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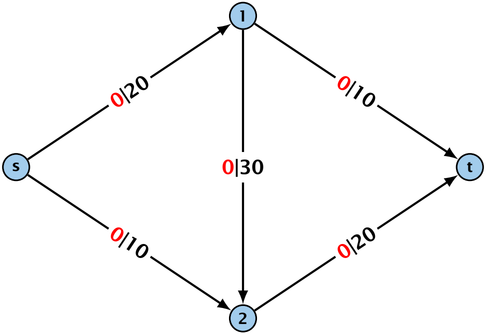

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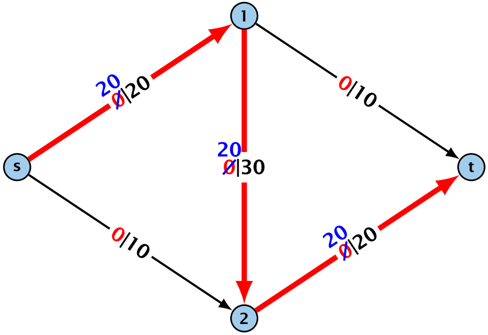

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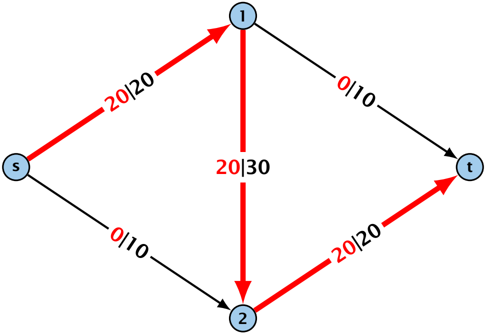

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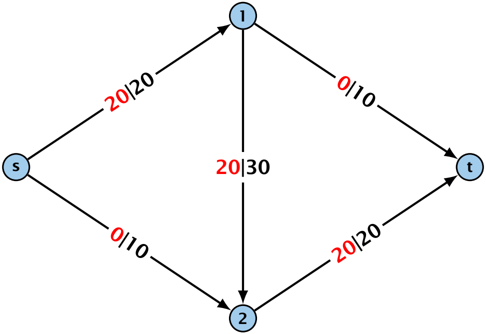

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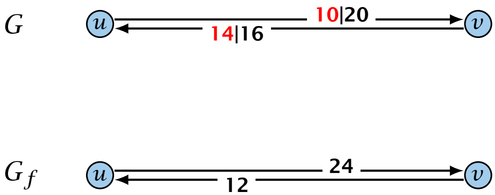

Augmenting Path Algorithm

Definition 1
An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 1 FordFulkerson$(G = (V, E, c))$

1: Initialize $f(e) \leftarrow 0$ for all edges.
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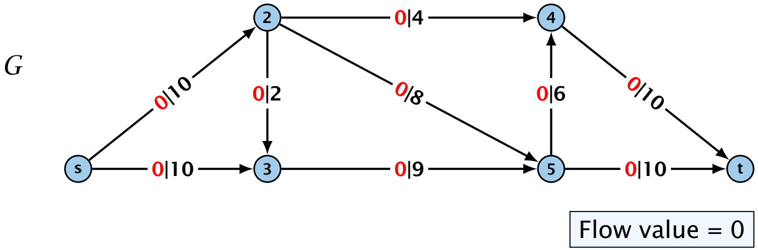

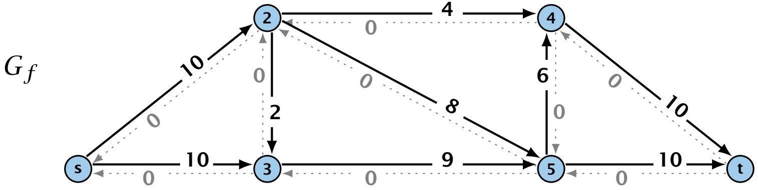

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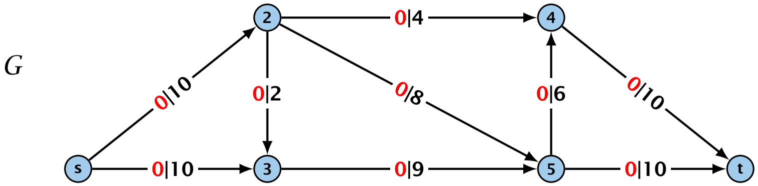
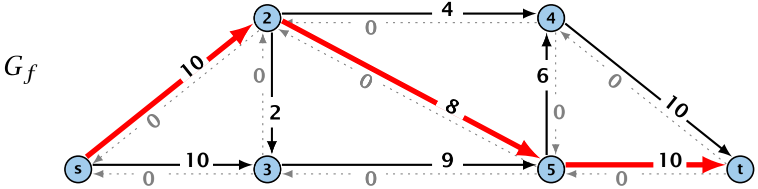


