A Fast Matching Algorithm

Algorithm 27 Bimatch-Hopcroft-Karp(G)

$M \leftarrow \emptyset$
repeat
    let $\mathcal{P} = \{P_1, \dots, P_k\}$ be maximal set of
    vertex-disjoint, shortest augmenting path w.r.t. M.
    $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
until $\mathcal{P} = \emptyset$
return M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis Hopcroft-Karp

Lemma 1

Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- The connected components of G are cycles and paths.
- The graph contains $k \triangleq |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. M.

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Analysis Hopcroft-Karp

- Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).
- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.
- Let P be an augmenting path in M'.

Lemma 2

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

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Analysis Hopcroft-Karp

Proof.

- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.
- Hence, the set contains at least $k+1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.
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Lemma 3

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

- If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.
- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \le k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \le |A| \le k\ell + |P| - 1$, and hence $|P| \ge \ell + 1$.

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Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Proof.

The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

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Analysis Hopcroft-Karp

Lemma 4
The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Proof.

- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|}+1) \leq \sqrt{|V|}$ additional augmentations.

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Lemma 4
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Analysis Hopcroft-Karp

Lemma 5
One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

construct a “level graph” G':

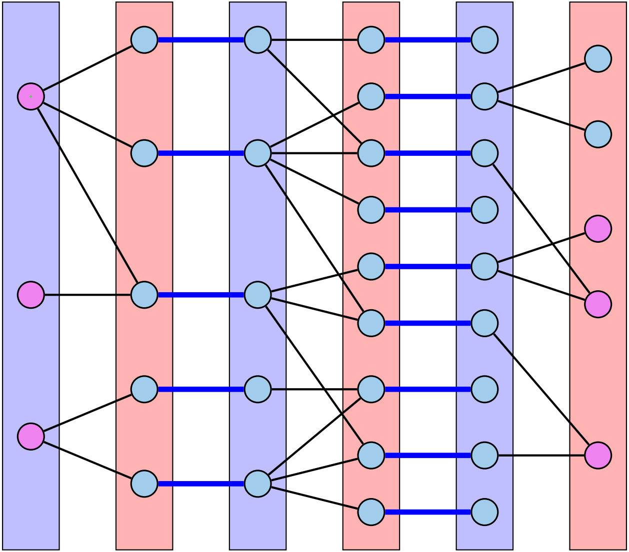
- construct Level 0 that includes all free vertices on left side L
- construct Level 1 containing all neighbors of Level 0
- construct Level 2 containing matching neighbors of Level 1
- construct Level 3 containing all neighbors of Level 2
- ...
- stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

Analysis Hopcroft-Karp

- a shortest augmenting path must go from Level 0 to the last layer constructed
- it can only use edges between layers
- construct a maxim**al** set of vertex disjoint augmenting path connecting the layers
- for this, go forward until you either reach a free vertex or you reach a "dead end" v
- if you reach a free vertex delete the augmenting path and all incident edges from the graph
- if you reach a dead end backtrack and delete v together with its incident edges

Analysis Hopcroft-Karp



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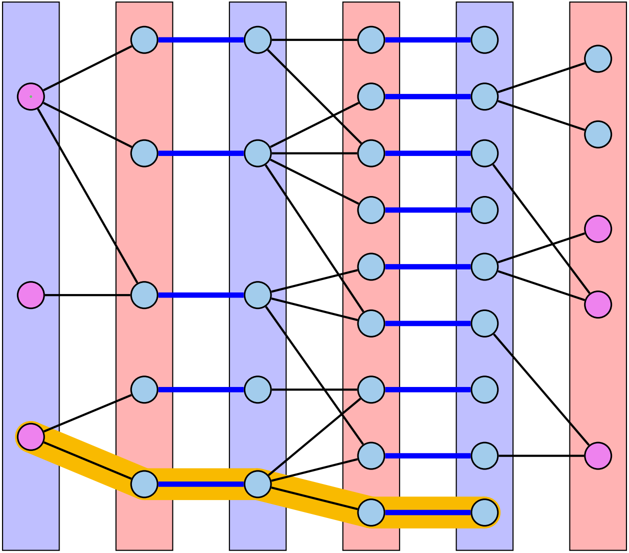

