

# Part III

## Data Structures

# Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a **[key, value]** pair.

- ▶ The **key** comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The **value** can be anything; it usually carries satellite information important for the application that uses the ADT.

# Dynamic Set Operations

- ▶  **$S$ . search( $k$ ):** Returns pointer to object  $x$  from  $S$  with  $\text{key}[x] = k$  or null.
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- ▶  $S$ . merge( $S'$ ): Sets  $S := S \cup S'$ . Requires  $S \cap S' = \emptyset$ .
- ▶  $S$ . split( $k, S'$ ):  
 $S := \{x \in S \mid \text{key}[x] \leq k\}$ ,  $S' := \{x \in S \mid \text{key}[x] > k\}$ .
- ▶  $S$ . concatenate( $S'$ ):  $S := S \cup S'$ .  
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- ▶  $S$ . decrease-key( $x, k$ ): Replace  $\text{key}[x]$  by  $k \leq \text{key}[x]$ .

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# Examples of ADTs

## Stack:

- ▶  **$S$ .push( $x$ )**: Insert an element.
- ▶  **$S$ .pop()**: Return the element from  $S$  that was inserted most recently; delete it from  $S$ .
- ▶  **$S$ .empty()**: Tell if  $S$  contains any object.

## Queue:

- ▶  $S$ .enqueue( $x$ ): Insert an element.
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## Priority-Queue:

- ▶  $S$ .insert( $x$ ): Insert an element.
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## 7 Dictionary

### Dictionary:

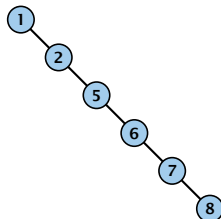
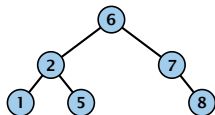
- ▶  **$S$ .insert( $x$ )**: Insert an element  $x$ .
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- ▶  **$S$ .search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node  $v$  have a smaller key-value than  $\text{key}[v]$  and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

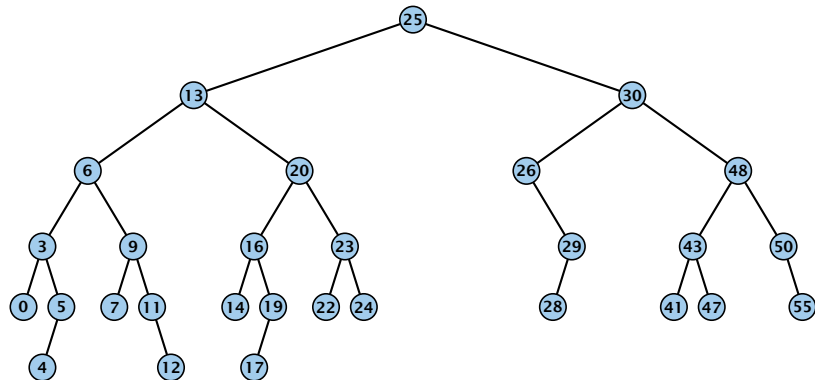


## 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶  $T.\text{insert}(x)$
- ▶  $T.\text{delete}(x)$
- ▶  $T.\text{search}(k)$
- ▶  $T.\text{successor}(x)$
- ▶  $T.\text{predecessor}(x)$
- ▶  $T.\text{minimum}()$
- ▶  $T.\text{maximum}()$

# Binary Search Trees: Searching

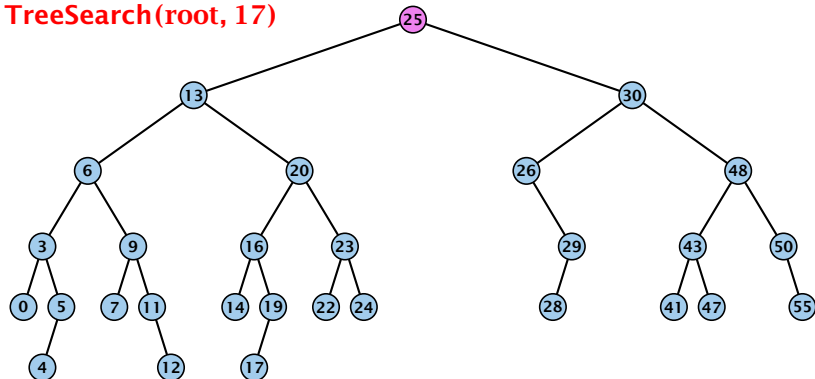


## Algorithm 1 TreeSearch( $x, k$ )

- 1: **if**  $x = \text{null}$  **or**  $k = \text{key}[x]$  **return**  $x$
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TreeSearch(root, 17)

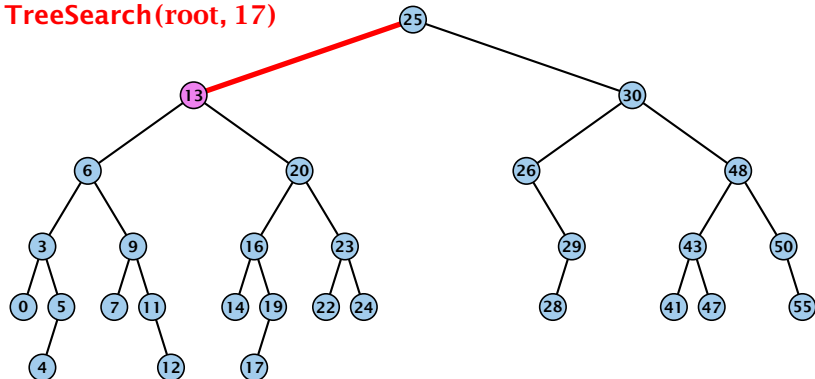


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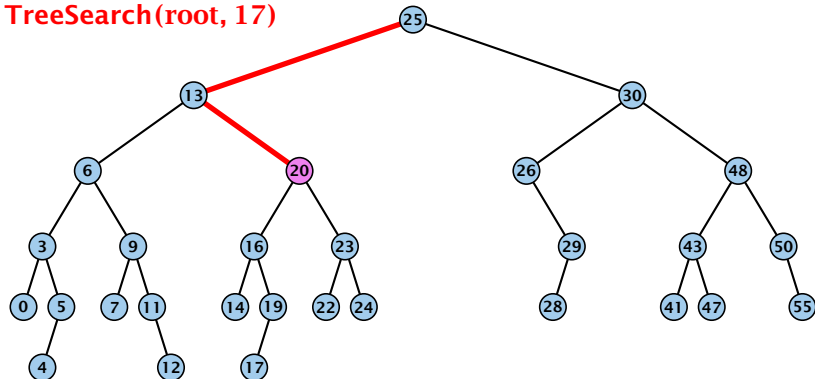


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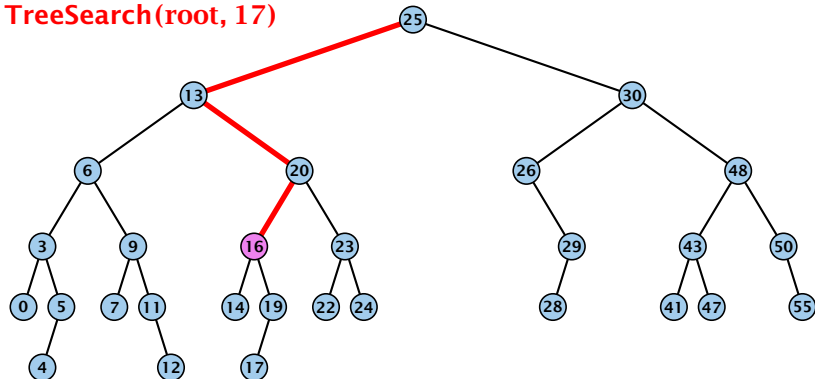
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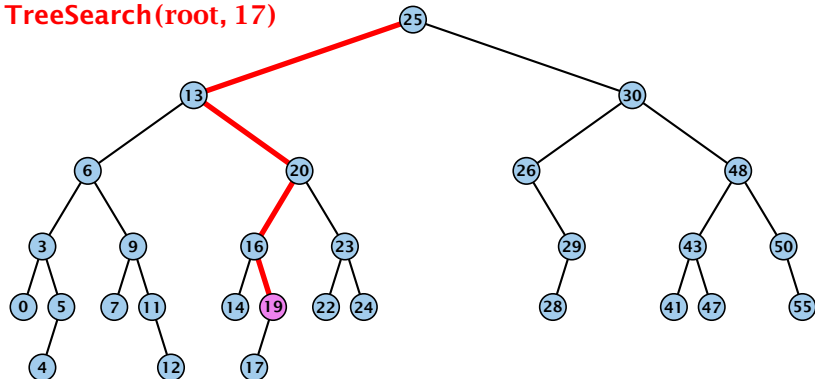


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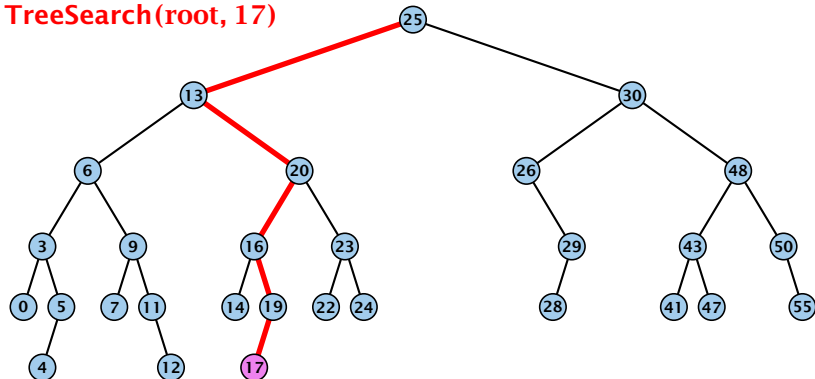


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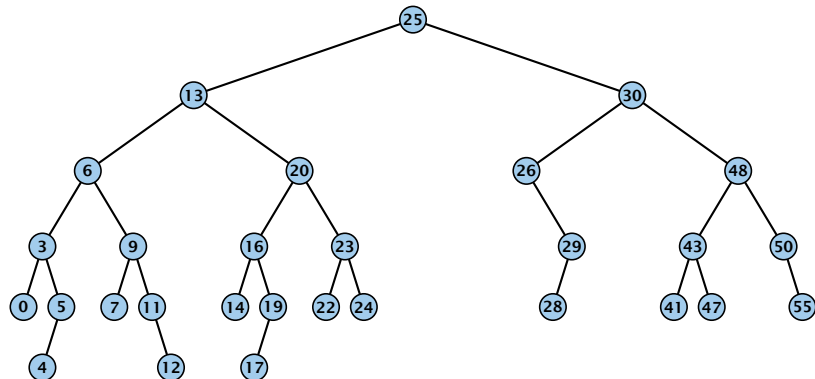
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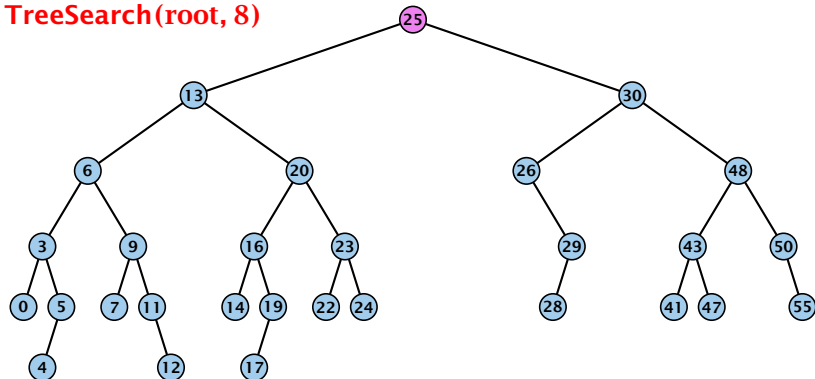


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# Binary Search Trees: Searching

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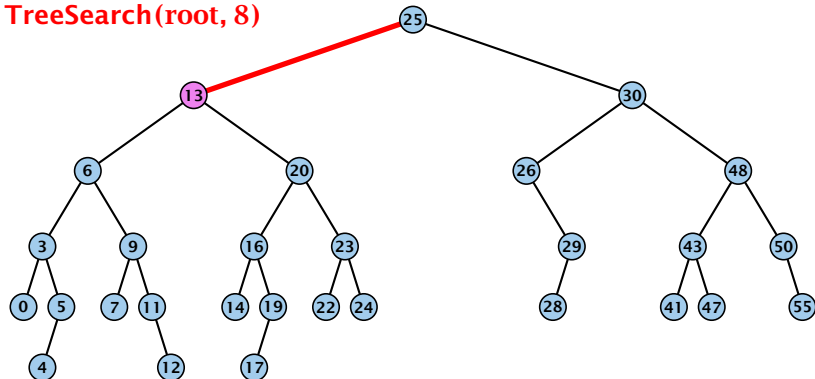


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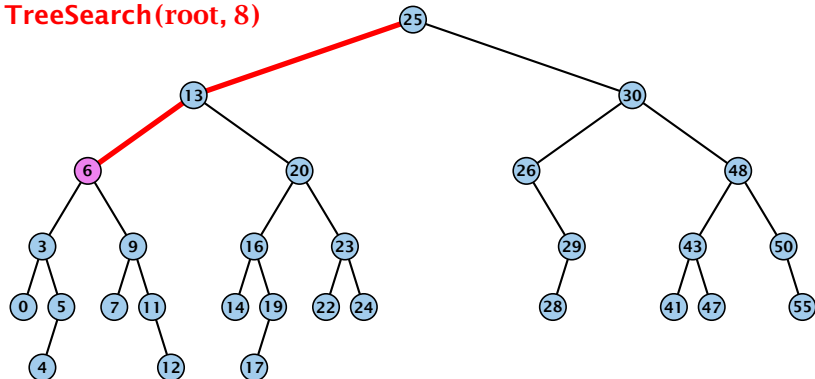


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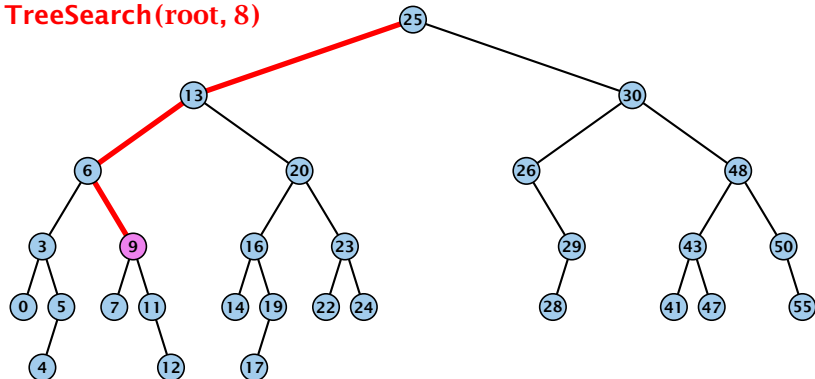


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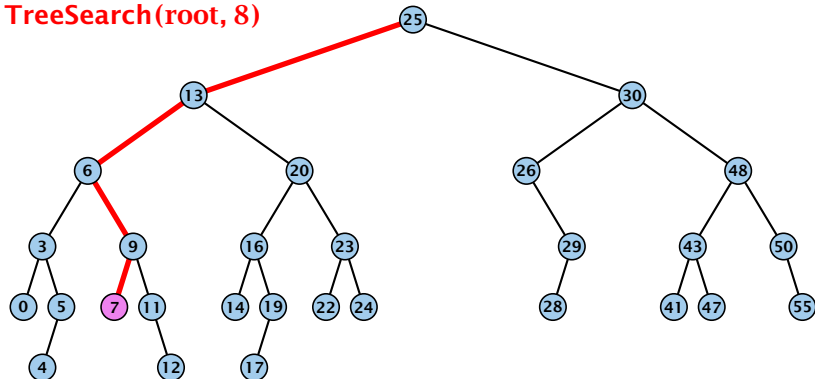
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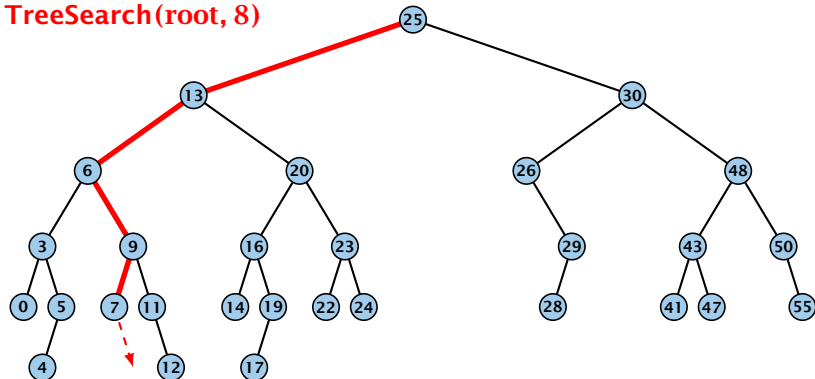


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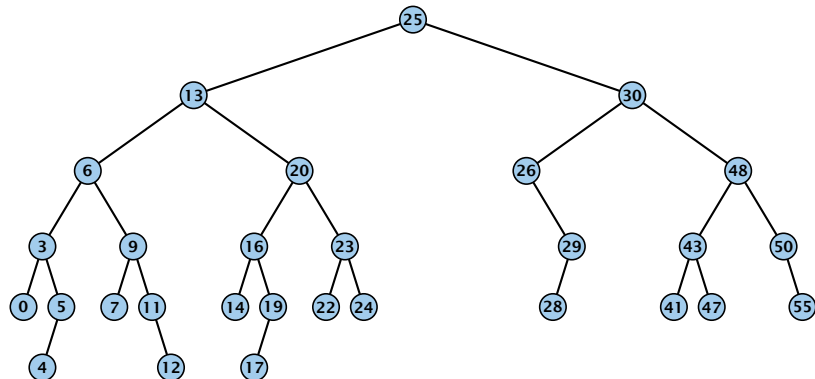
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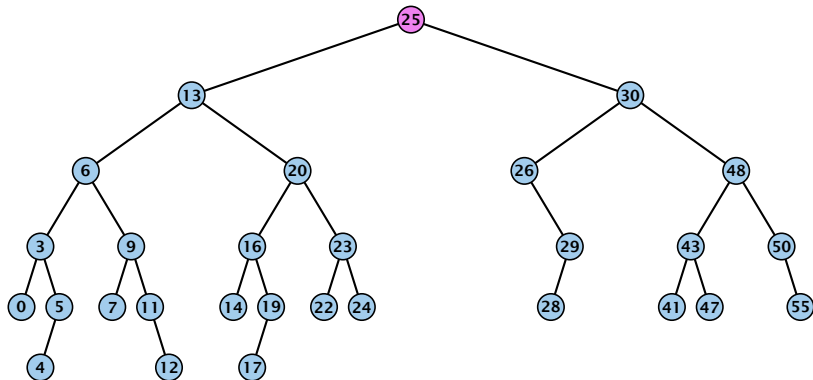
# Binary Search Trees: Minimum



## Algorithm 2 TreeMin( $x$ )

- 1: **if**  $x = \text{null}$  **or**  $\text{left}[x] = \text{null}$  **return**  $x$
- 2: **return**  $\text{TreeMin}(\text{left}[x])$