Flow value = 0



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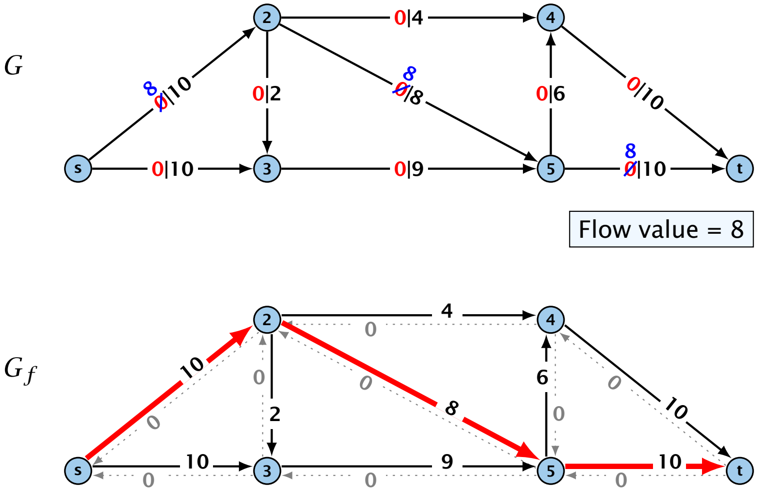
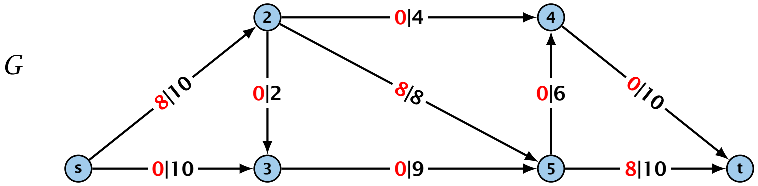


Flow value = 8

Augmenting Path Algorithm



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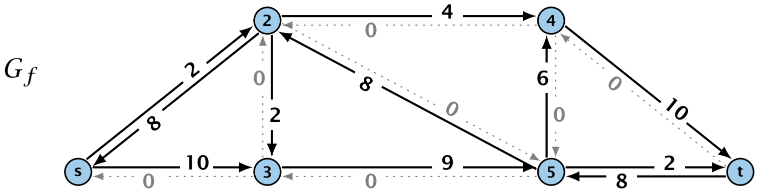

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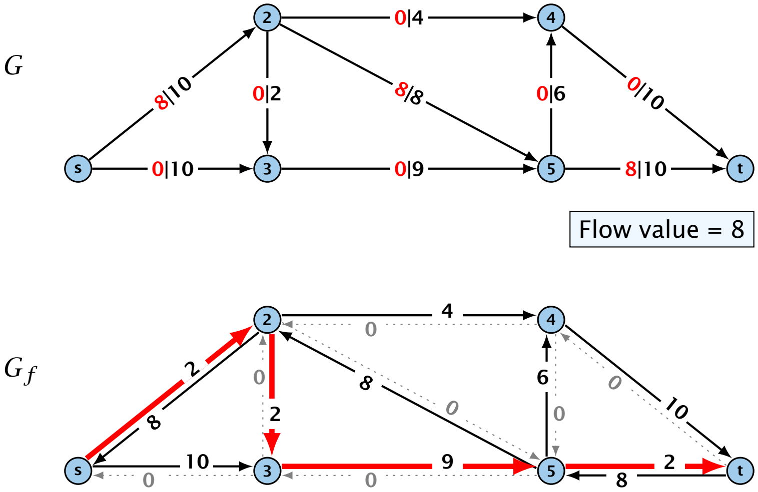

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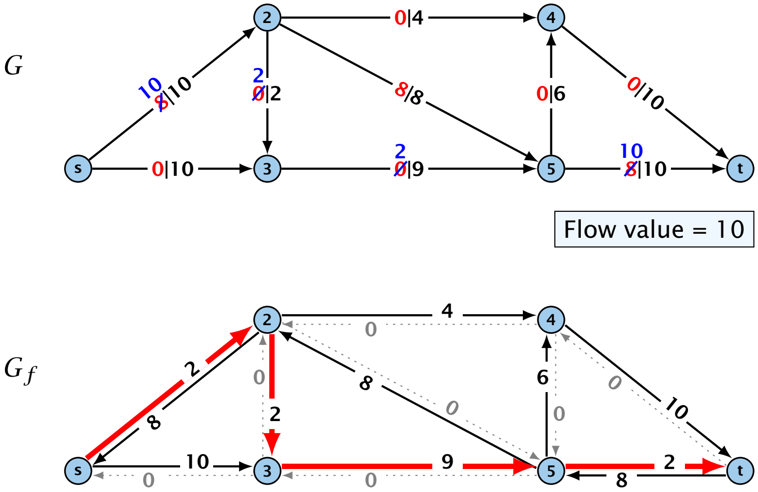