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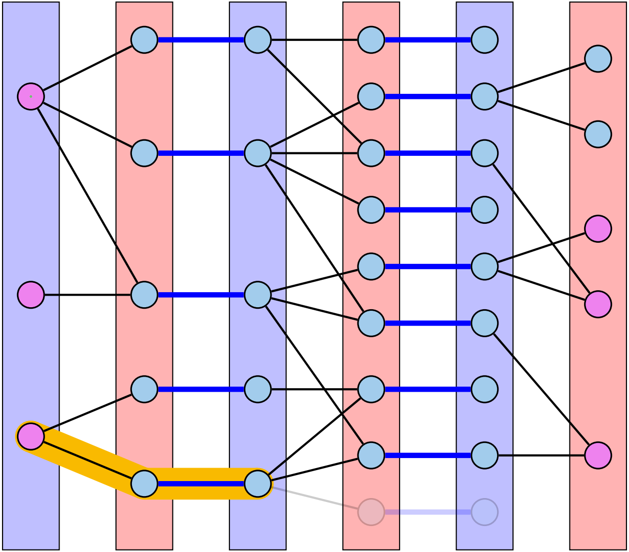

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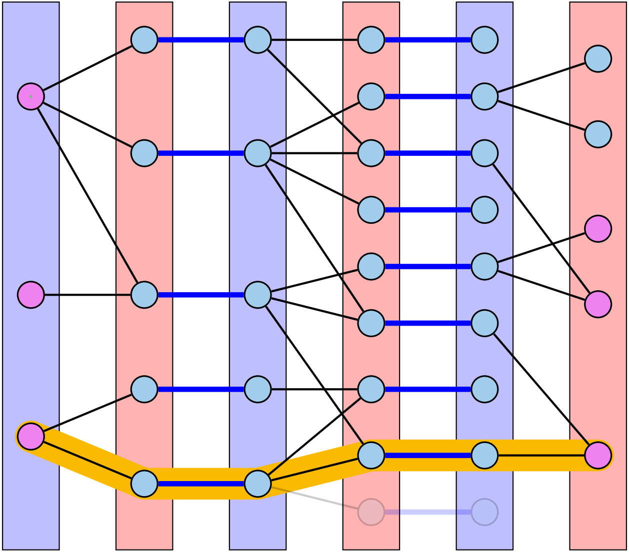

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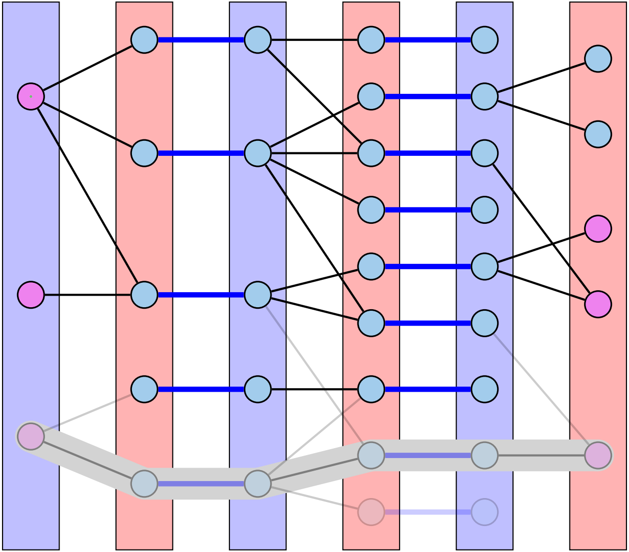

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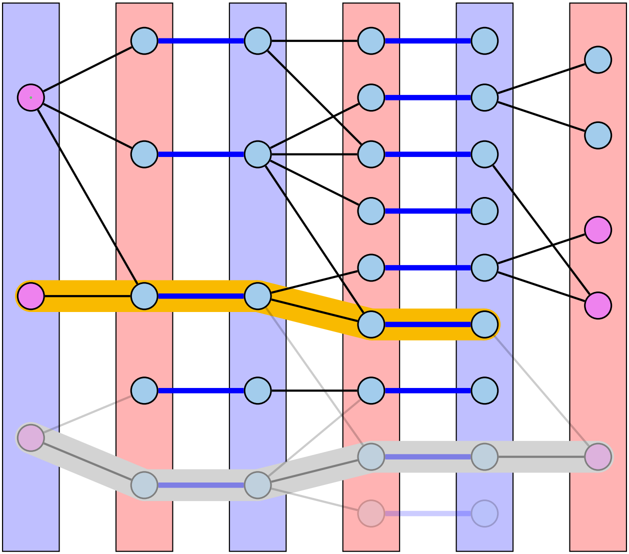

