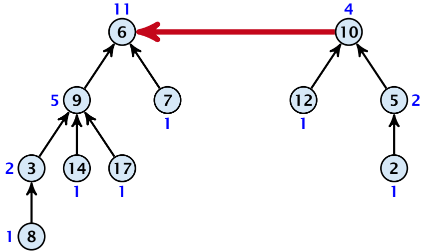

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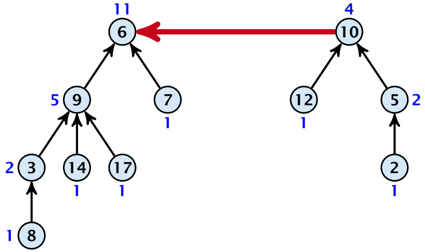

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

Implementation via Trees

Lemma 3

The running time (non-amortized!!!) for $\mathrm{find}(x)$ *is* $\mathcal{O}(\log n)$.

Proof.

- When we attach a tree with root c to become a child of a tree with root p, then $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, where size denotes the value of the size-field right after the operation.
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find(x):

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Note that the size-fields now only give an upper bound on the size of a sub-tree.

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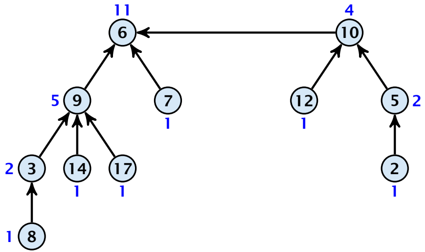

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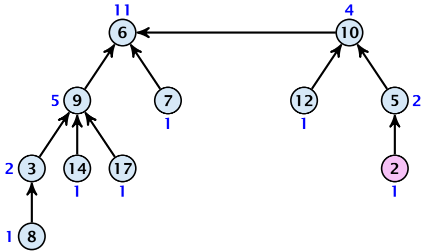

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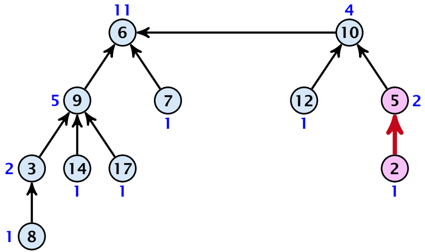

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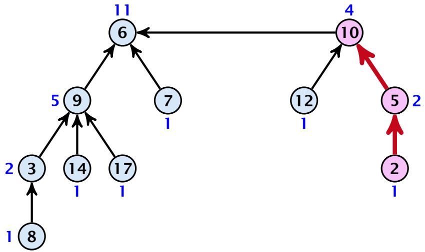

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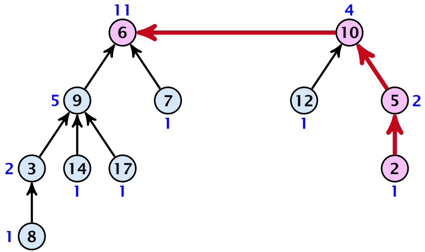

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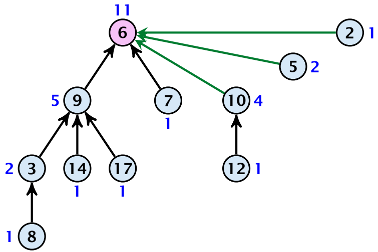

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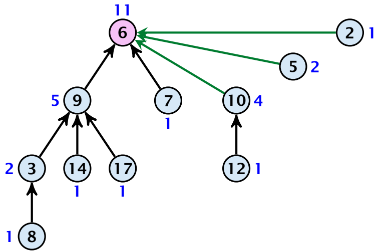

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Amortized Analysis

Definitions:

- $\mathrm{size}(v)$ = the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find operations.

- $\mathrm{rank}(v) = \lfloor \log(\mathrm{size}(v)) \rfloor$.
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The rank of a parent must be strictly larger than the rank of a child.

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- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. $\square$

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Theorem 6

Union find with path compression fulfills the following amortized running times:

- $\operatorname{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\operatorname{find}(x) : \mathcal{O}(\log^*(n))$
- $\operatorname{union}(x, y) : \mathcal{O}(\log^*(n))$

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

- A node with rank $\text{rank}(v)$ is in rank group $\log^*(\text{rank}(v))$.
- The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.
- A rank group $g \geq 1$ contains ranks $\text{tow}(g-1)+1, \dots, \text{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
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Accounting Scheme:

- create an account for every find-operation
- create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
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This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x-1, 1) & \text{if } y = 0 \\ A(x-1, A(x, y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- $A(0, y) = y + 1$
- $A(1, y) = y + 2$
- $A(2, y) = 2y + 3$
- $A(3, y) = 2^{y+3} - 3$
- $A(4, y) = \underbrace{2^{2^{2}}}_{y+3 \text{ times}} - 3$

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