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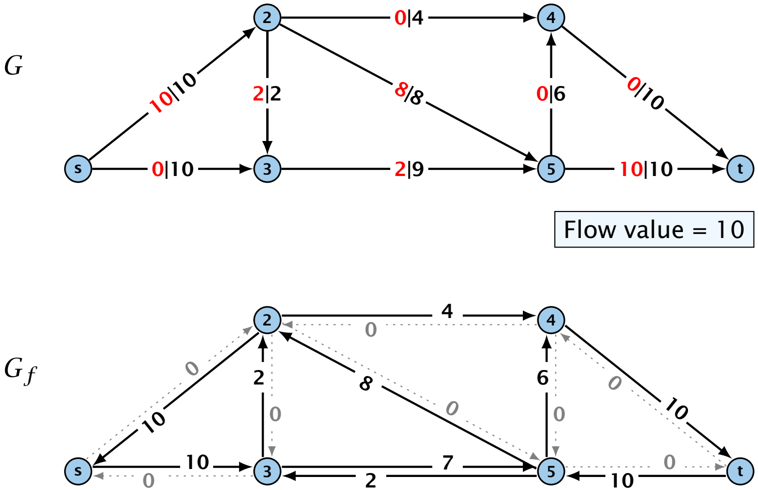

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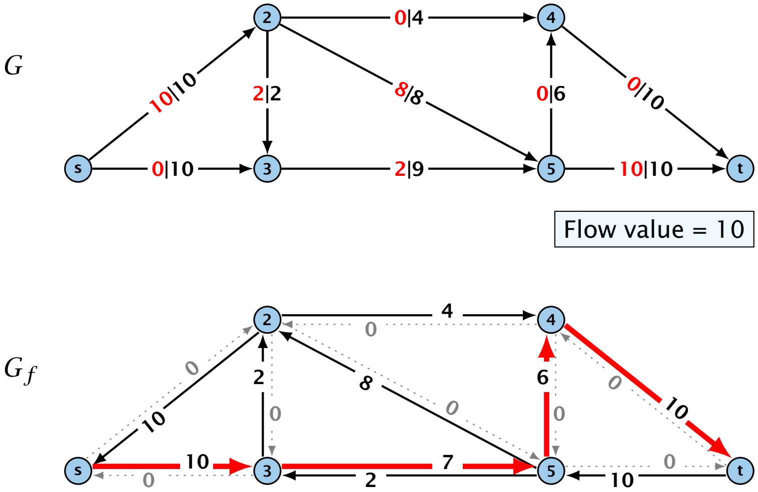

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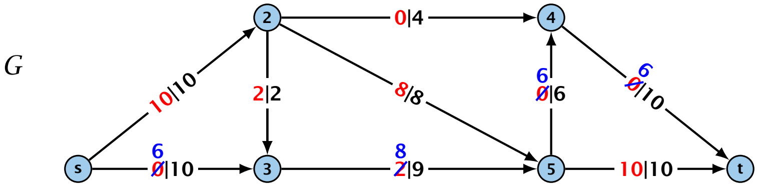

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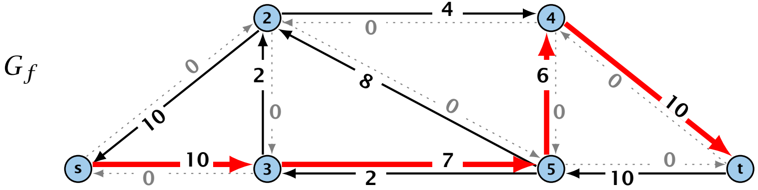

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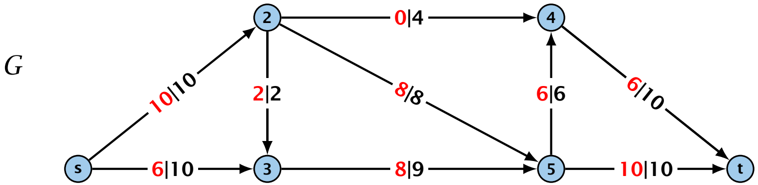

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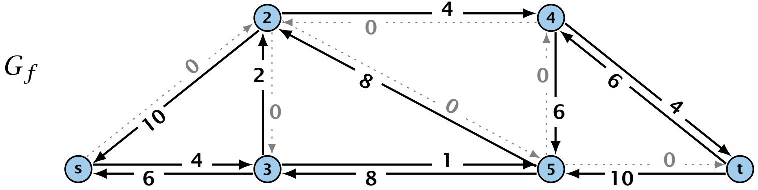

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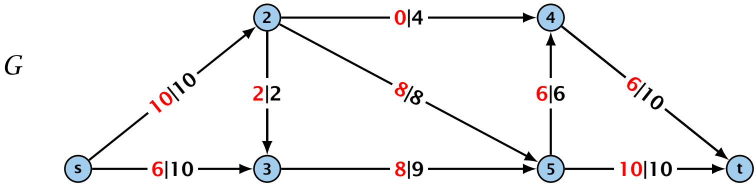

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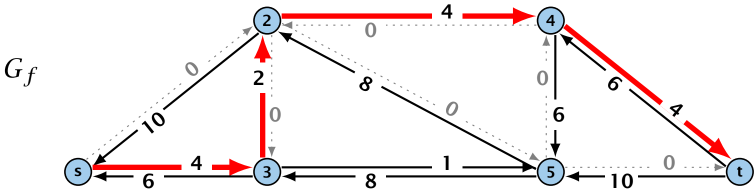
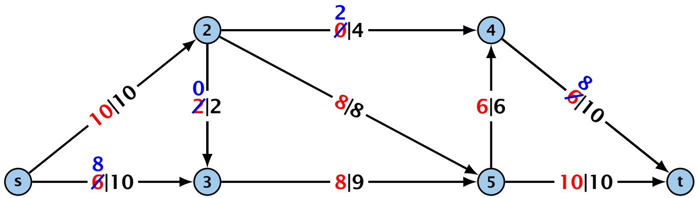