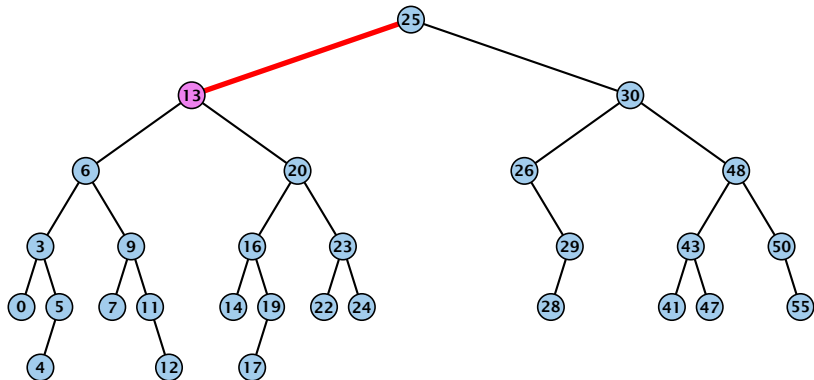
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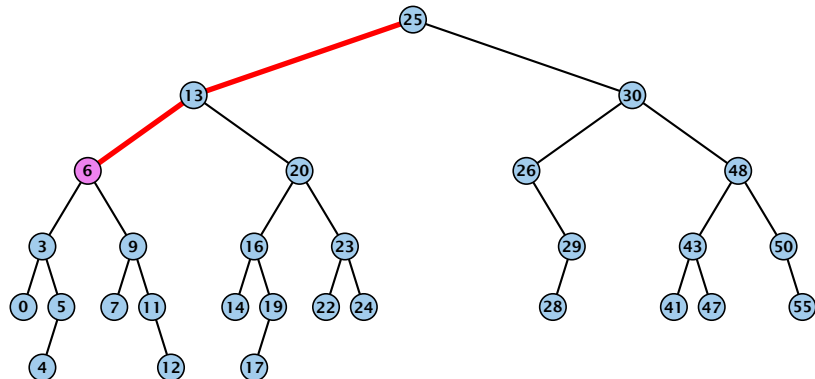
# Binary Search Trees: Minimum



## Algorithm 2 TreeMin( $x$ )

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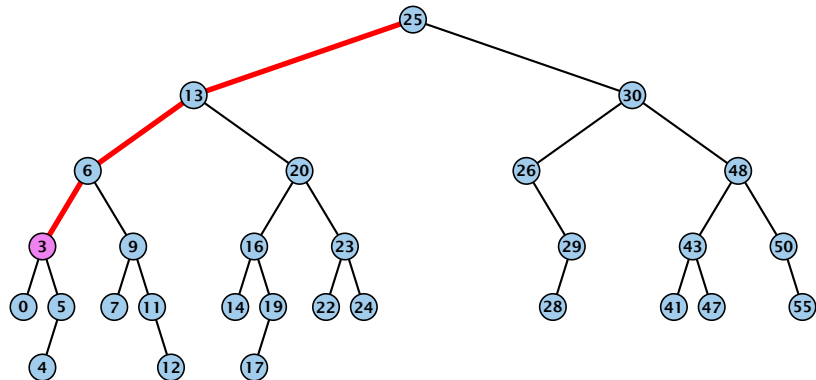
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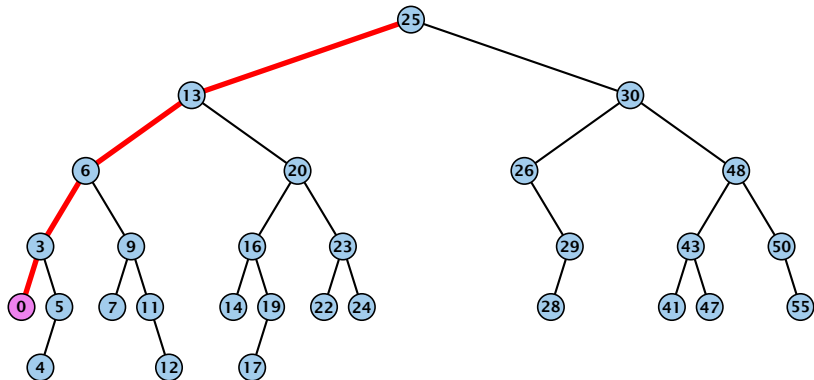
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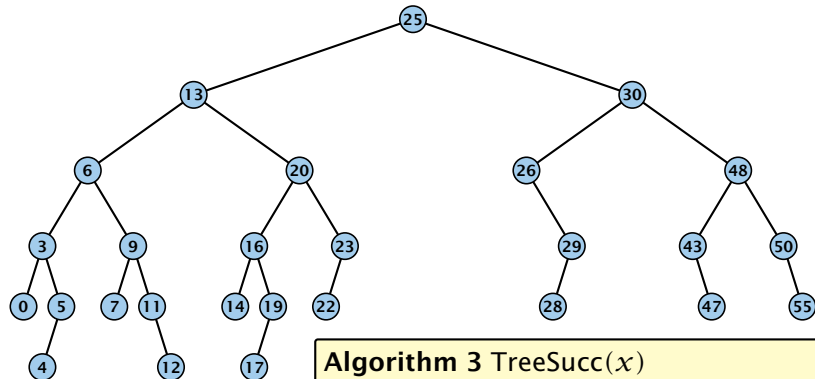


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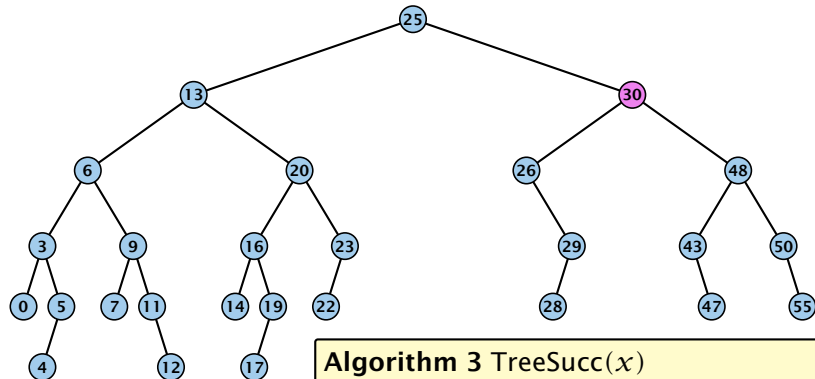
# Binary Search Trees: Successor



## Algorithm 3 TreeSucc( $x$ )

- 1: **if** right[ $x$ ]  $\neq$  null **return** TreeMin(right[ $x$ ])
- 2:  $y \leftarrow$  parent[ $x$ ]
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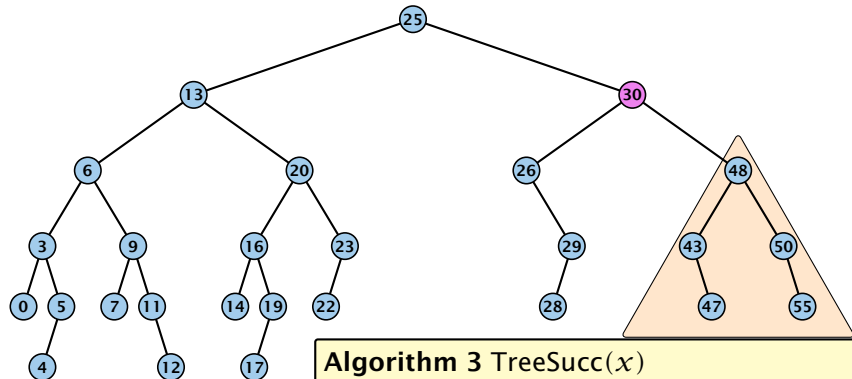
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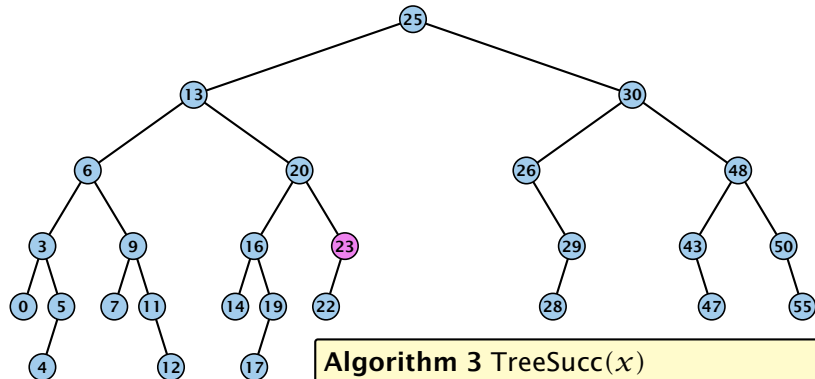
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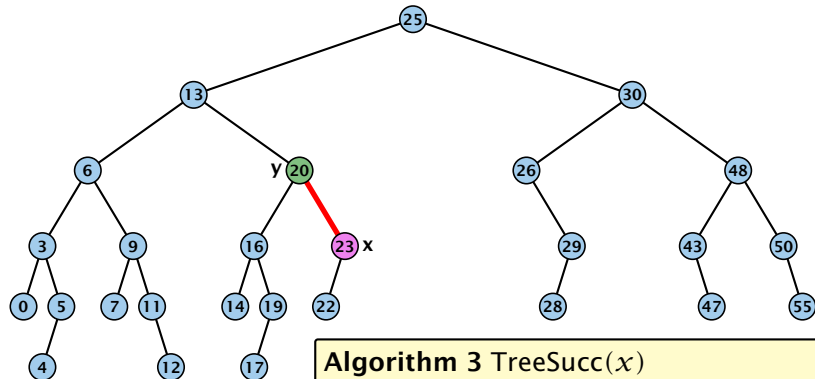
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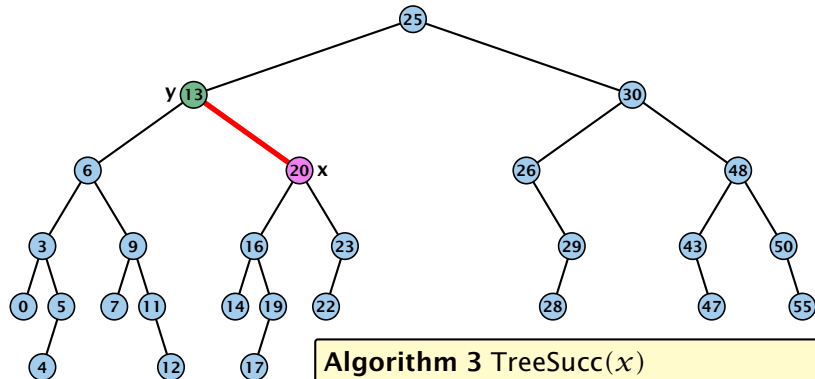
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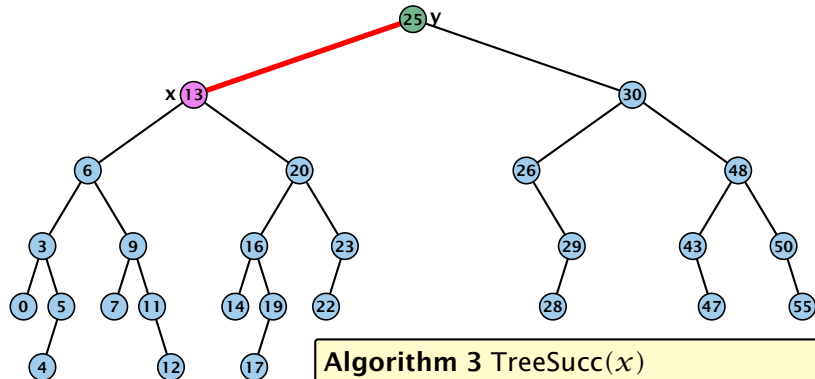
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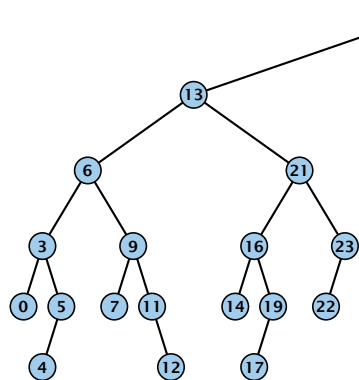
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## Binary Search Trees: Insert



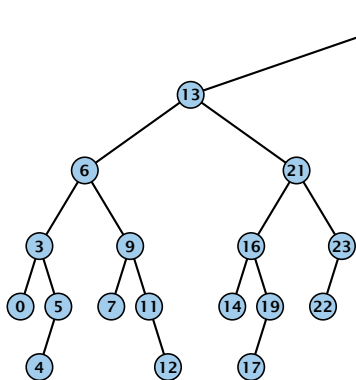
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1: if  $x = \text{null}$  then  
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3:   return;  
4: if  $\text{key}[x] > \text{key}[z]$  then  
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# Binary Search Trees: Insert

Insert element **not** in the tree.

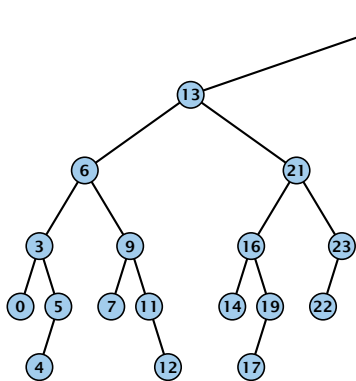


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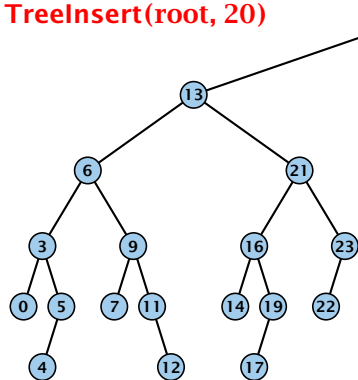
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**TreeInsert**(root, 20)



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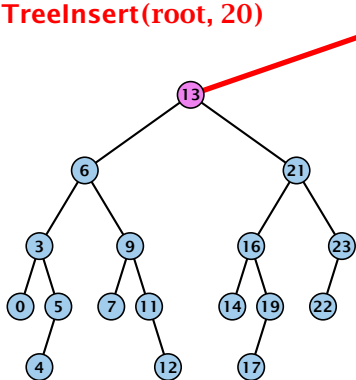
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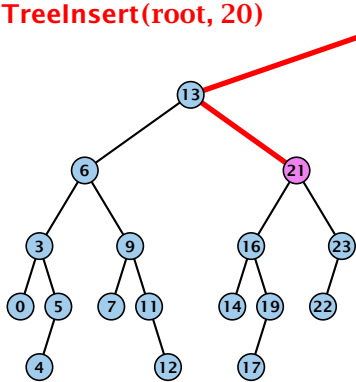
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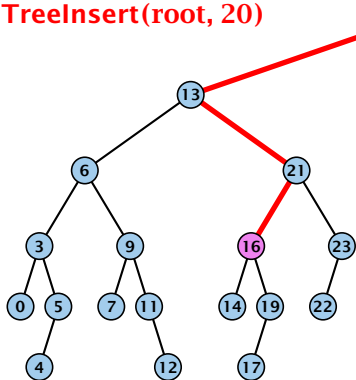
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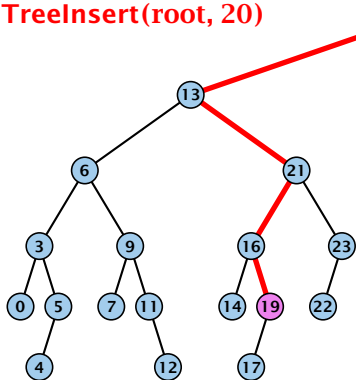
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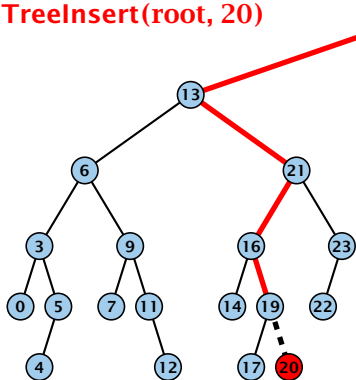
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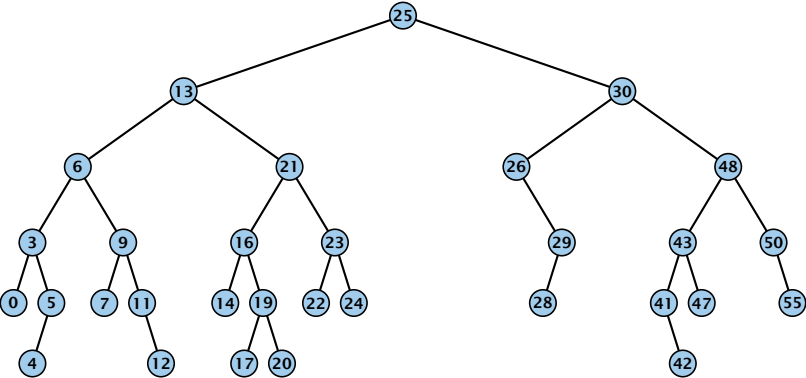
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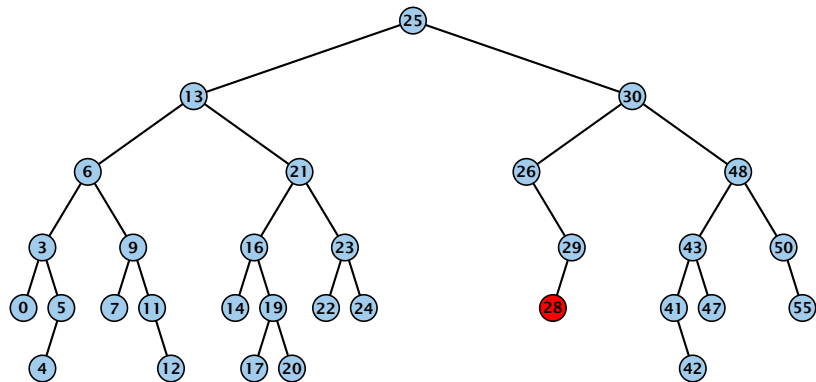
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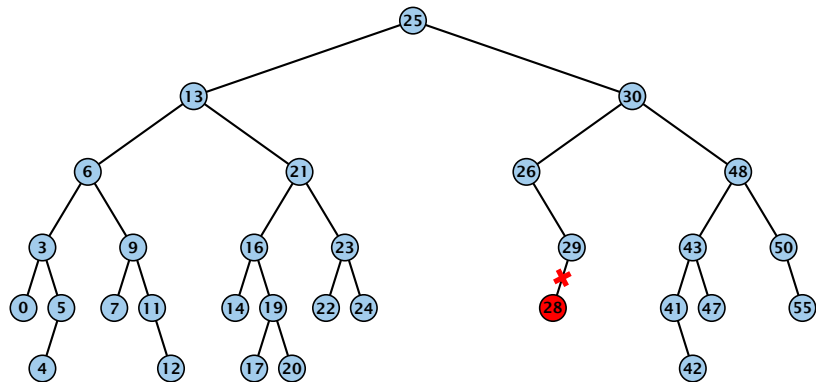


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Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to **null**.