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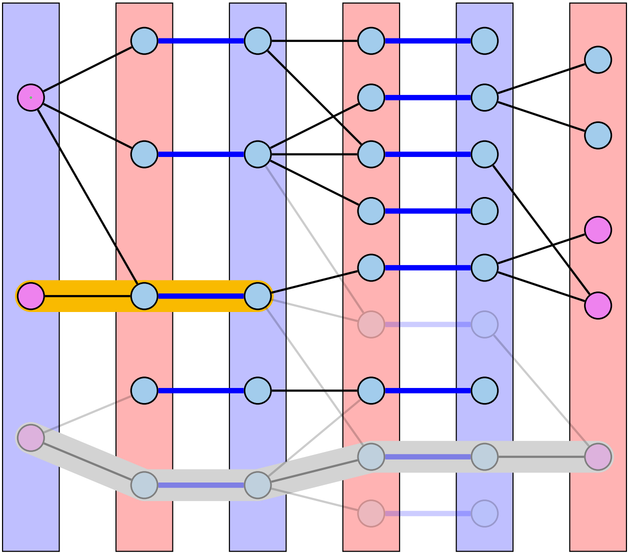

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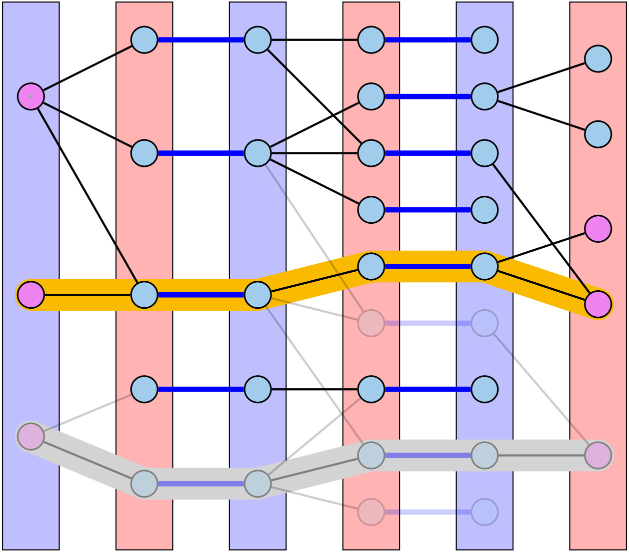

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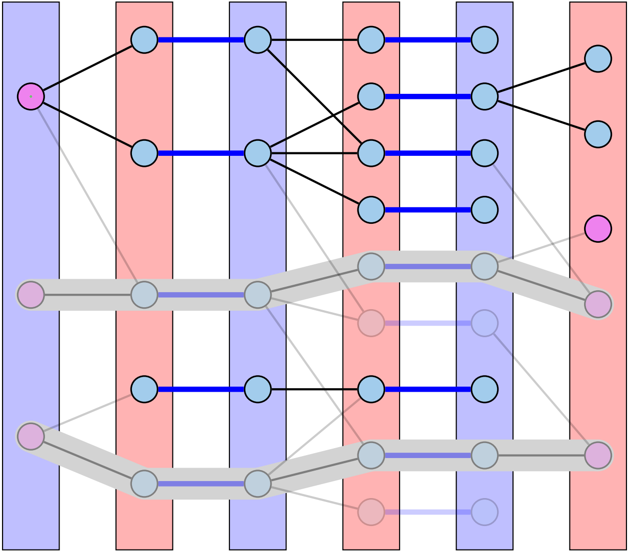