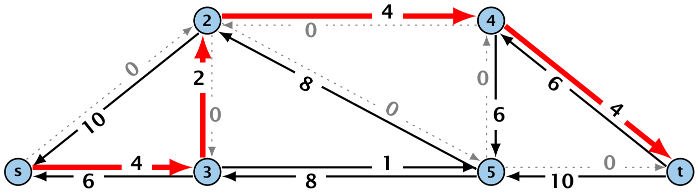
Augmenting Path Algorithm

G



Flow value = 18

G_f



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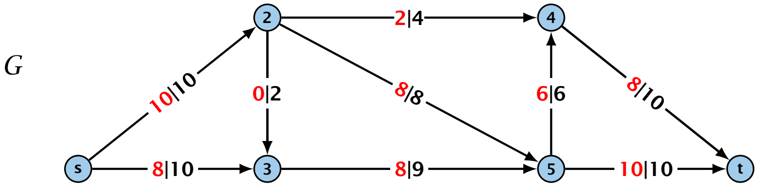

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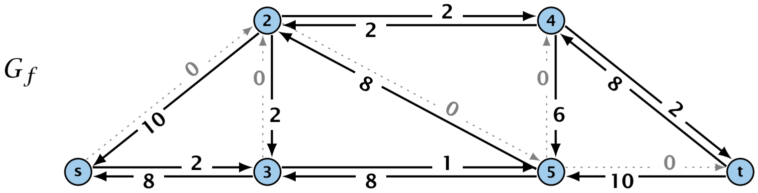

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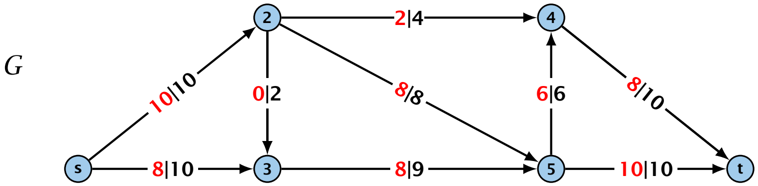

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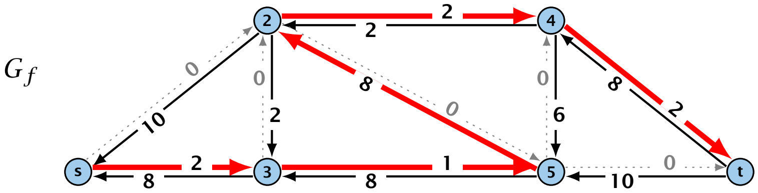

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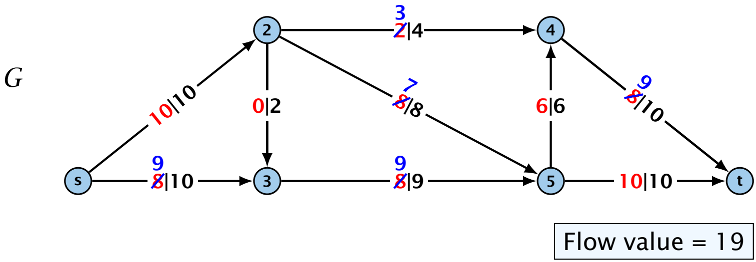

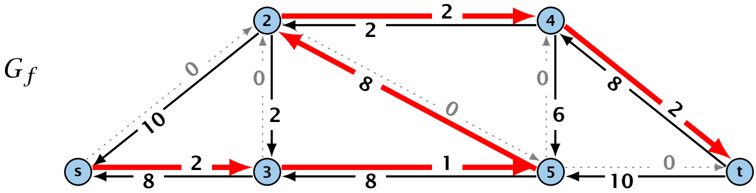

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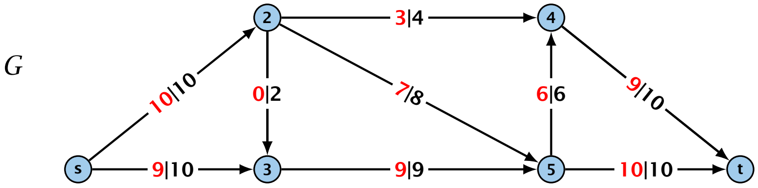

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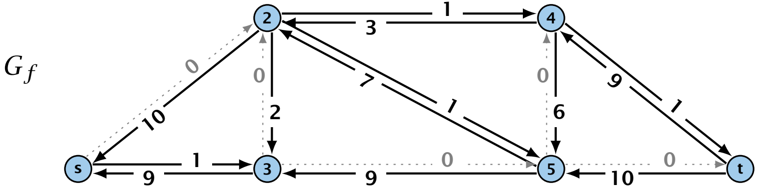