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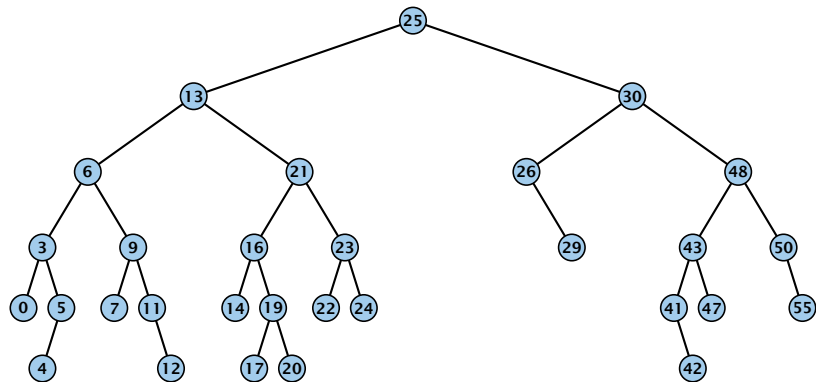


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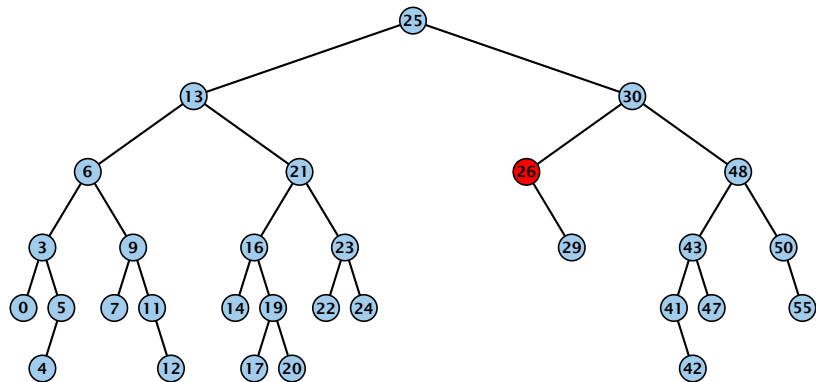


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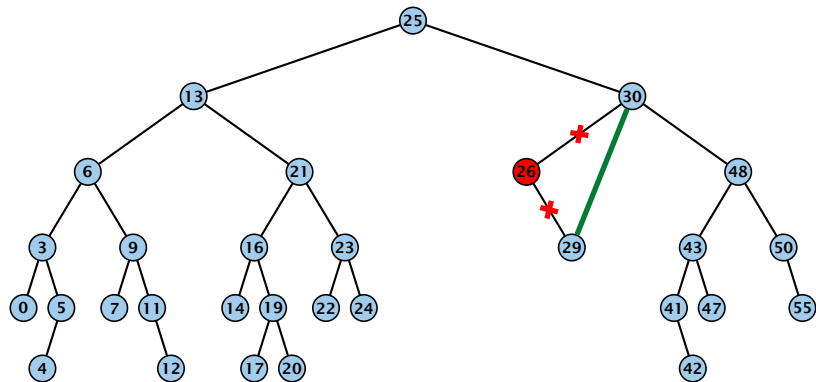


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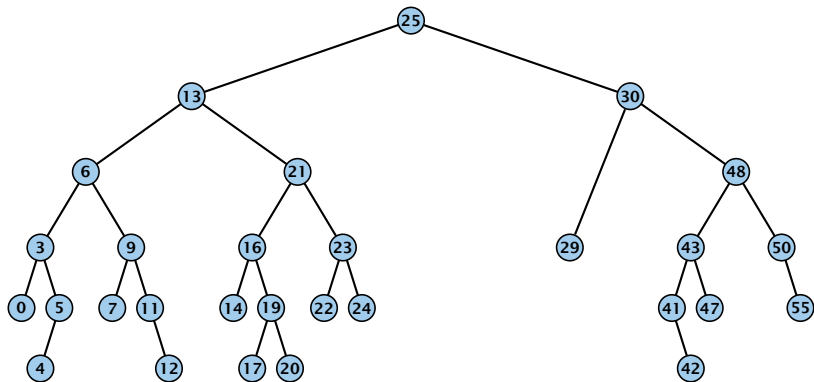


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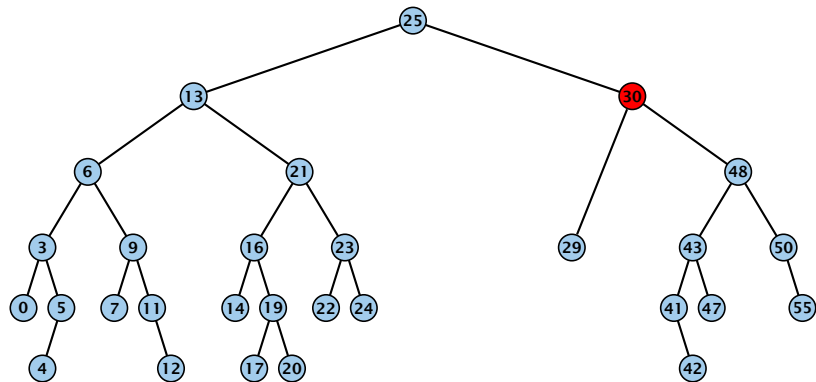


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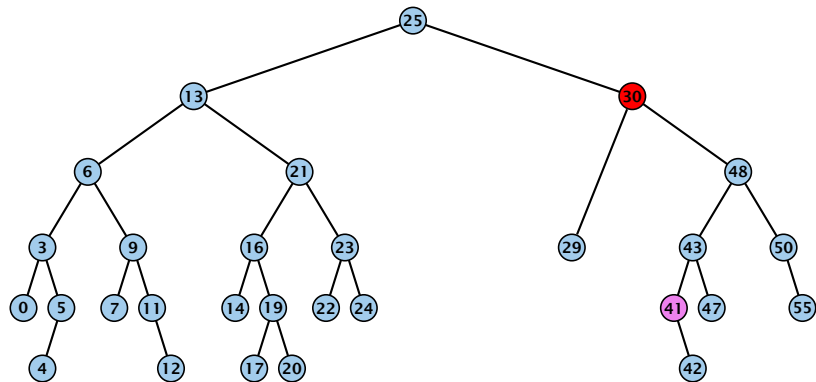
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Element has two children

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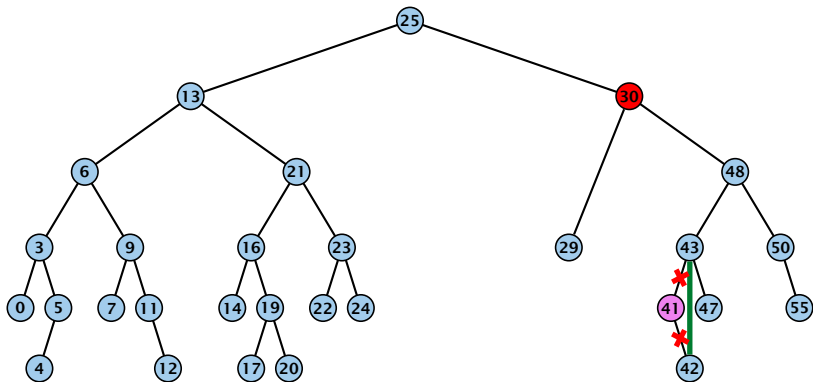


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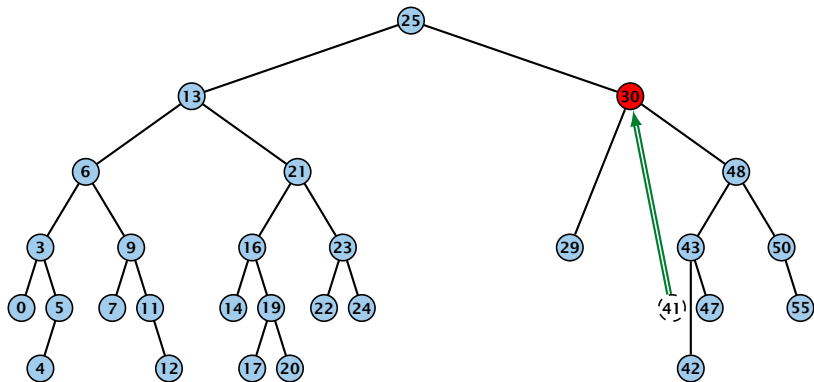


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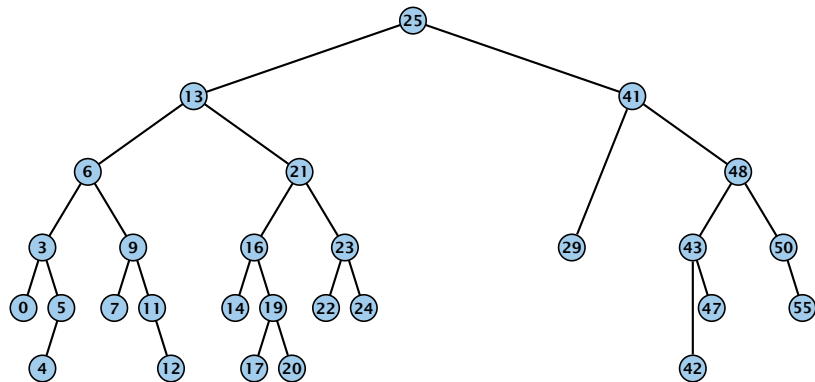


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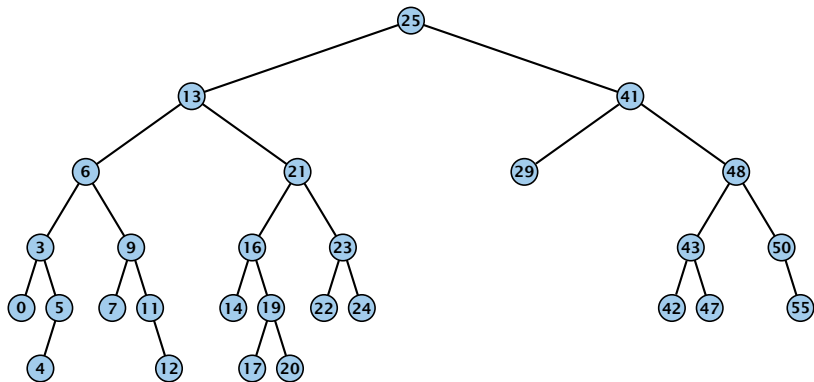


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# Binary Search Trees: Delete

## Algorithm 9 TreeDelete( $z$ )

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:     if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:     if parent[ $y$ ] = null then
7:       root[ $T$ ]  $\leftarrow x$ 
8:     else
9:       if  $y = \text{left}[\text{parent}[y]]$  then
10:        left[parent[ $y$ ]]  $\leftarrow x$ 
11:       else
12:        right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to  $x$

# Balanced Binary Search Trees

All operations on a binary search tree can be performed in time  $\mathcal{O}(h)$ , where  $h$  denotes the height of the tree.

However the height of the tree may become as large as  $\Theta(n)$ .

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With each insert- and delete-operation perform **local** adjustments to guarantee a height of  $\mathcal{O}(\log n)$ .

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

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All operations on a binary search tree can be performed in time  $\mathcal{O}(h)$ , where  $h$  denotes the height of the tree.

However the height of the tree may become as large as  $\Theta(n)$ .

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With each insert- and delete-operation perform **local** adjustments to guarantee a height of  $\mathcal{O}(\log n)$ .

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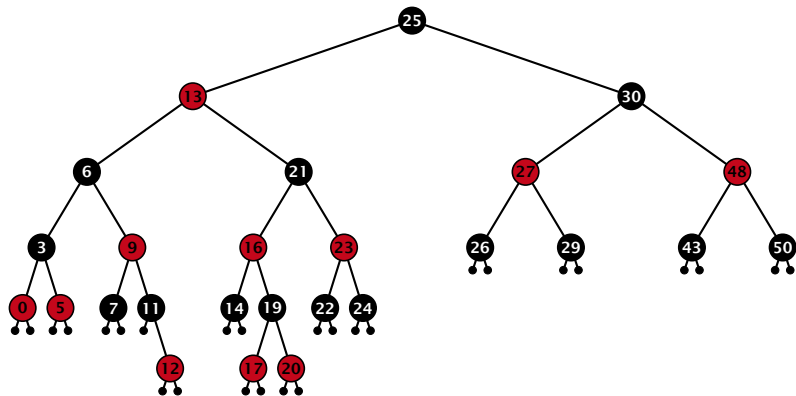
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# Red Black Trees: Example



## 7.2 Red Black Trees

### Lemma 2

A red-black tree with  $n$  internal nodes has height at most  $\mathcal{O}(\log n)$ .

### Definition 3

The black height  $\text{bh}(v)$  of a node  $v$  in a red black tree is the number of black nodes on a path from  $v$  to a leaf vertex (not counting  $v$ ).

We first show:

### Lemma 4

A sub-tree of black height  $\text{bh}(v)$  in a red black tree contains at least  $2^{\text{bh}(v)} - 1$  internal vertices.

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## 7.2 Red Black Trees

### Proof of Lemma 4.

Induction on the height of  $v$ .

base case ( $\text{height}(v) = 0$ )

The number of (maximum distance from) a node in the subtree rooted at  $v$  is 0, then  $b(v)$  is 0.

The black height of  $v$  is 0.

The subtree rooted at  $v$  contains 0 nodes, so  $b(v) = \text{black height}(v)$ .

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The root's maximum distance from a leaf node in the

subtree rooted at  $v$  is then  $\text{height}(v) = 0$ .

The black height of  $v$  is also  $\text{height}(v) = 0$ .

The subtrees rooted at  $v$ 's children have height

at most  $\text{height}(v) = 0$ .

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- ▶ If  $\text{height}(v)$  (maximum distance btw.  $v$  and a node in the sub-tree rooted at  $v$ ) is 0 then  $v$  is a leaf.
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### Proof (cont.)

#### induction step

Assume  $x$  is a node with left child  $l$  and right child  $r$ .

$l$  has two children with strictly smaller height.

These children (and  $r$ ) either have two children

by induction hypothesis both of whose children contain at least one internal vertex.

They are separated from  $x$  by

edges



## 7.2 Red Black Trees

### Proof (cont.)

#### induction step

- ▶ Suppose  $v$  is a node with  $\text{height}(v) > 0$ .
- ▶  $v$  has two children with strictly smaller height.
- ▶ These children ( $c_1, c_2$ ) either have  $\text{bh}(c_i) = \text{bh}(v)$  or  $\text{bh}(c_i) = \text{bh}(v) - 1$ .
- ▶ By induction hypothesis both sub-trees contain at least  $2^{\text{bh}(v)-1} - 1$  internal vertices.
- ▶ Then  $T_v$  contains at least  $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$  vertices.



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### Proof of Lemma 2.

Let  $h$  denote the height of the red-black tree, and let  $P$  denote a path from the root to the furthest leaf.

At least half of the nodes on  $P$  must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least  $h/2$ .

The tree contains at least  $2^{h/2} - 1$  internal vertices. Hence,  
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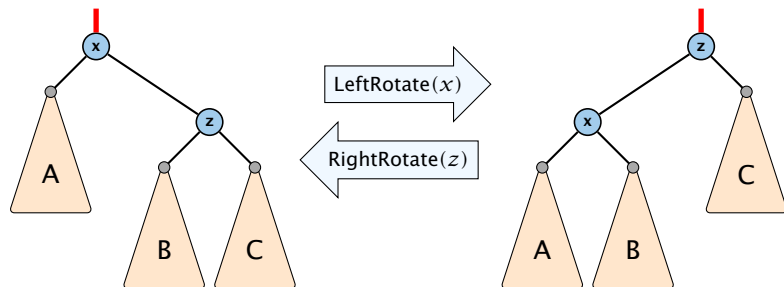
## 7.2 Red Black Trees

We need to adapt the insert and delete operations so that the red black properties are maintained.

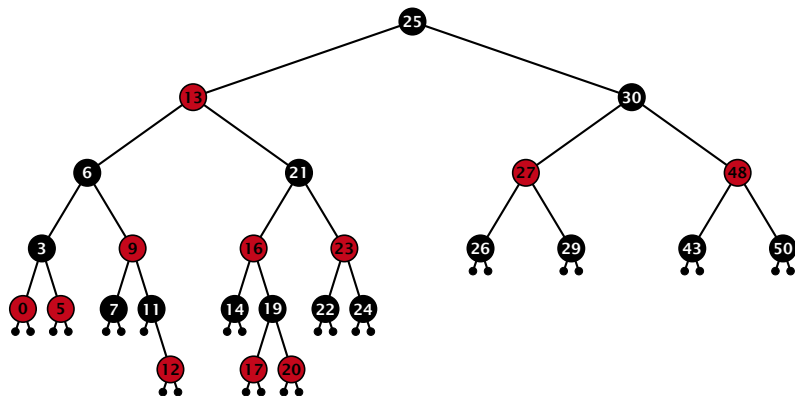


# Rotations

The properties will be maintained through rotations:



# Red Black Trees: Insert

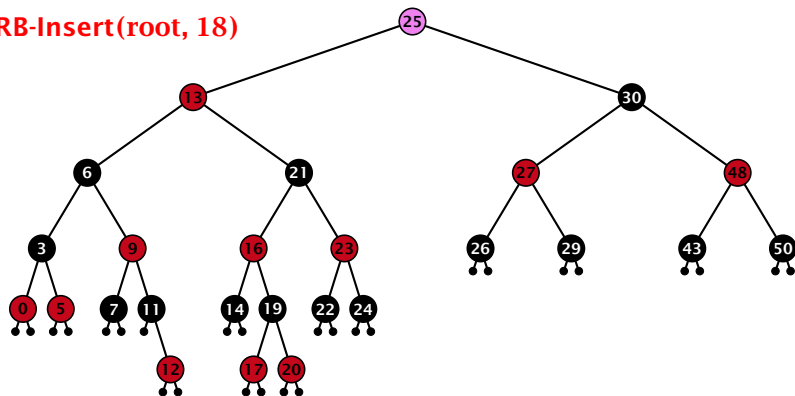


## Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

# Red Black Trees: Insert

RB-Insert(root, 18)

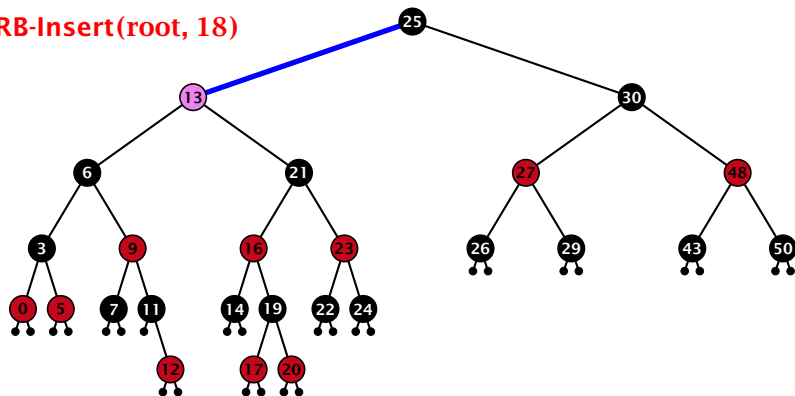


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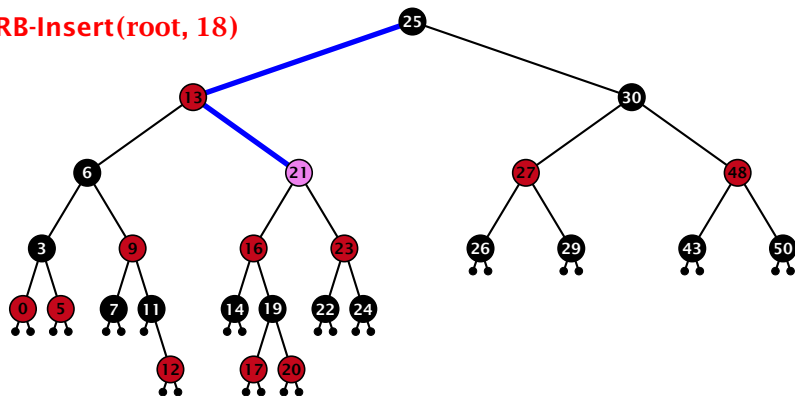