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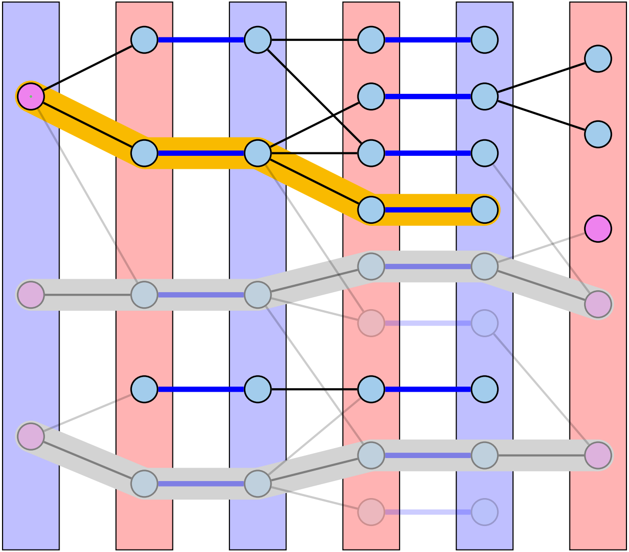

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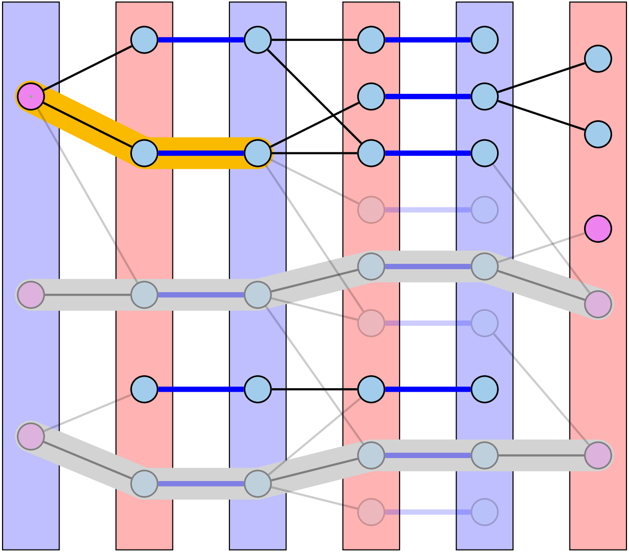

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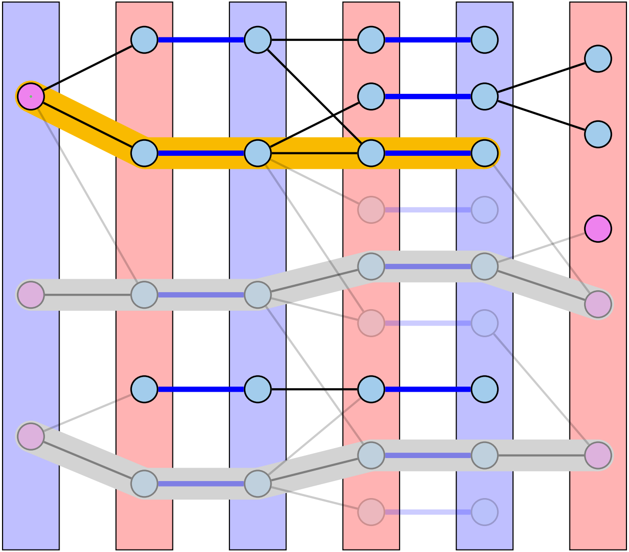

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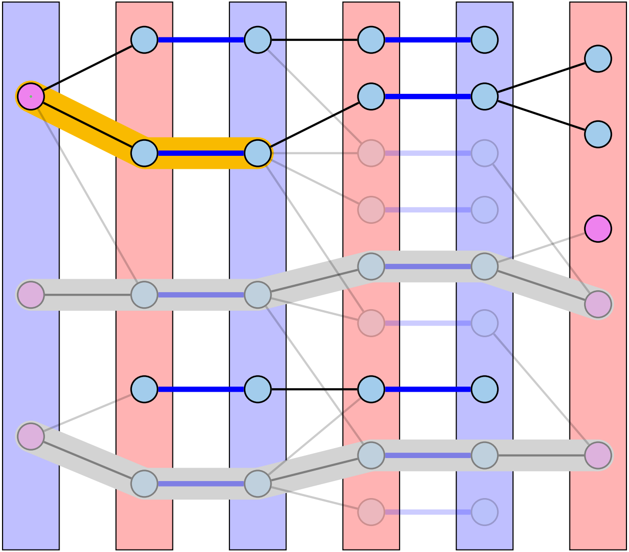