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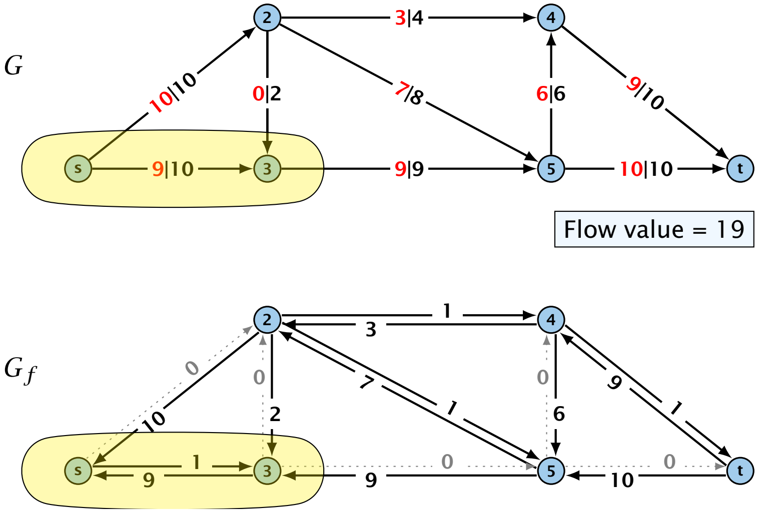

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*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 3
The value of a maximum flow is equal to the value of a minimum cut.

Proof.
Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

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Assumption:
All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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The algorithm terminates in at most $\mathrm{val}(f^*) \leq nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

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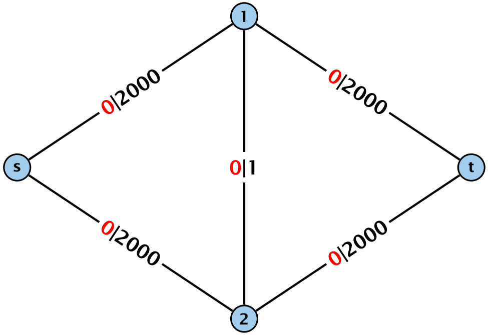
A Bad Input

Problem: The running time may not be polynomial.



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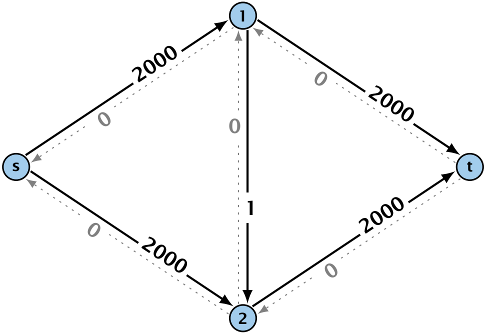

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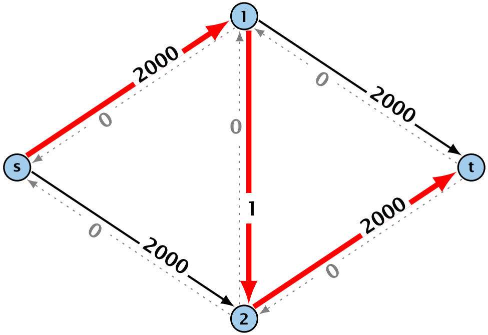

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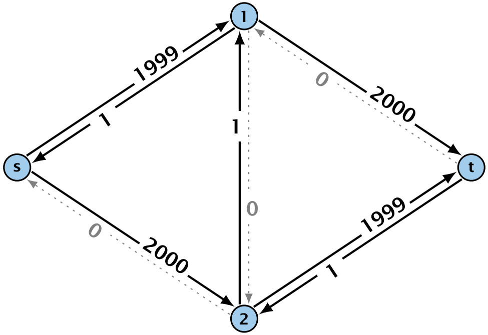

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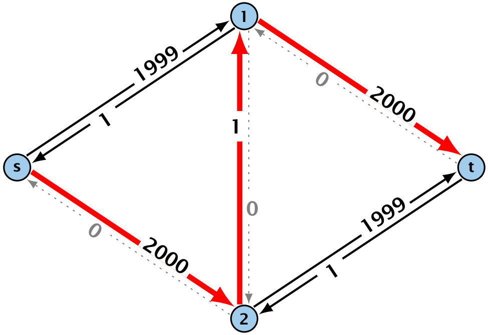

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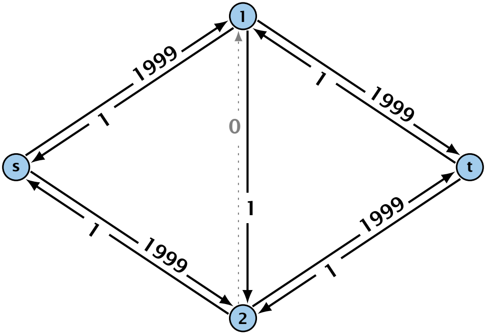