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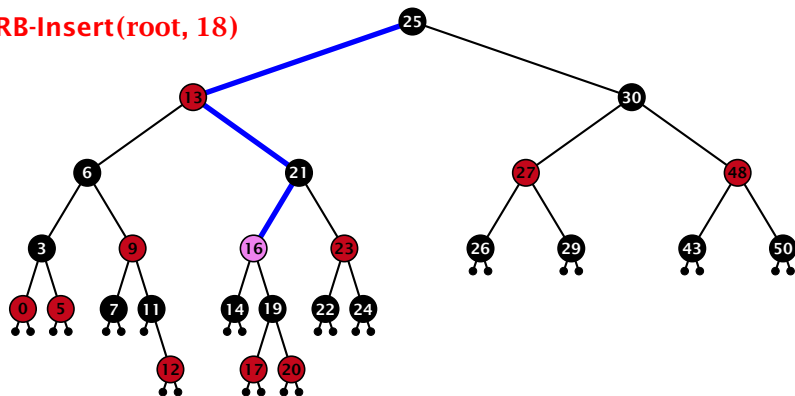


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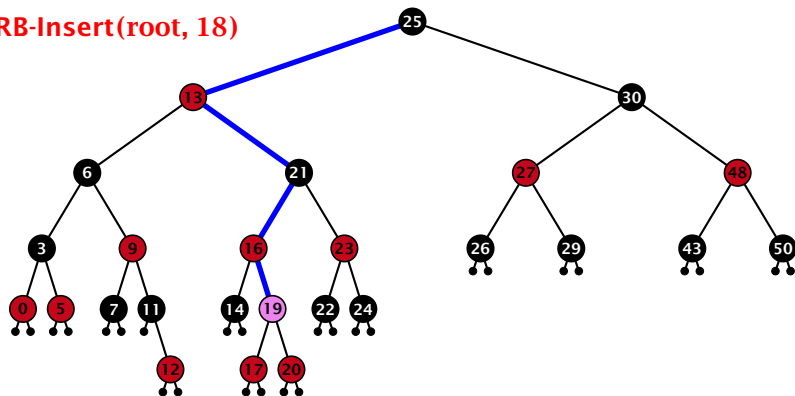


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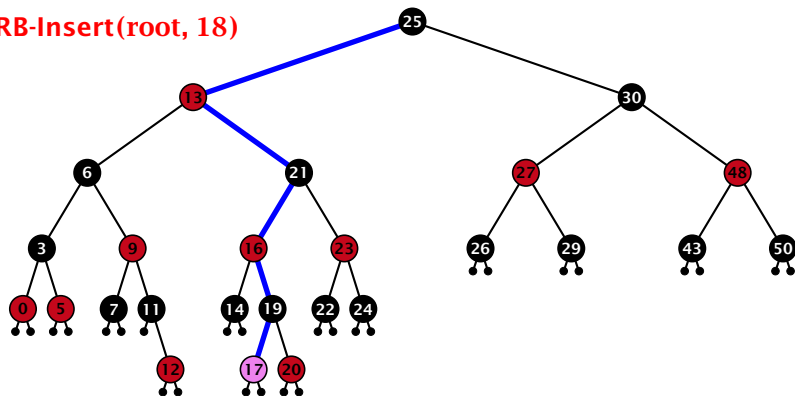


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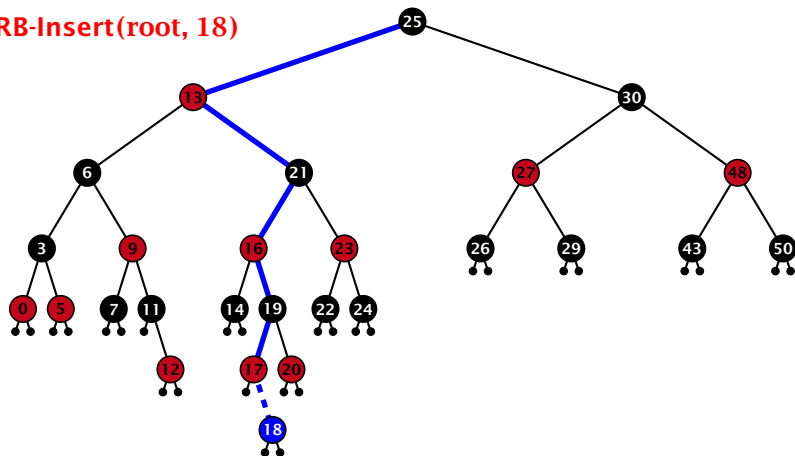
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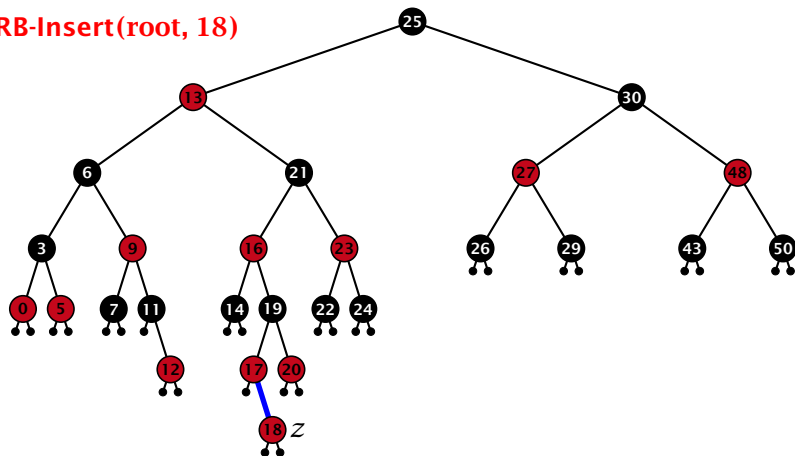


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## Invariant of the fix-up algorithm:

- ▶  $z$  is a red node
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  - either both of them are red (most important case)
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If  $z$  has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
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# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:   else Case 2: uncle black
8:     if  $z$  = right[parent[ $z$ ]] then
9:        $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:    col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:    RightRotate(gp[ $z$ ]);
12:   else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
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# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

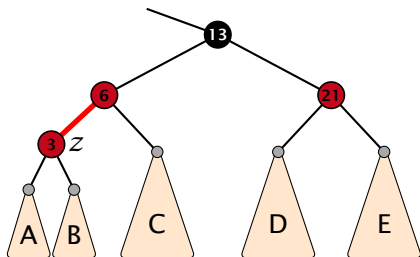
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3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
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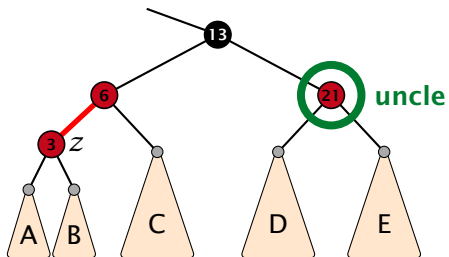
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## Case 1: Red Uncle

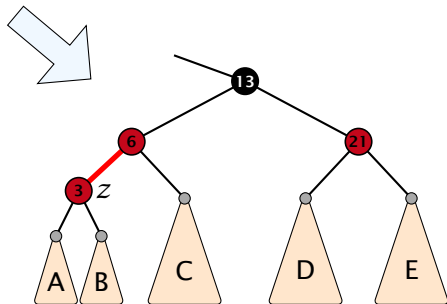
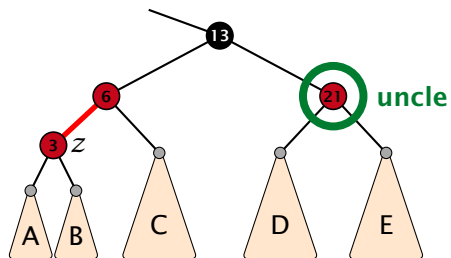


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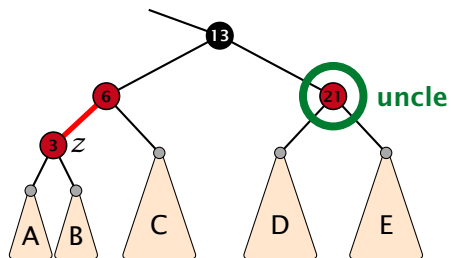




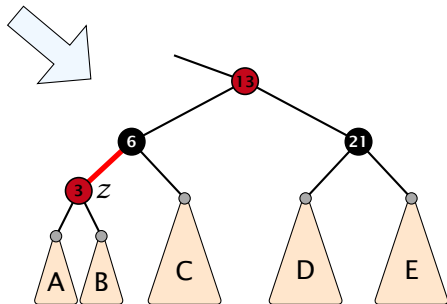
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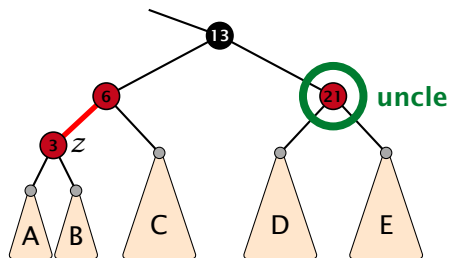
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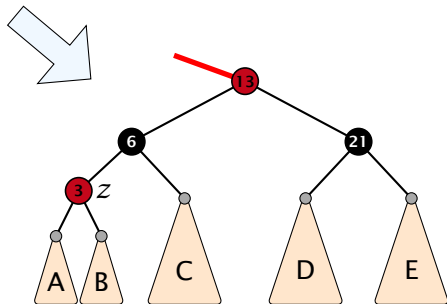
1. recolour



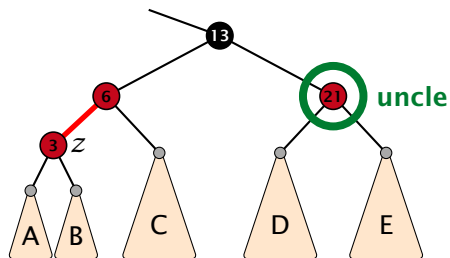
## Case 1: Red Uncle



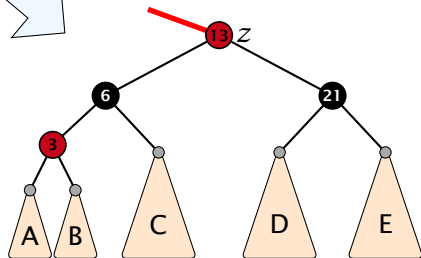
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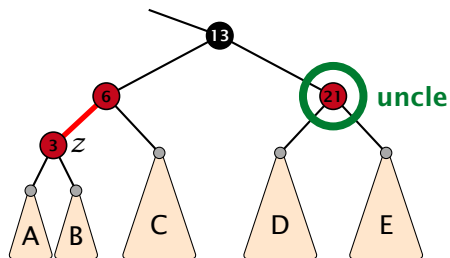
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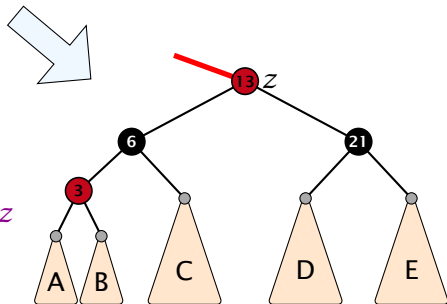
1. recolour
2. move  $z$  to grand-parent



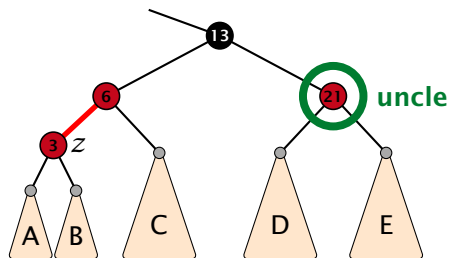
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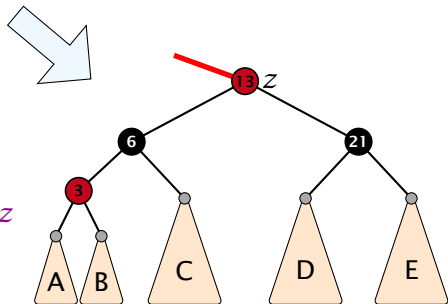
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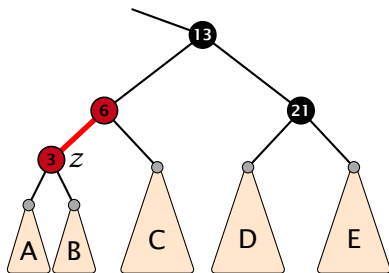


1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$
4. you made progress



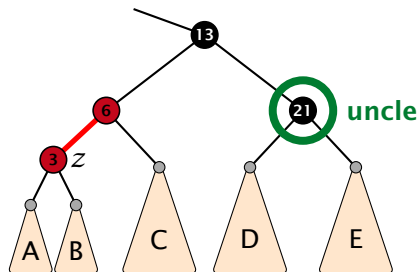
## Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



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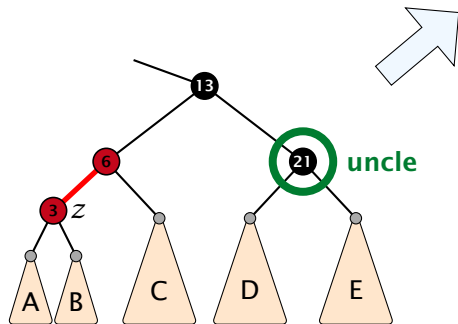
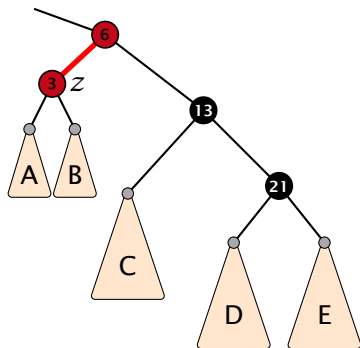
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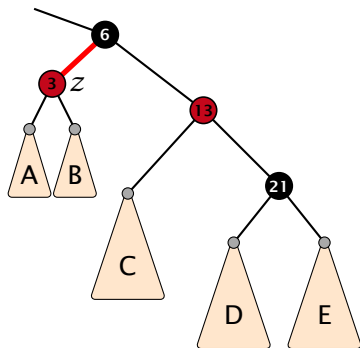
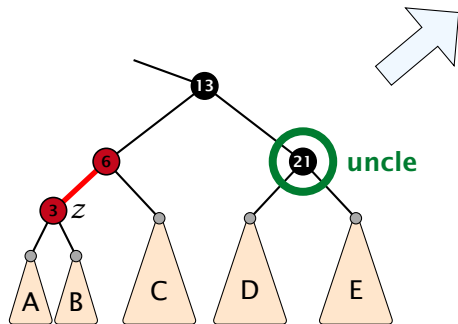
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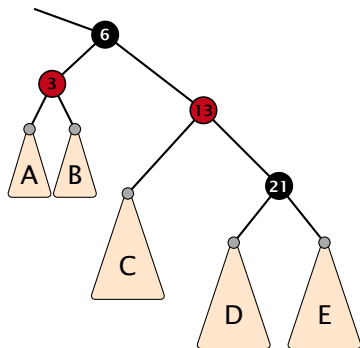
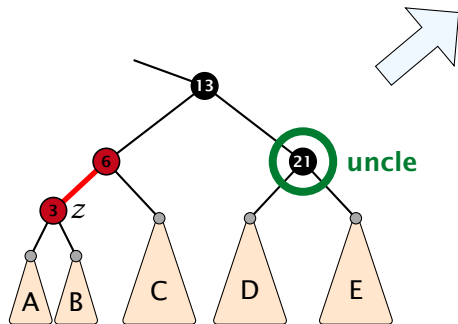
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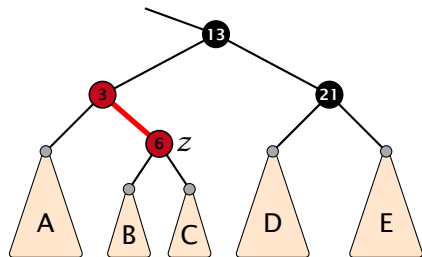
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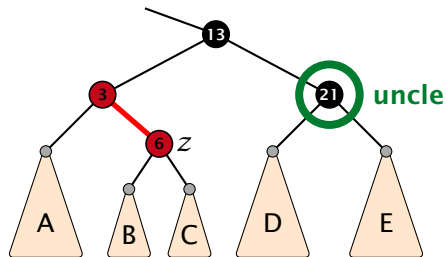
## Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



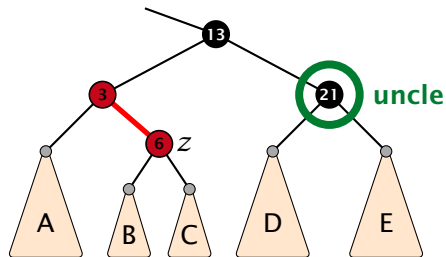
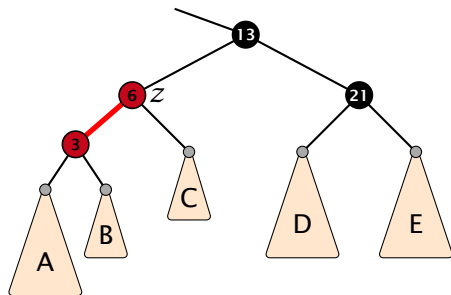
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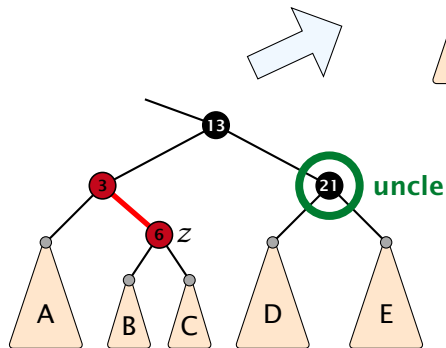
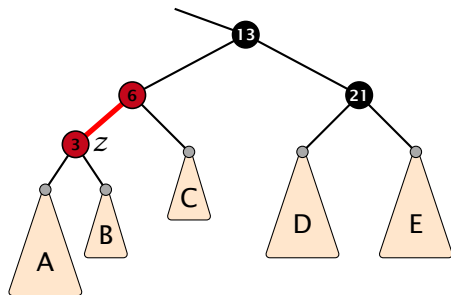
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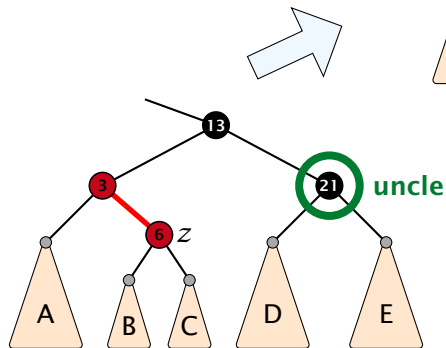
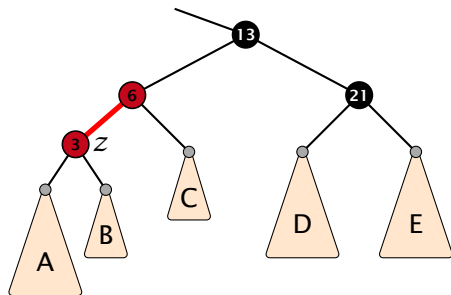
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# Red Black Trees: Insert

## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Performing Case 1 at most  $\mathcal{O}(\log n)$  times and every other case at most once, we get a red-black tree. Hence  $\mathcal{O}(\log n)$  re-colorings and at most 2 rotations.