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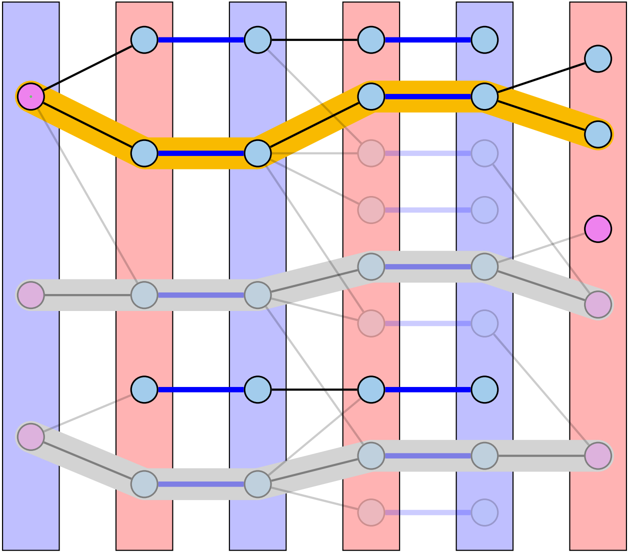

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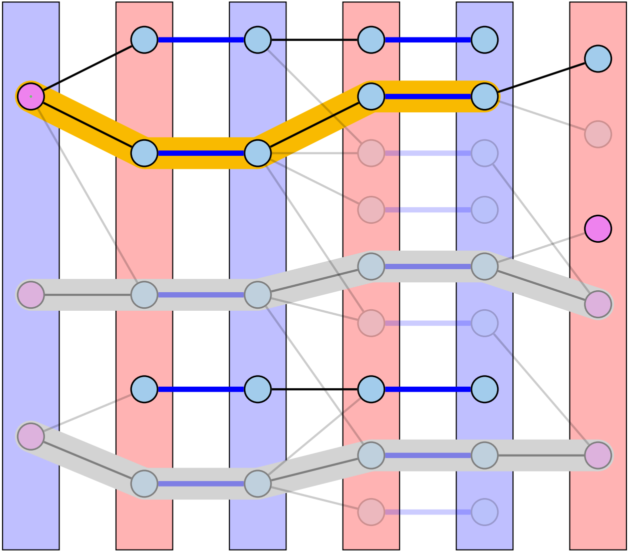

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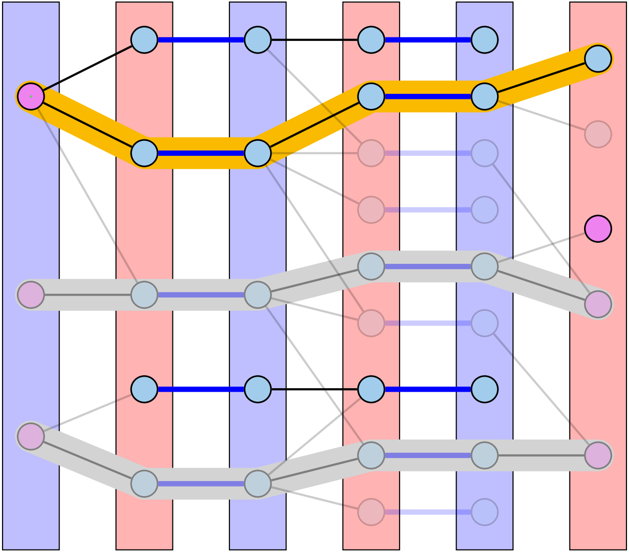

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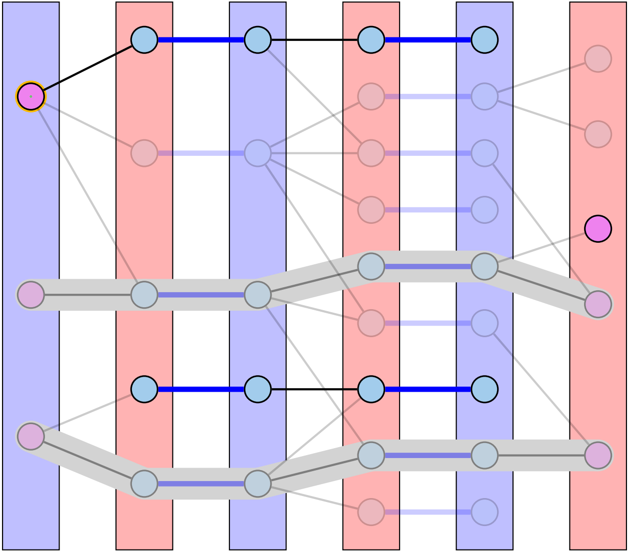