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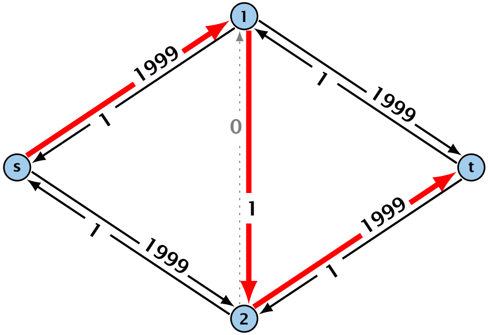

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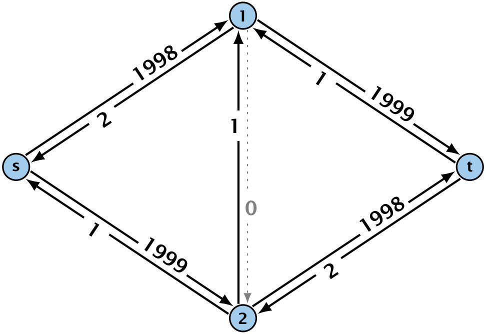

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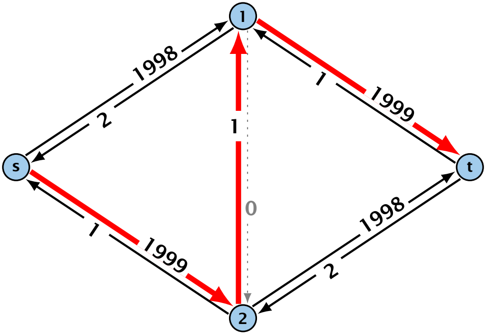

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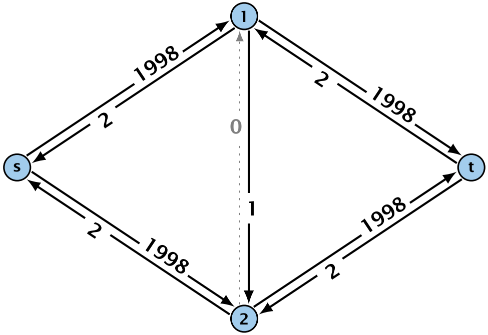

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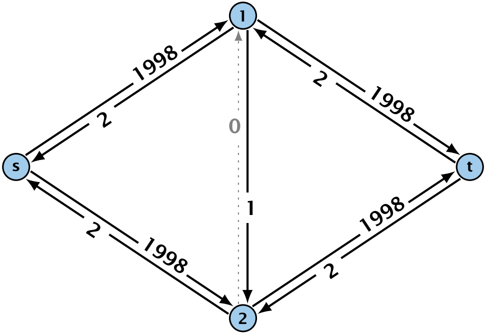

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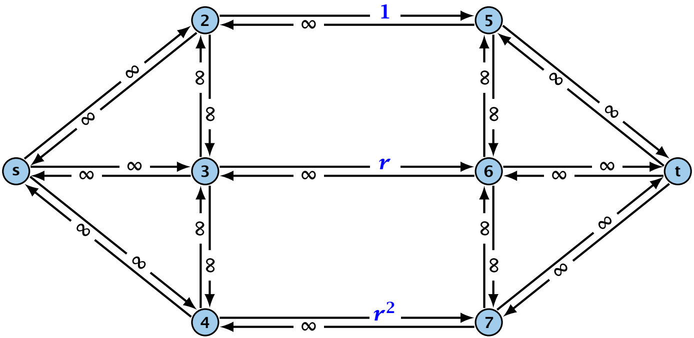

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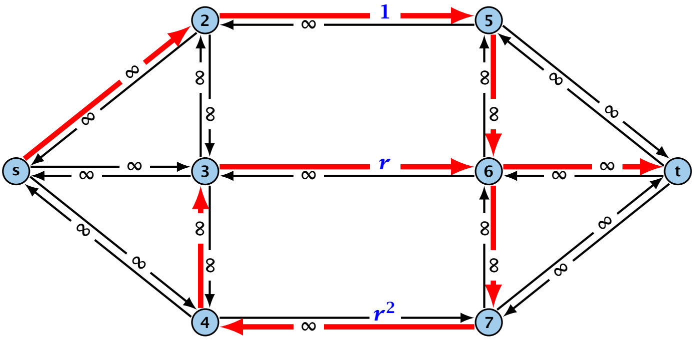

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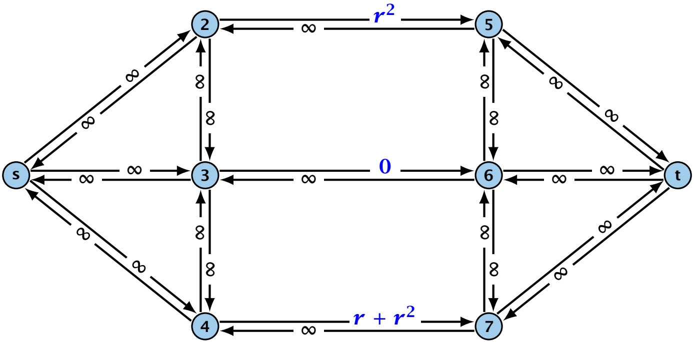