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First do a standard delete.

If the spliced out node  $x$  was red everything is fine.

If it was black there may be the following problems.

1. Parent and child of  $x$  were red; two adjacent red vertices.

2. If you delete the root, the root may now be red.

3. Every path from an ancestor of  $x$  to a descendant leaf of  $x$  changes the number of black nodes. Black height property might be violated.

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2.  $x$  was the root, the root can't have two children.

3.  $x$  was the root, its removal of a leaf decreased height.

4.  $x$  was the root, its removal decreased the number of black nodes, Black Height property.

5.  $x$  was not the root.

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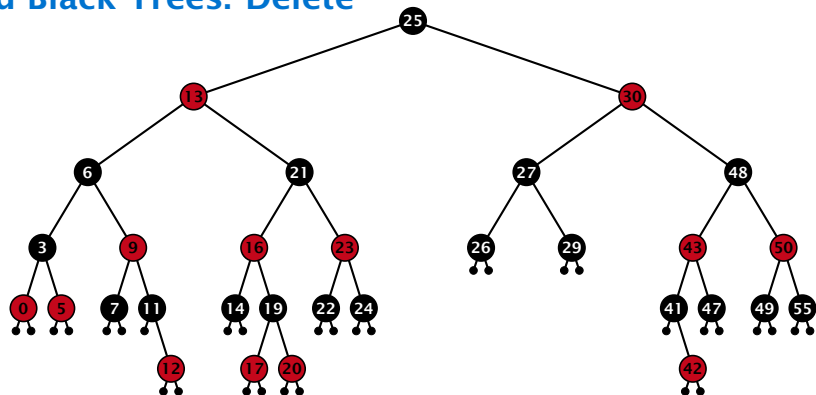
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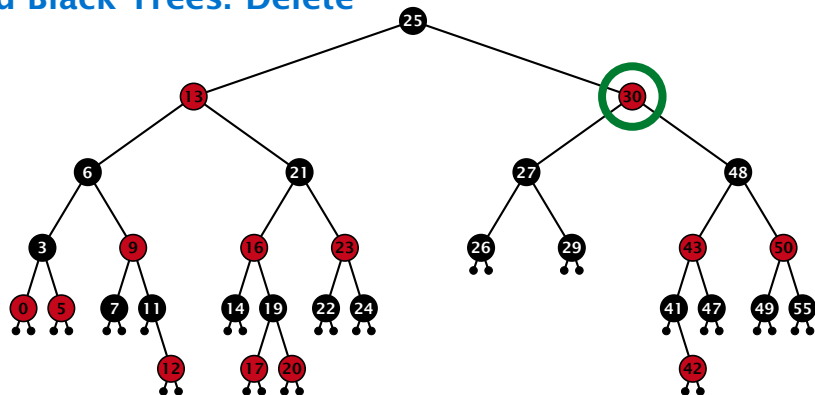
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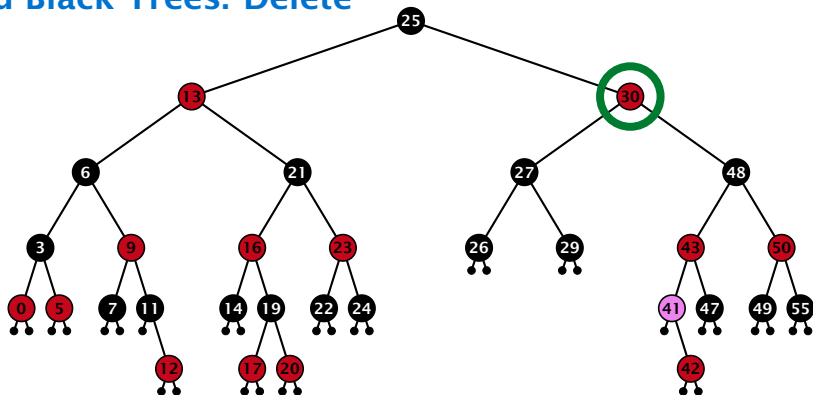


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Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

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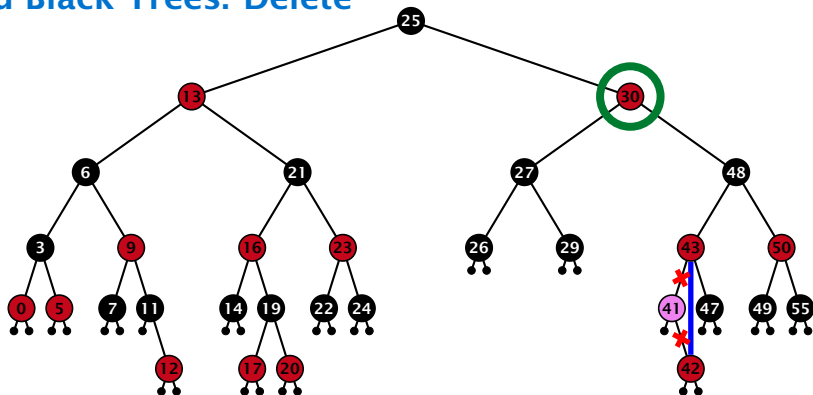


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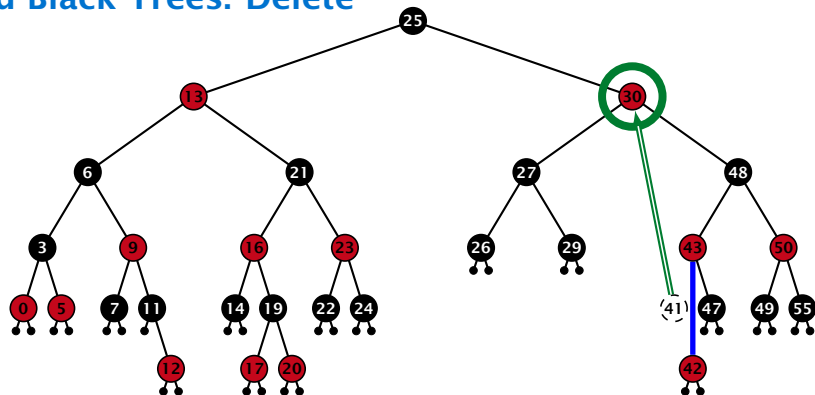


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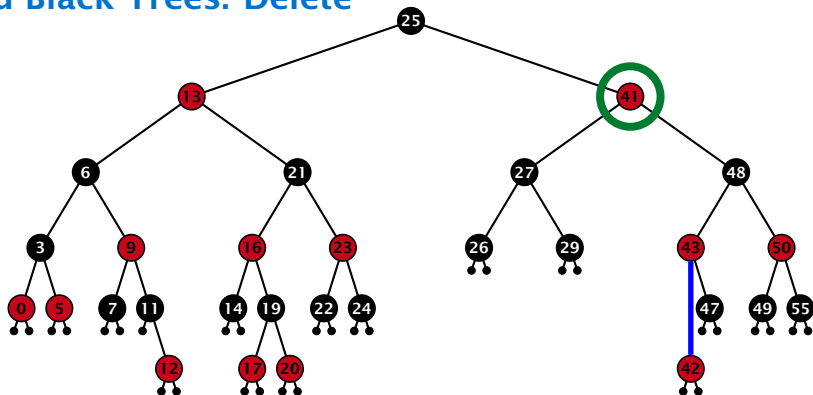
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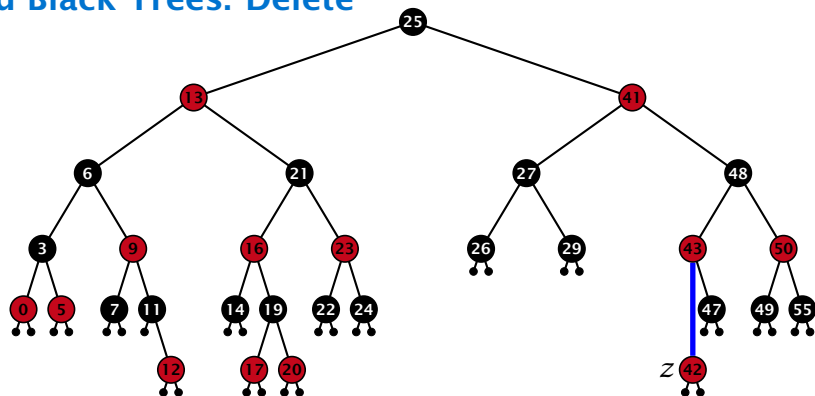


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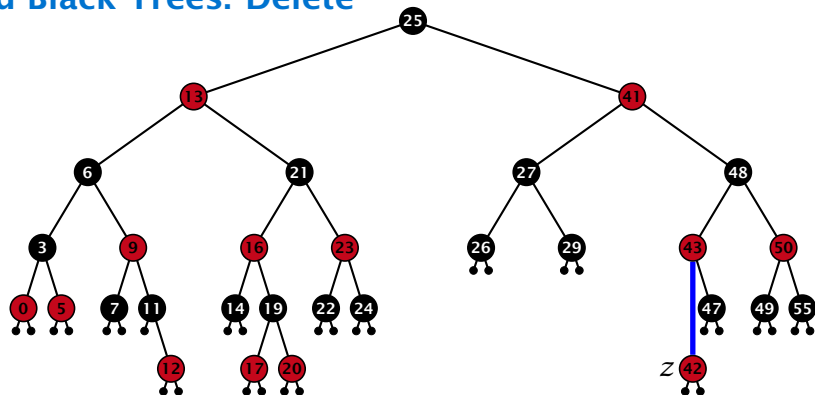
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### Delete:

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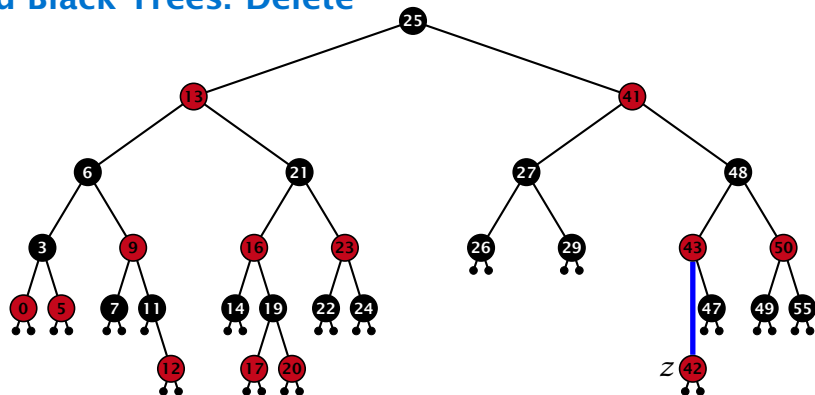
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## Invariant of the fix-up algorithm

- ▶ the node  $z$  is black
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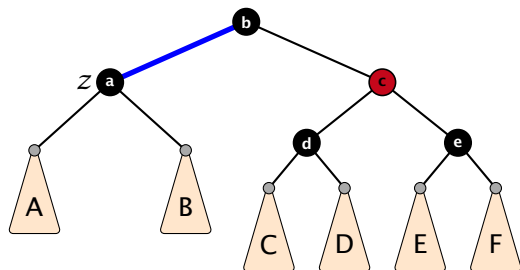
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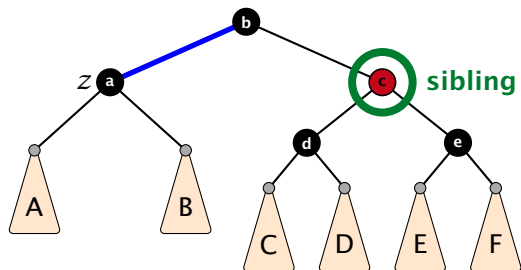


1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black  
(and parent of  $z$  is red)
4. Case 2 (special),  
or Case 3, or Case 4





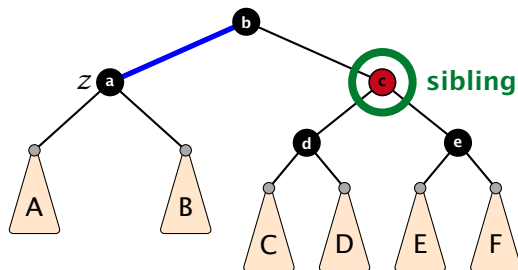
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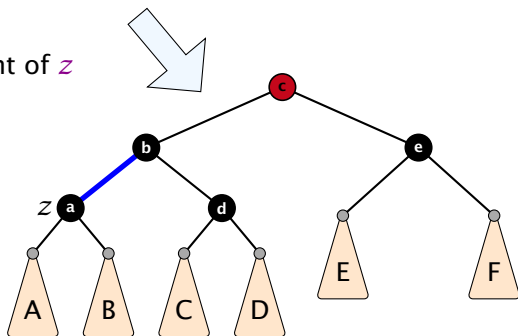


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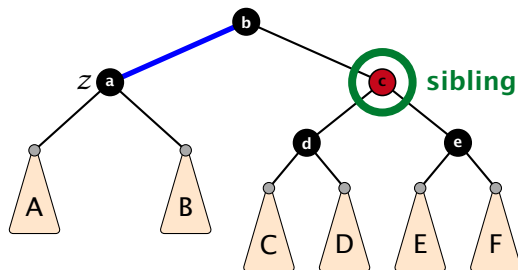
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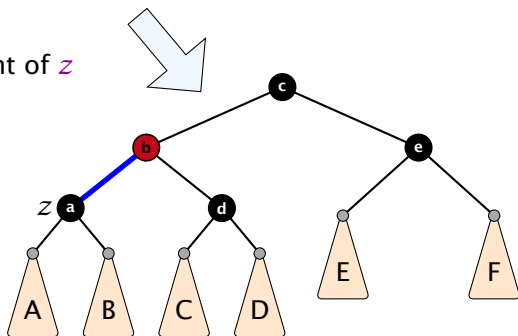
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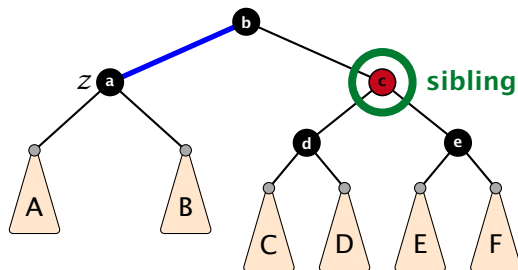
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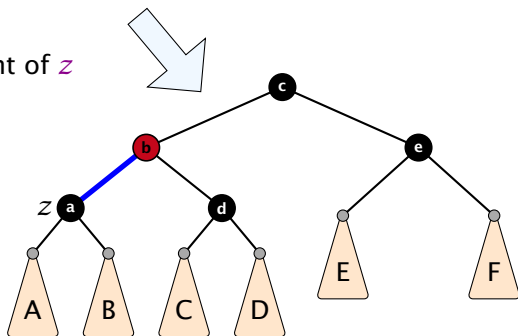
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)
4. Case 2 (special), or Case 3, or Case 4



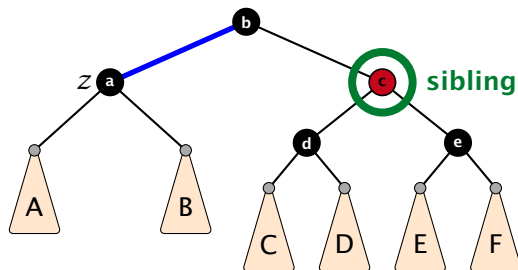
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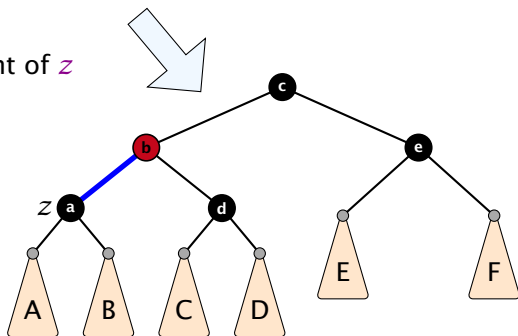
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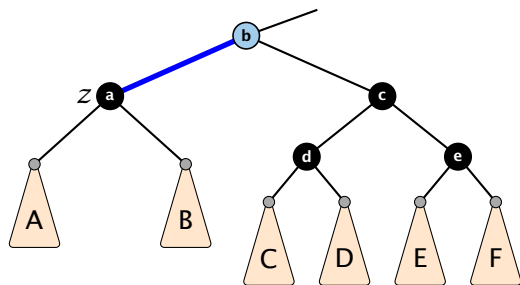
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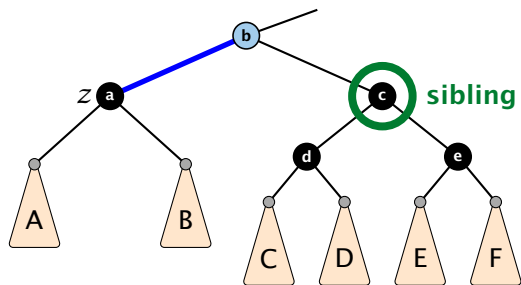
## Case 2: Sibling is black with two black children



1. re-color node  $c$
2. move fake black unit upwards
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5. if  $b$  is red we color it black and are done



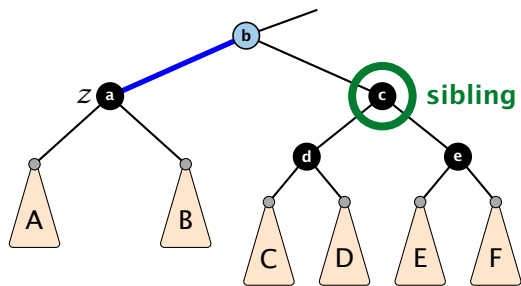
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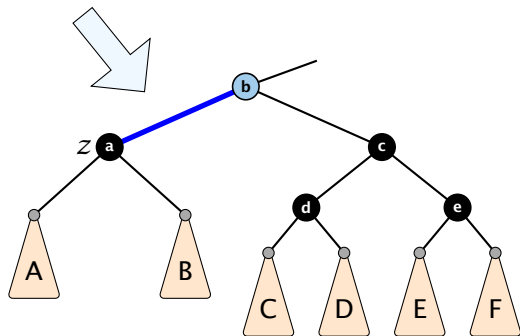
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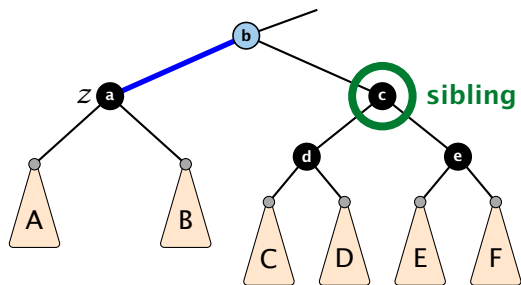


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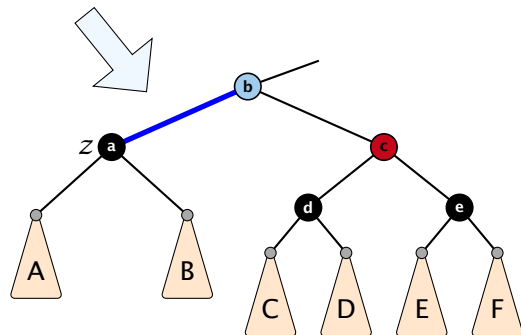