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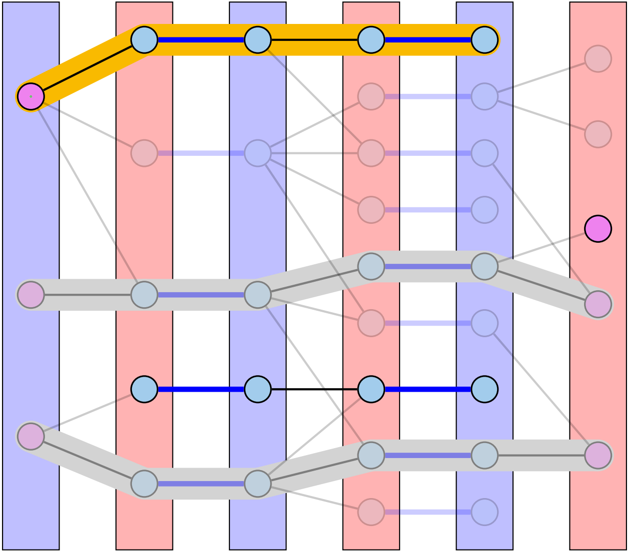

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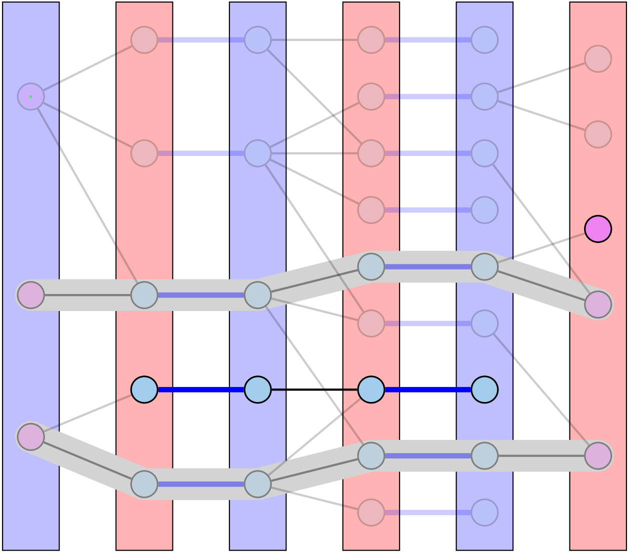

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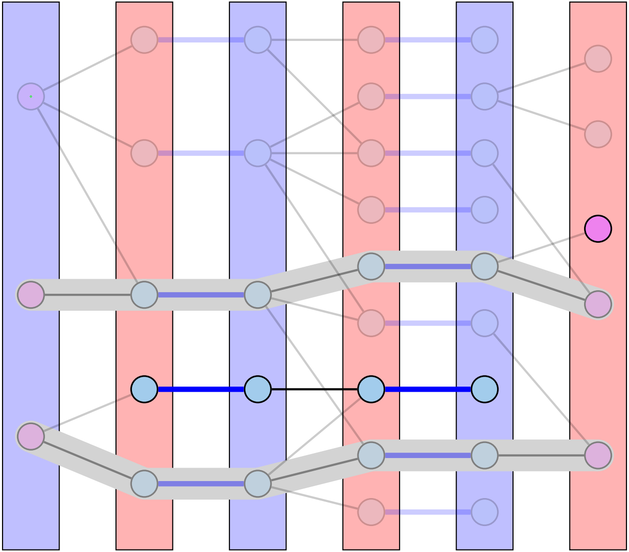

Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $\mathcal{O}(mn)$

- a search (successful or unsuccessful) takes time $\mathcal{O}(n)$
- a search deletes at least one edge from the level graph

there are at most n phases

Time: $\mathcal{O}(mn^2)$.

Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$

- an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases

- after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph
- hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.