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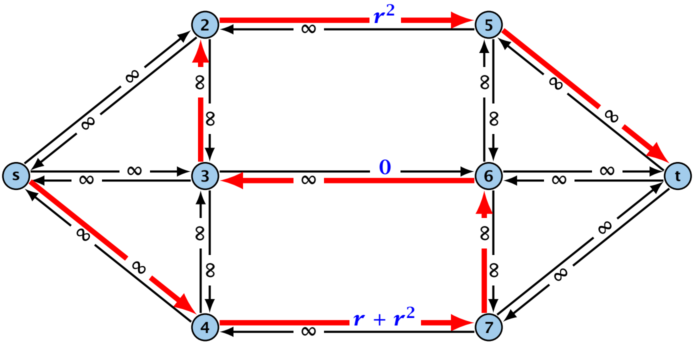

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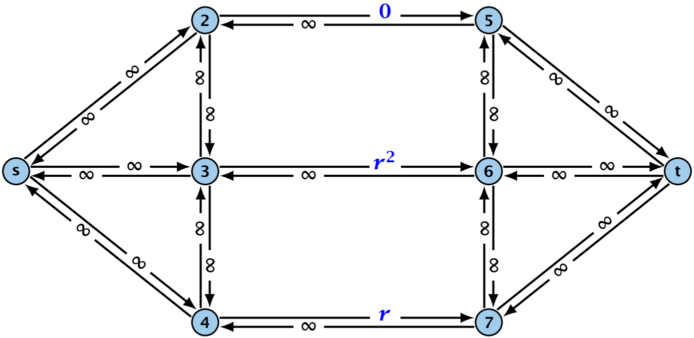

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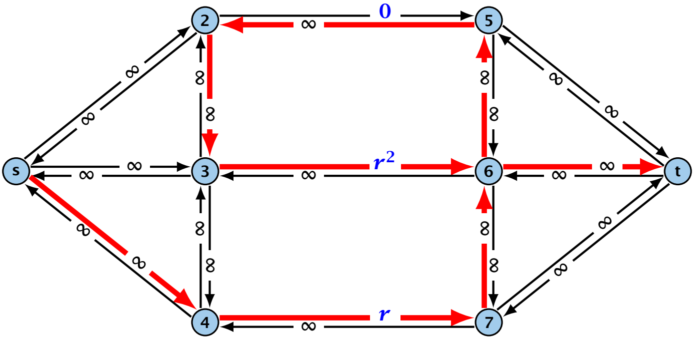

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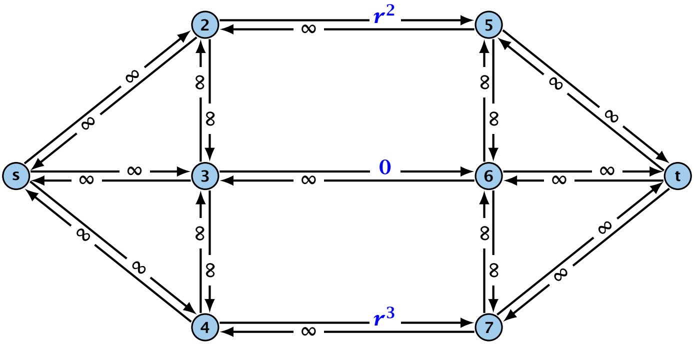

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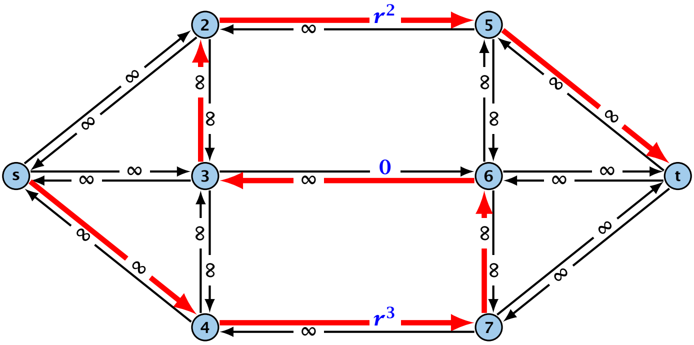

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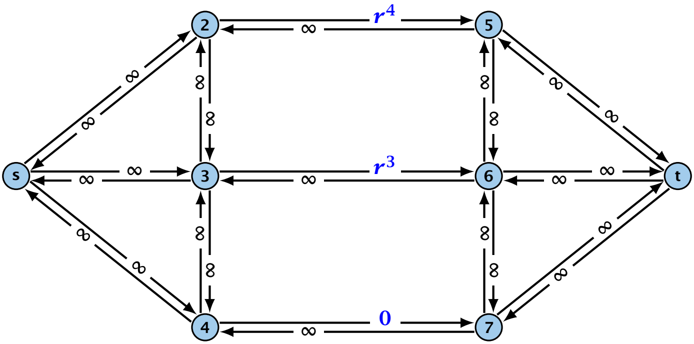

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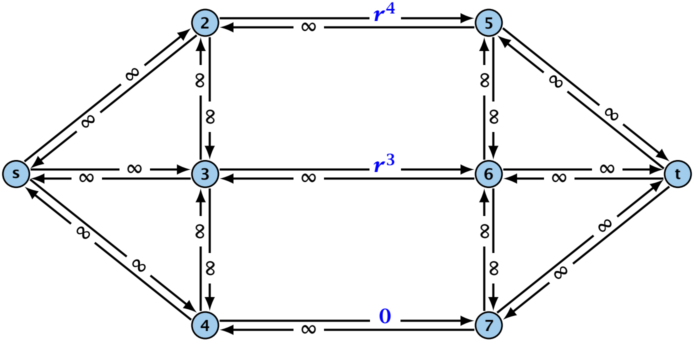

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

Lemma 6

The length of the shortest augmenting path never decreases.