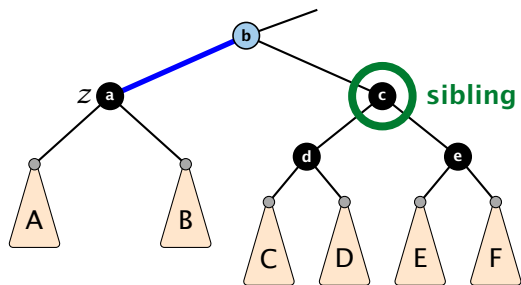
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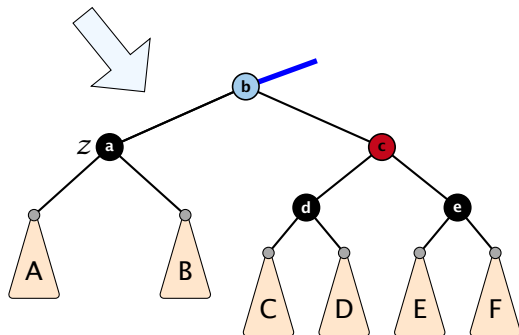
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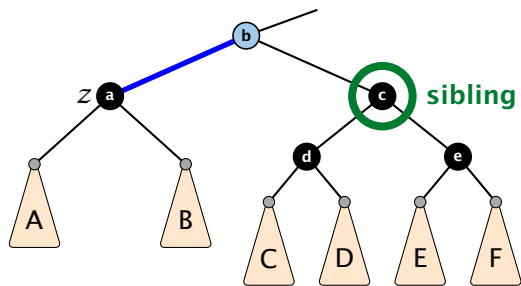
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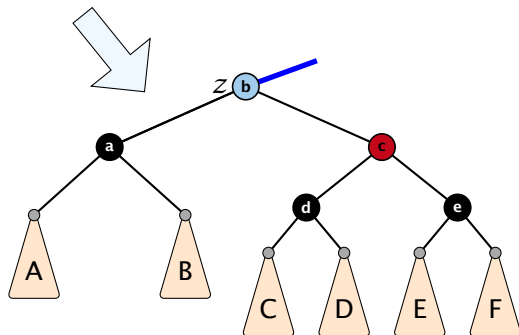
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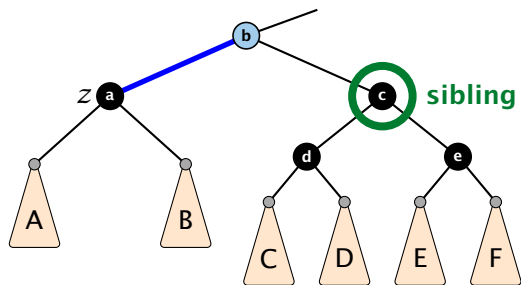
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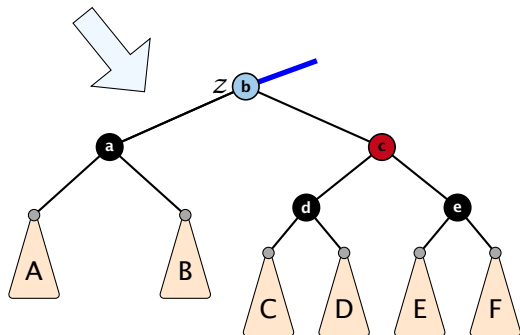
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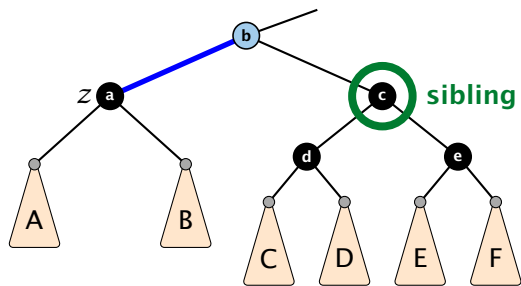
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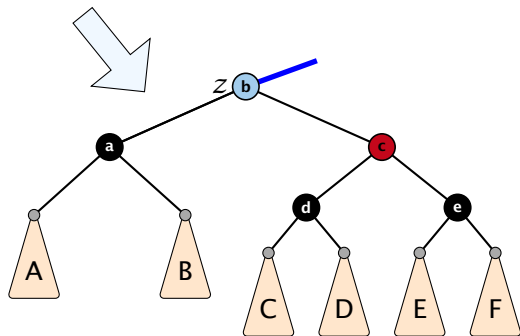
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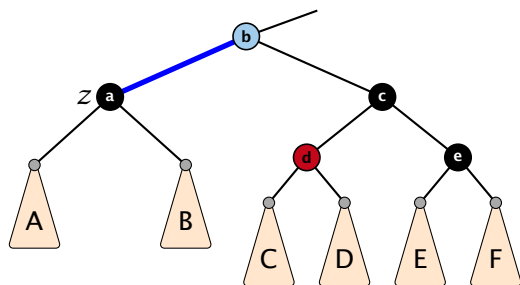


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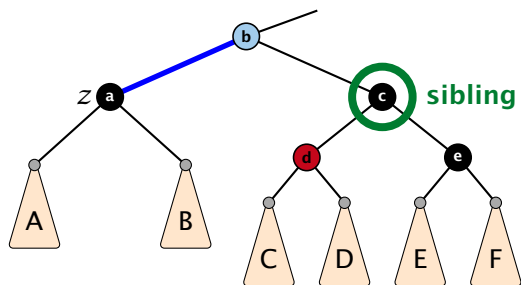
## Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor  $c$  and  $d$
3. new sibling is black with red right child (Case 4)



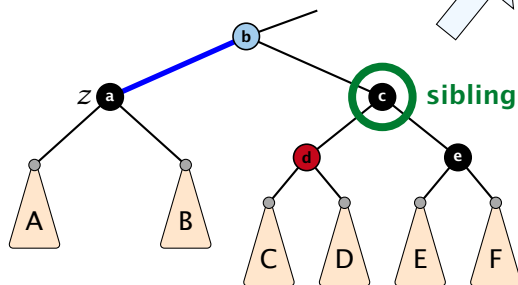
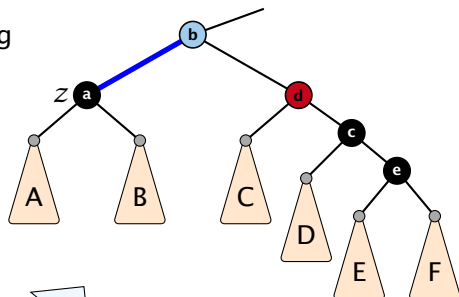
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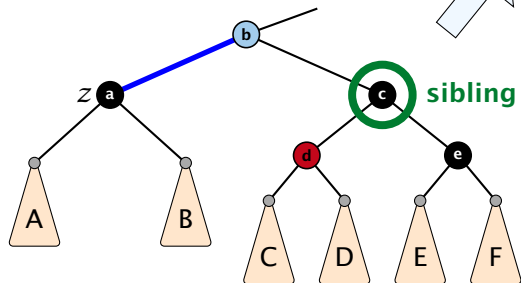
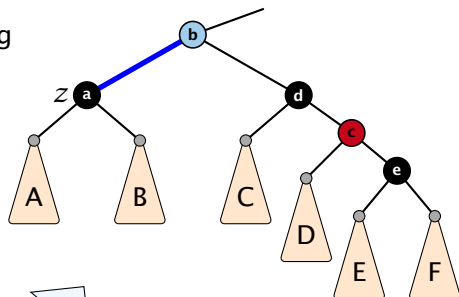
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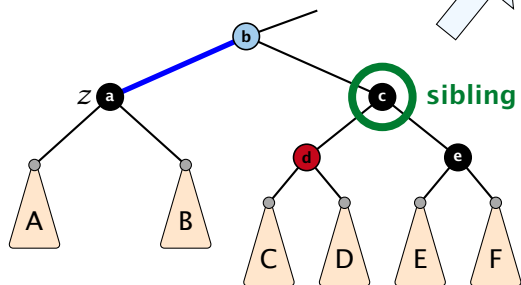
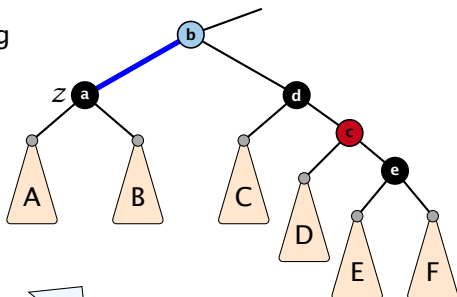
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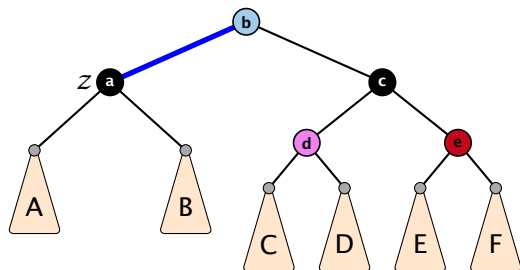


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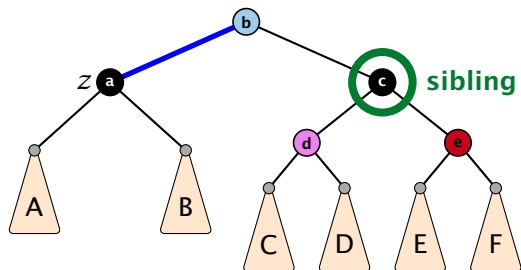
## Case 4: Sibling is black with red right child



1. left-rotate around  $b$
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4. you have a valid red black tree



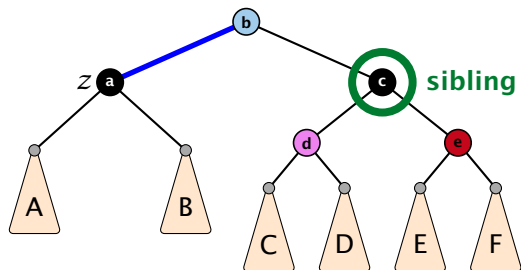
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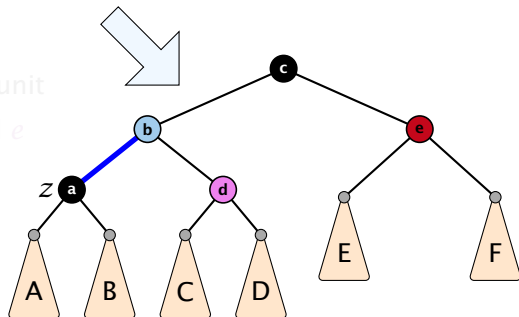
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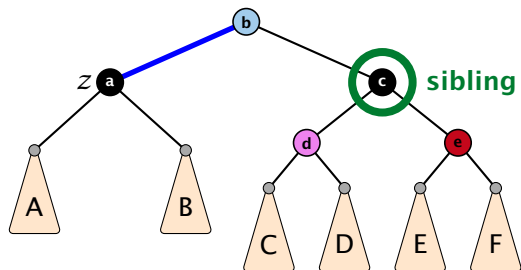
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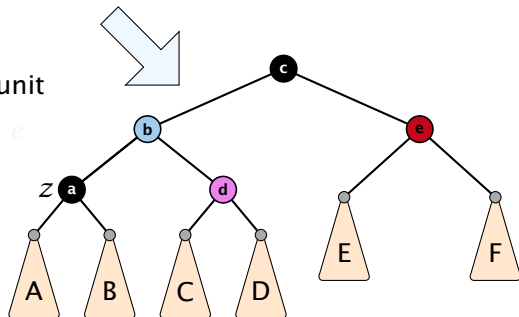
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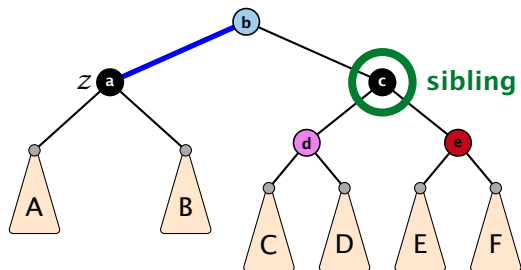
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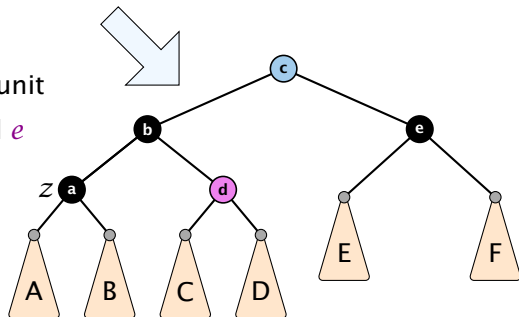
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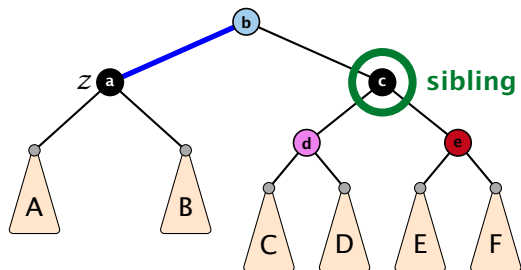
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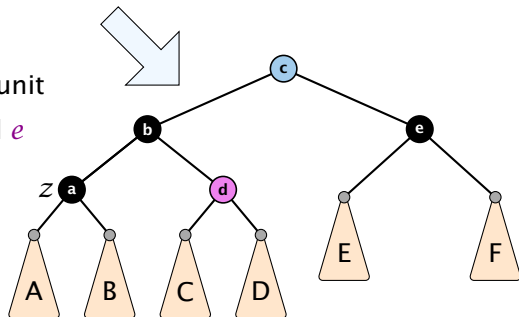
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## Running time:

- ▶ only Case 2 can repeat; but only  $h$  many steps, where  $h$  is the height of the tree
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Performing Case 2 at most  $\mathcal{O}(\log n)$  times and every other step at most once, we get a red black tree. Hence,  $\mathcal{O}(\log n)$  re-colorings and at most 3 rotations.

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# Splay Trees

## Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

## Splay Trees:

- after access, an element is moved to the root (splay)
- repeated accesses are faster
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## Splay Trees:

- when accessing an element is moved to the root (splay op)
- repeated accesses are faster
- amortized  $O(\log n)$  per op
- can be used to change the tree



# Splay Trees

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## Splay Trees:

What happens if elements are inserted in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 10?

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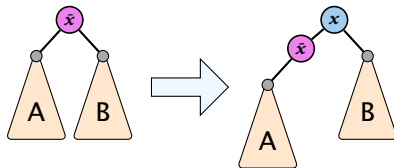
## **find( $x$ )**

- ▶ search for  $x$  according to a search tree
- ▶ let  $\tilde{x}$  be last element on search-path
- ▶ splay( $\tilde{x}$ )

# Splay Trees

## insert( $x$ )

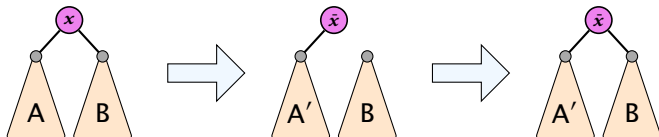
- ▶ search for  $x$ ;  $\bar{x}$  is last visited element during search (successor or predecessor of  $x$ )
- ▶ splay( $\bar{x}$ ) moves  $\bar{x}$  to the root
- ▶ insert  $x$  as new root



# Splay Trees

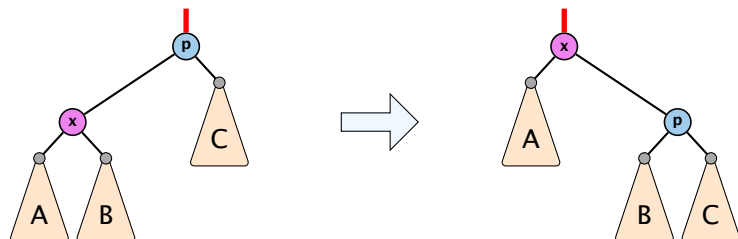
## delete( $x$ )

- ▶ search for  $x$ ; splay( $x$ ); remove  $x$
- ▶ search largest element  $\bar{x}$  in  $A$
- ▶ splay( $\bar{x}$ ) (on subtree  $A$ )
- ▶ connect root of  $B$  as right child of  $\bar{x}$





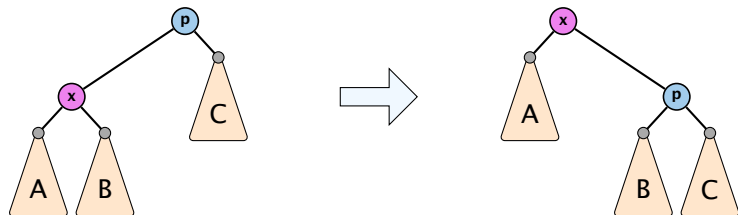
# Move to Root



## How to bring element to root?

- ▶ one (bad) option: `moveToRoot( $x$ )`
- ▶ iteratively do rotation around parent of  $x$  until  $x$  is root
- ▶ if  $x$  is left child do right rotation otw. left rotation

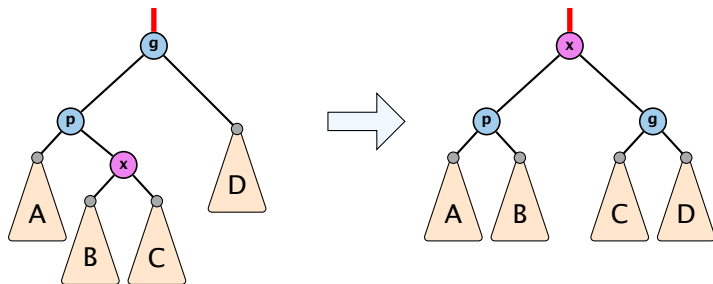
## Splay: Zig Case



**better option splay( $x$ ):**

- ▶ zig case: if  $x$  is child of root do left rotation or right rotation around parent

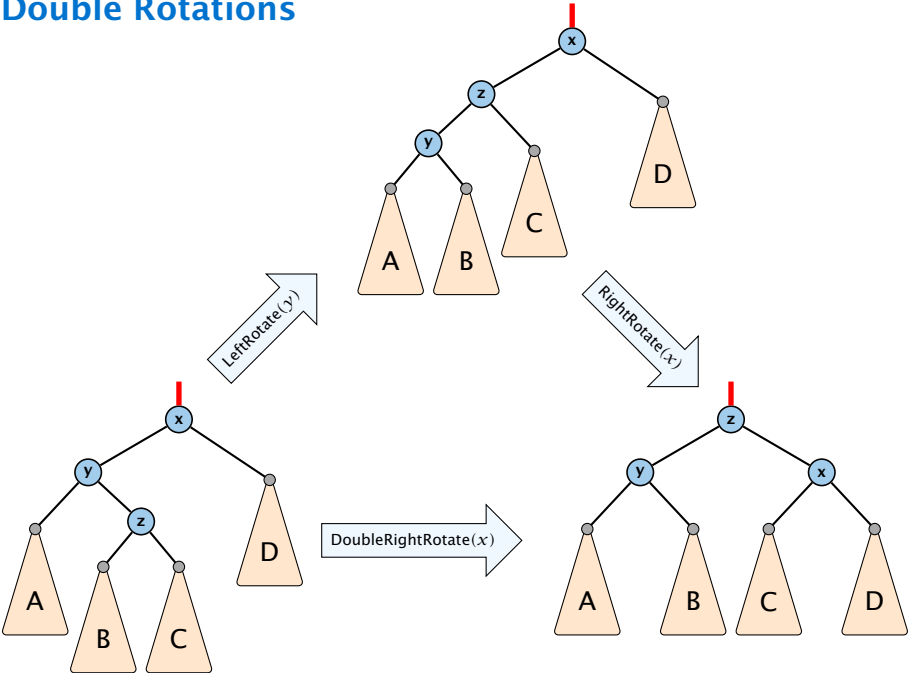
## Splay: Zigzag Case



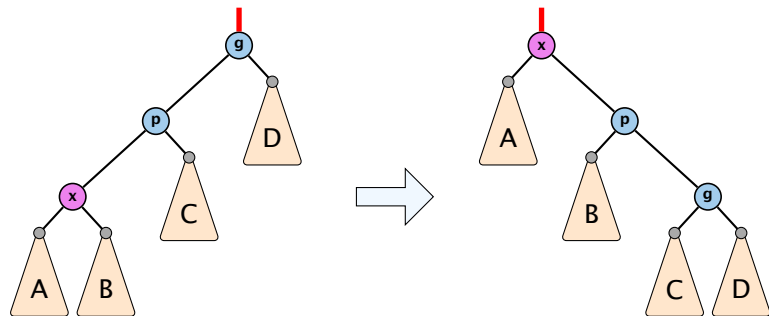
### better option $\text{splay}(x)$ :

- ▶ zigzag case: if  $x$  is right child and parent of  $x$  is left child (or  $x$  left child parent of  $x$  right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

# Double Rotations



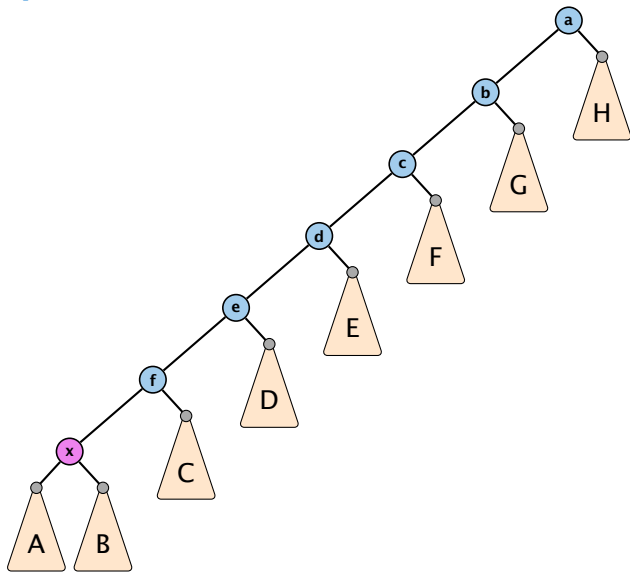
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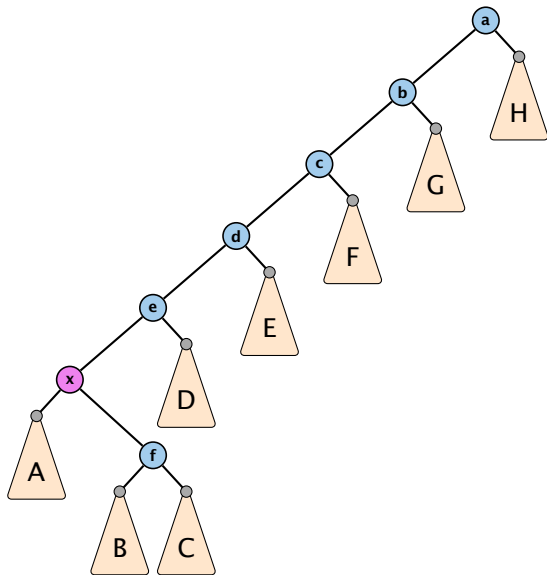
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- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

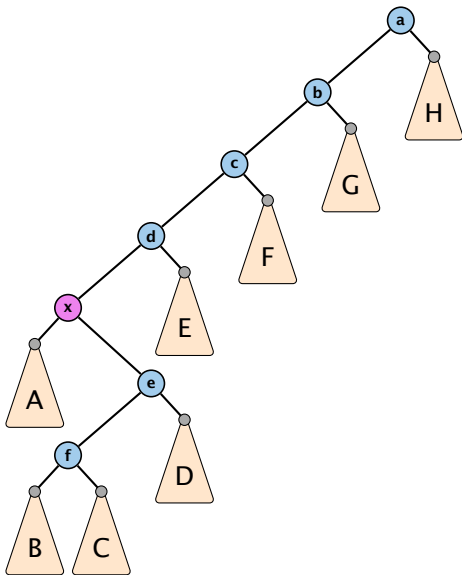
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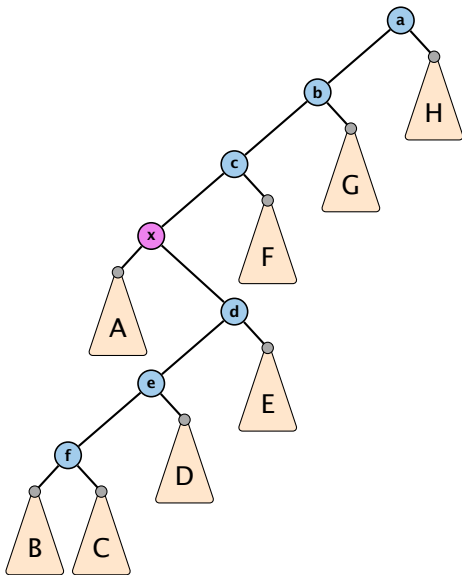


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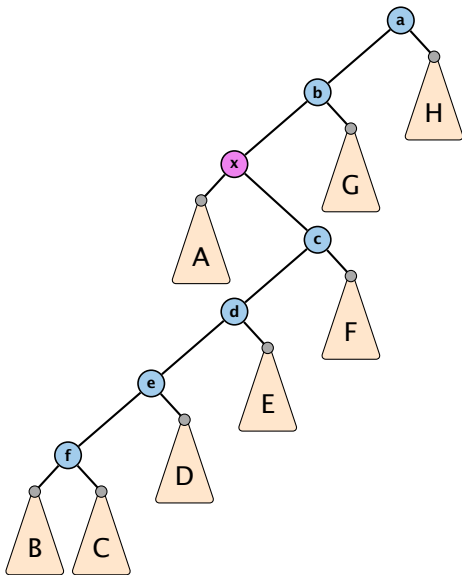




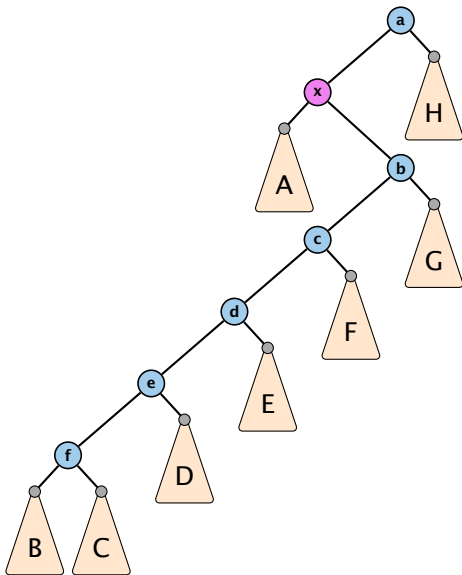
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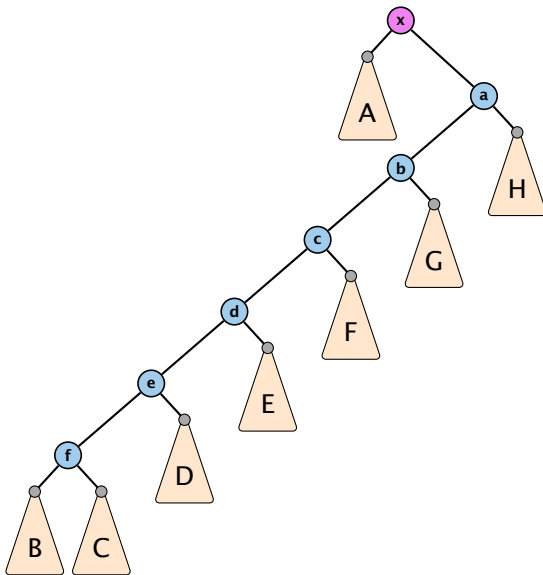
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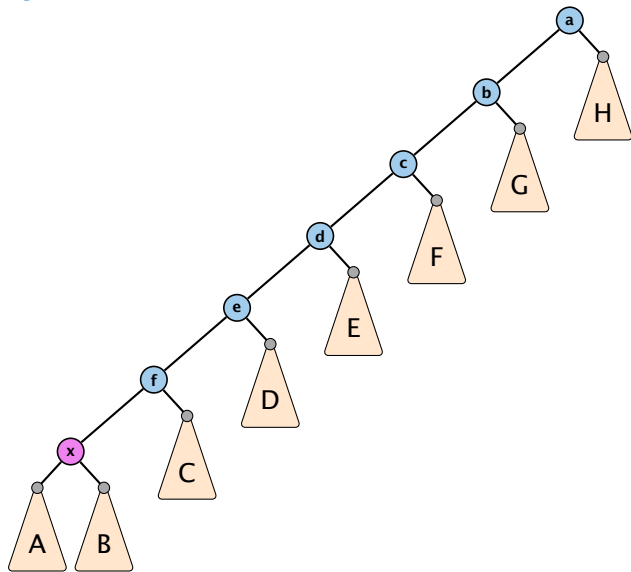
# Splay vs. Move to Root



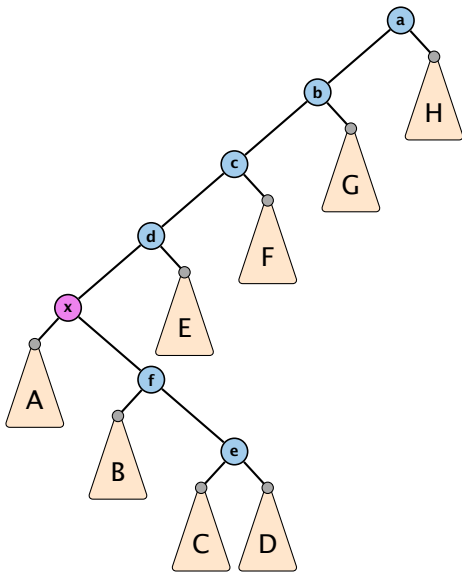
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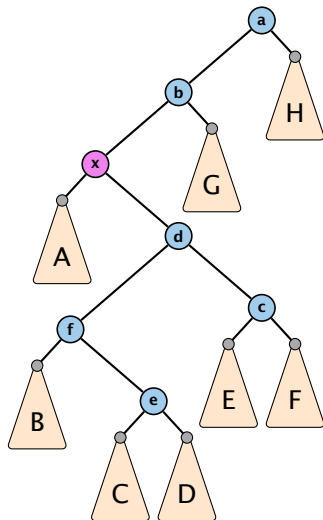
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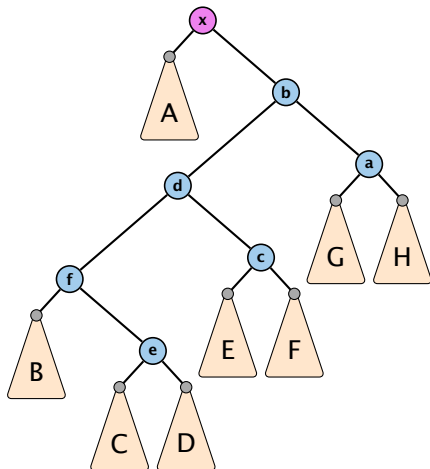
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# Static Optimality

Suppose we have a sequence of  $m$  find-operations.  $\text{find}(x)$  appears  $h_x$  times in this sequence.

The cost of a **static** search tree  $T$  is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is  $\mathcal{O}(\text{cost}(T_{\min}))$ , where  $T_{\min}$  is an **optimal static search tree**.

# Dynamic Optimality

Let  $S$  be a sequence with  $m$  find-operations.

Let  $A$  be a data-structure based on a search tree:

- ▶ the cost for accessing element  $x$  is  $1 + \text{depth}(x)$ ;
- ▶ after accessing  $x$  the tree may be re-arranged through rotations;

## Conjecture:

A splay tree that only contains elements from  $S$  has cost  $\mathcal{O}(\text{cost}(A, S))$ , for processing  $S$ .

## Lemma 5

*Splay Trees have an **amortized** running time of  $\mathcal{O}(\log n)$  for all operations.*

# Amortized Analysis

## Definition 6

A data structure with operations  $op_1(), \dots, op_k()$  has amortized running times  $t_1, \dots, t_k$  for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most  $n$  elements, and let  $k_i$  denote the number of occurrences of  $op_i()$  within this sequence. Then the actual running time must be at most  $\sum_i k_i \cdot t_i(n)$ .

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Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

# Example: Stack

## Stack

- ▶  $S.\text{push}()$
- ▶  $S.\text{pop}()$
- ▶  $S.\text{mulpop}(k)$ : removes  $k$  items from the stack. If the stack currently contains less than  $k$  items it empties the stack.
- ▶ The user has to ensure that  $\text{pop}$  and  $\text{mulpop}$  do not generate an underflow.

## Actual cost:

- ▶  $S.\text{push}()$ : cost 1.
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## Example: Binary Counter

### Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

Incrementing an  $n$ -bit binary counter may require to examine  $n$ -bits, and maybe change them.

### Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
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## Example: Binary Counter

Choose potential function  $\Phi(x) = k$ , where  $k$  denotes the number of ones in the binary representation of  $x$ .

**Amortized cost:**

Let  $z$  denote the number of consecutive ones in the least significant bit positions. An increment involves  $z$  operations, and one  $\text{O}(1)$  operation.

Hence, the amortized cost is

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Hence, the amortized cost is  $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$ .

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# Splay Trees

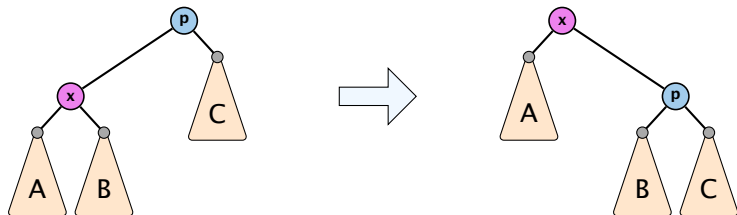
## potential function for splay trees:

- ▶ size  $s(x) = |T_x|$
- ▶ rank  $r(x) = \log_2(s(x))$
- ▶  $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

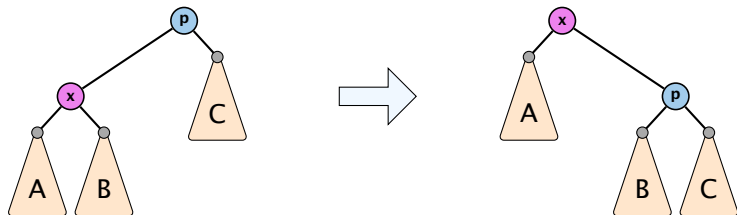
## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

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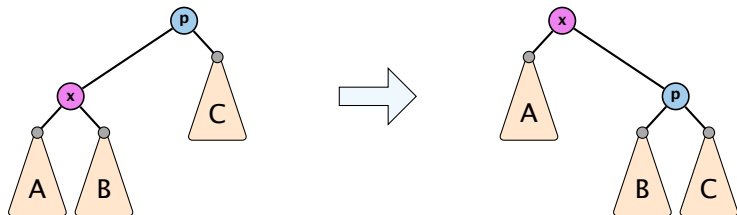
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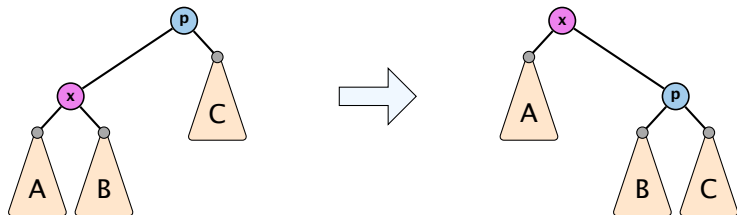
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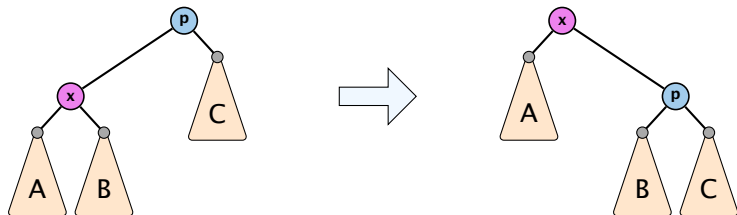
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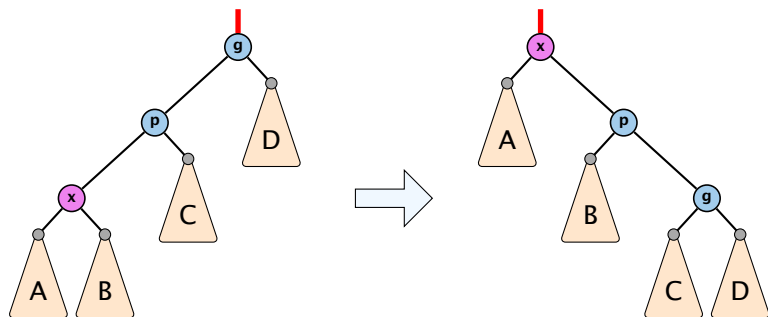
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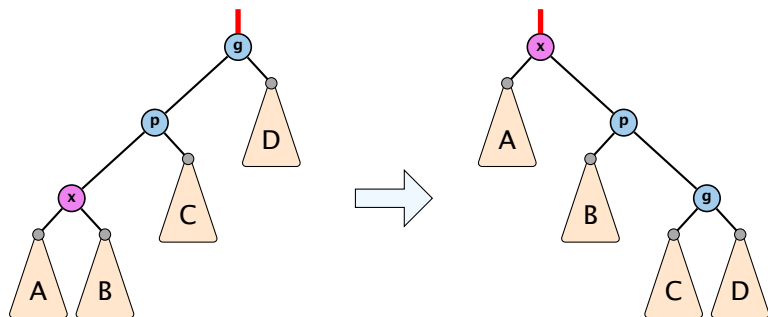
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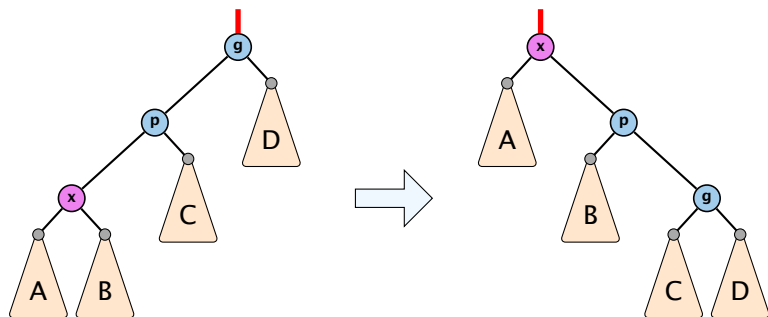