Lemma 7

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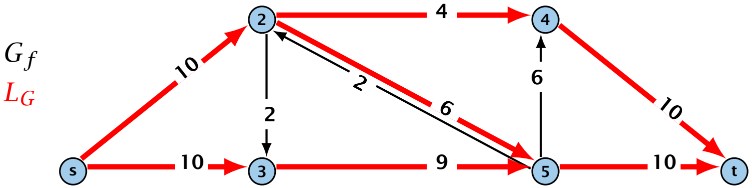

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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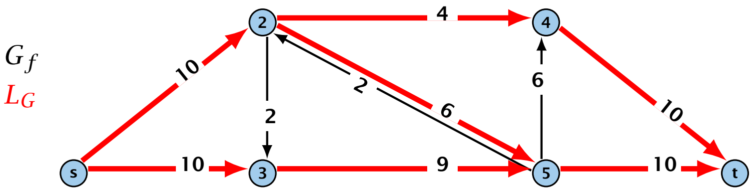

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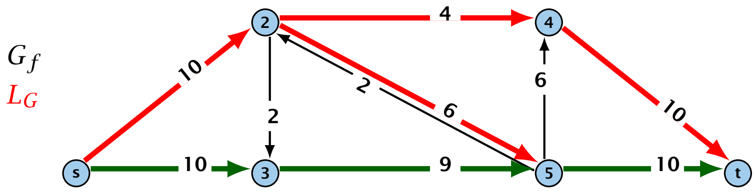

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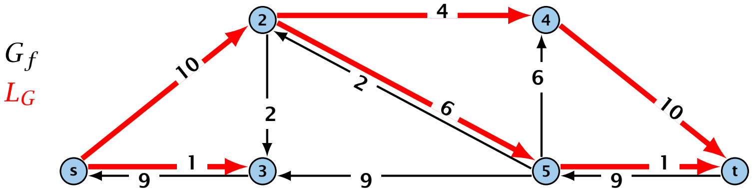

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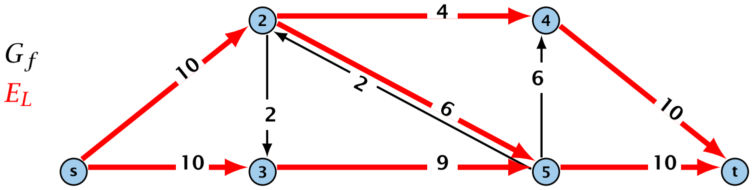

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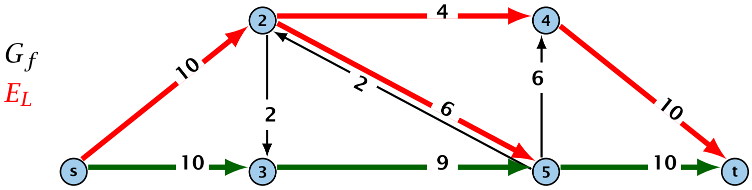

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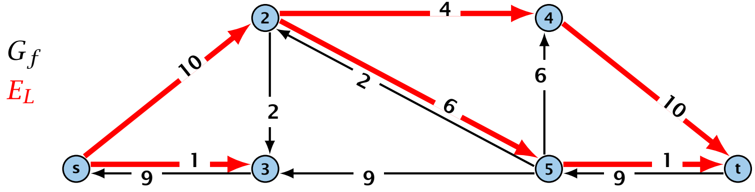

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The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 10 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

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We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s-t path using only edges from E_L is a shortest augmenting path.

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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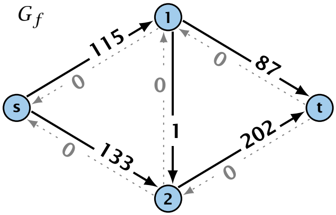

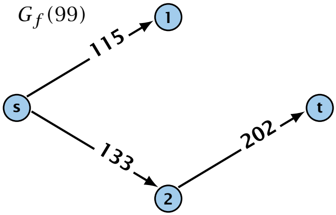

Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path P in $G_f(\Delta)$ **do**
        $f \leftarrow$ augment(f, c, P)
        update$(G_f(\Delta))$
    $\Delta \leftarrow \Delta/2$
return f

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- this means we have a maximum flow.

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- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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Theorem 14

We need $\mathcal{O}(m \log C)$ *augmentations. The algorithm can be implemented in time* $\mathcal{O}(m^2 \log C)$.