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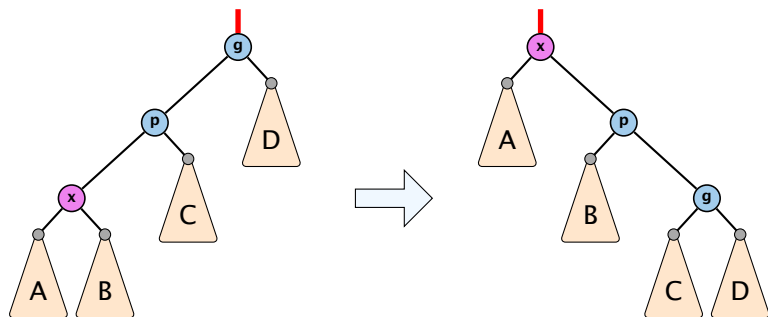
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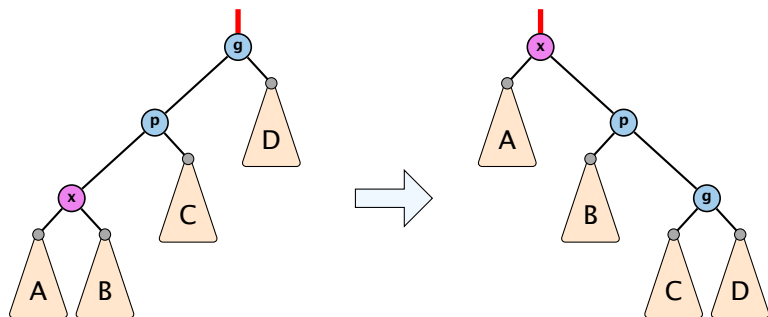
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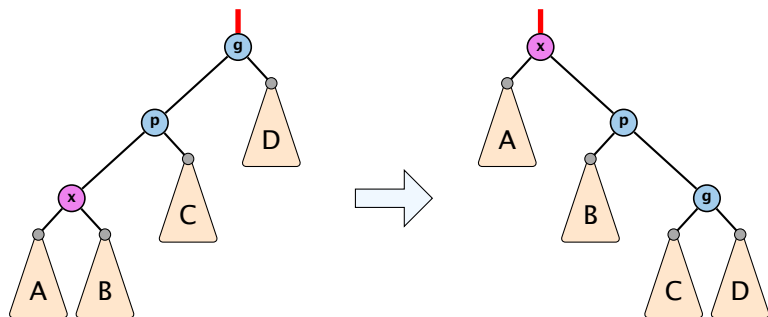
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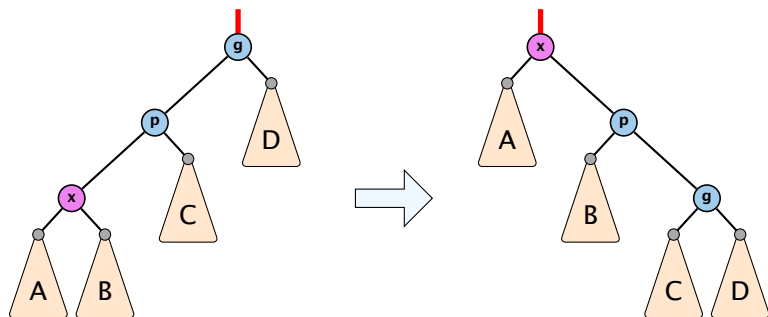
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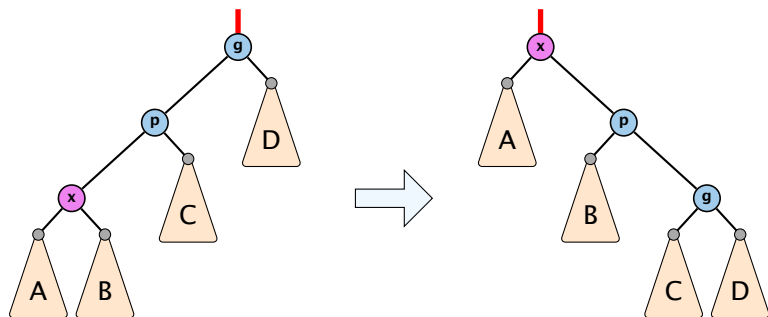
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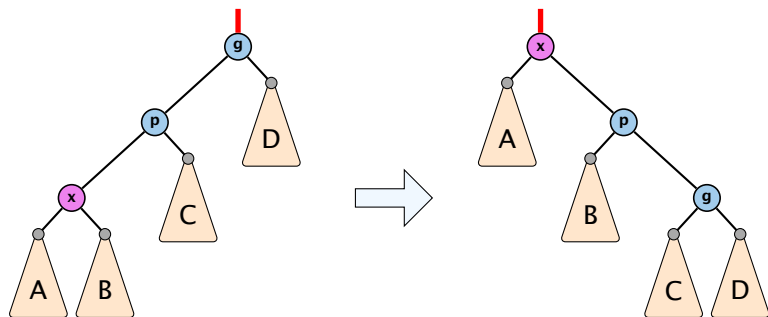
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## Splay: Zigzig Case



$$\frac{1}{2} (r(x) + r'(g) - 2r'(x))$$

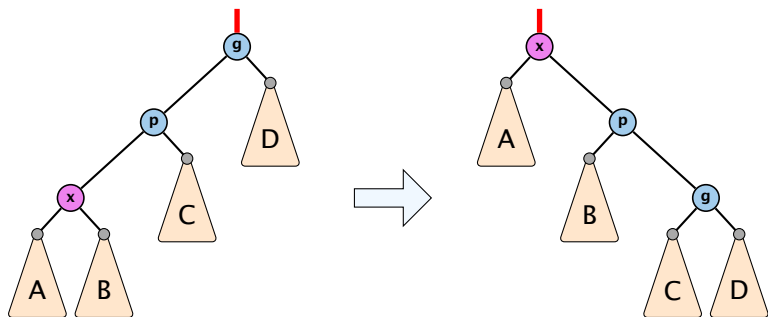
$$= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2 \log(s'(x)))$$

$$= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right)$$

$$\leq \log\left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1$$

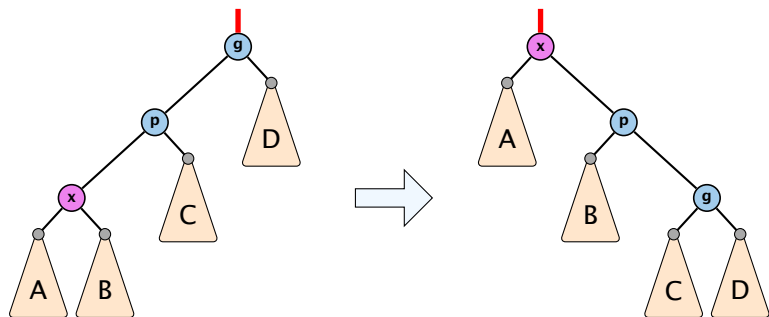


## Splay: Zigzig Case



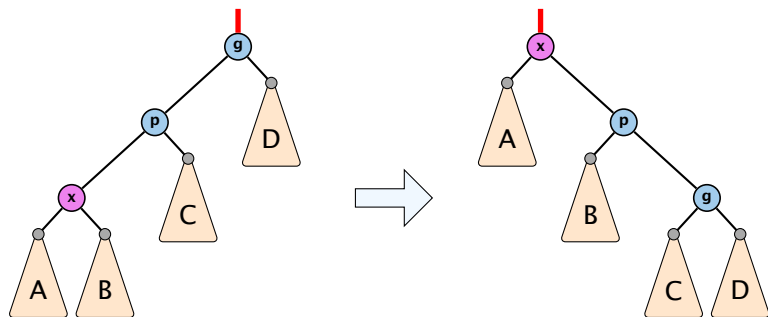
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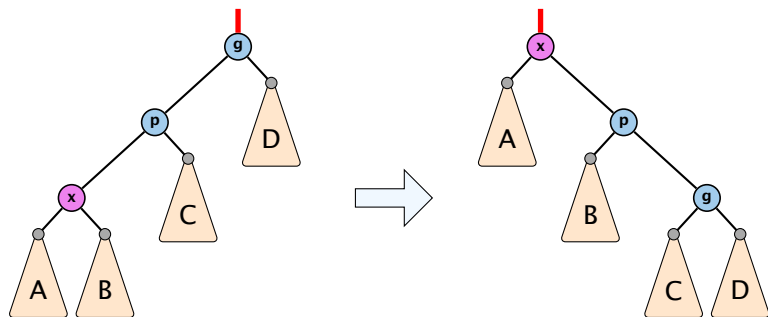
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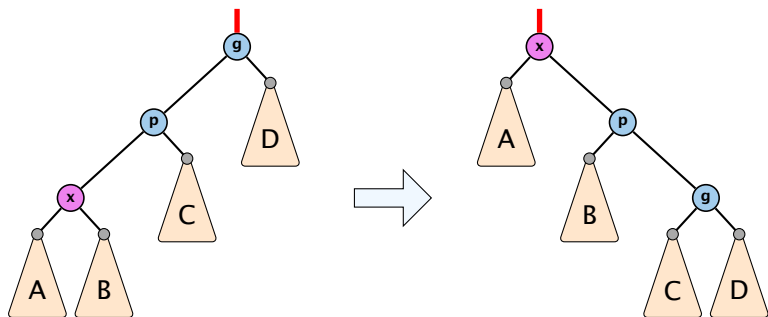
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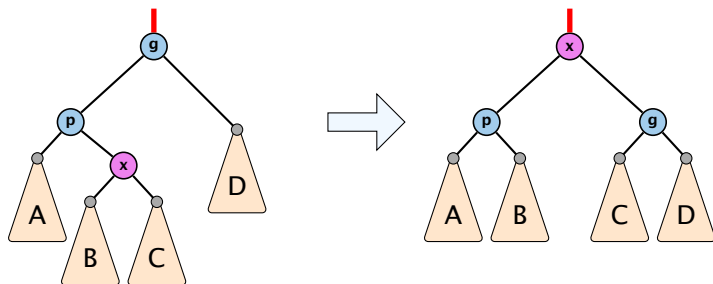
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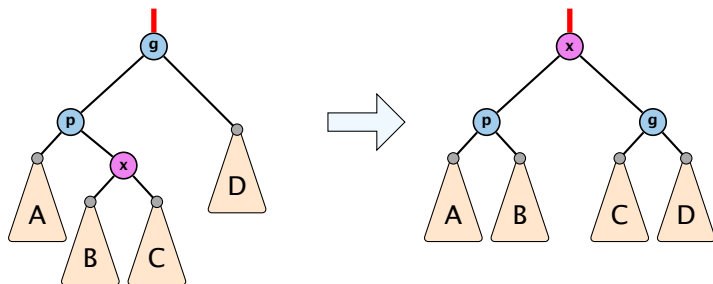
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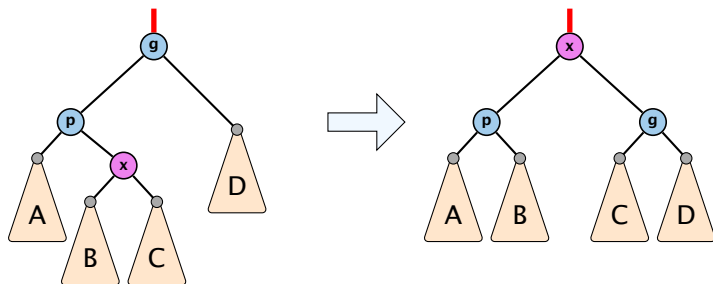
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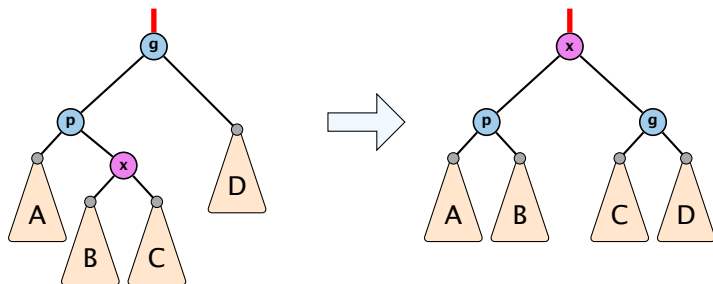
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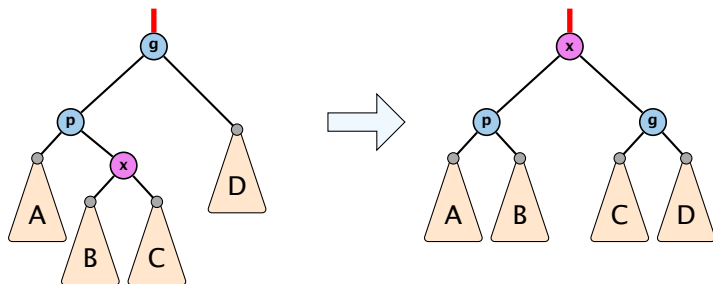


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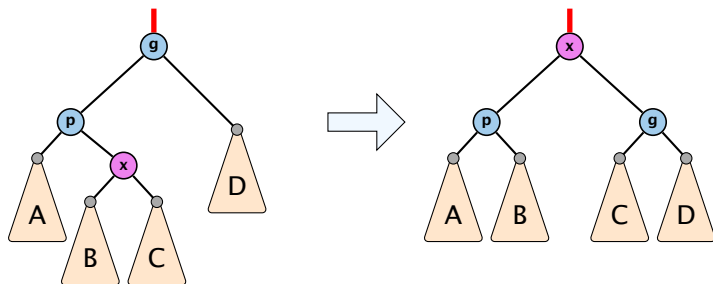
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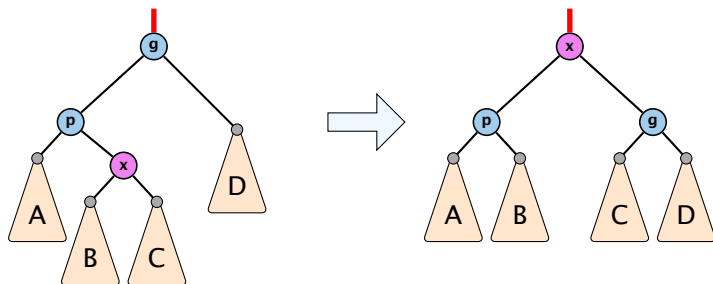
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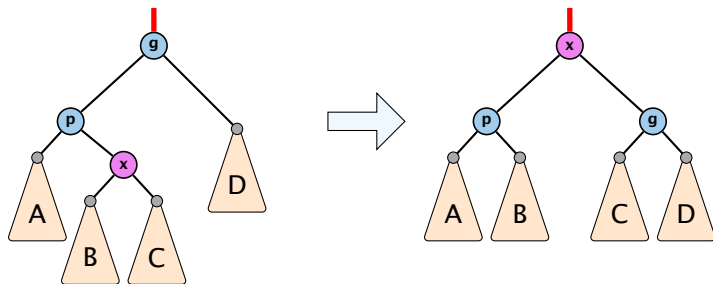
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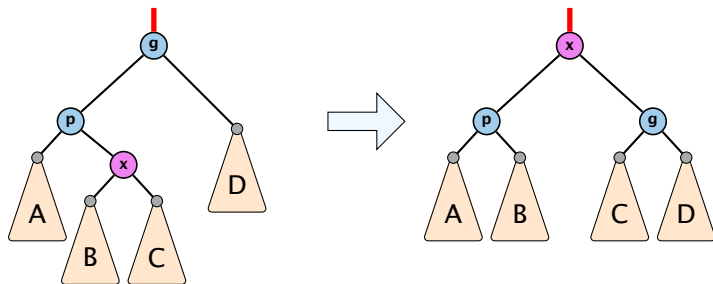
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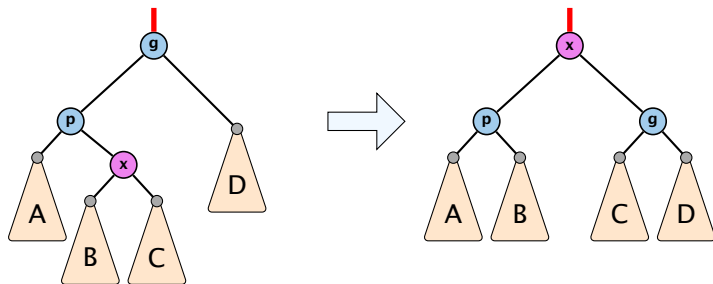
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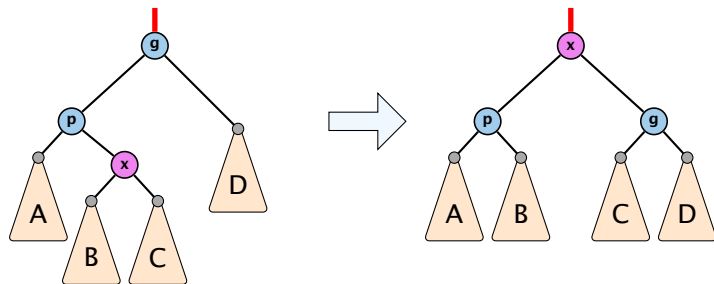
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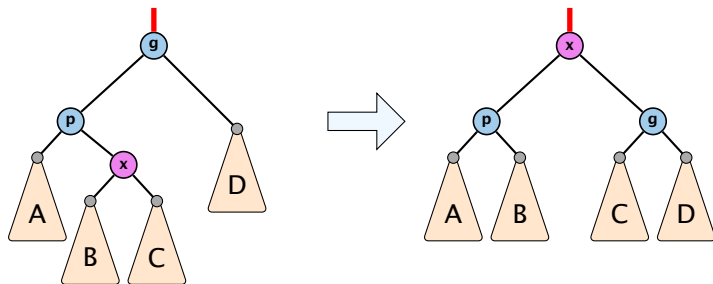
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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + r(\text{root}) - r_0(x) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
- ▶ **Search( $k$ )**: search for element with key  $k$ .
- ▶ **Delete( $x$ )**: delete element referenced by pointer  $x$ .
- ▶ **find-by-rank( $\ell$ )**: return the  $\ell$ -th element; return “error” if the data-structure contains less than  $\ell$  elements.

Augment an existing data-structure instead of developing a new one.

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## 7.4 Augmenting Data Structures

### How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node  $v$  the size of the sub-tree rooted at  $v$ .
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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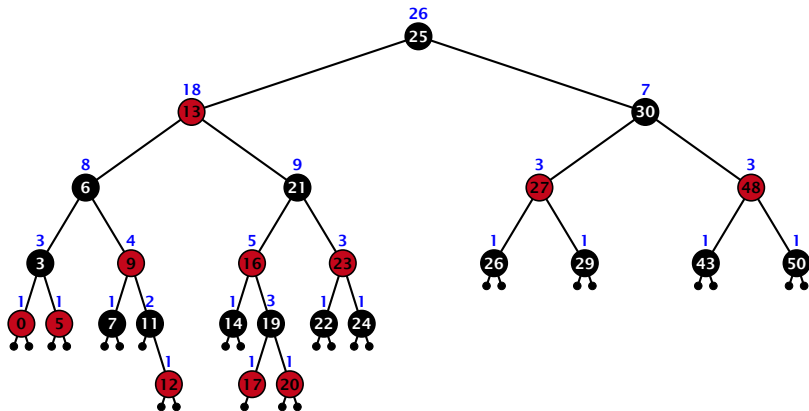
4. How does find-by-rank work?

Find-by-rank( $k$ ) := Select( $\text{root}, k$ ) with

**Algorithm 11** Select( $x, i$ )

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

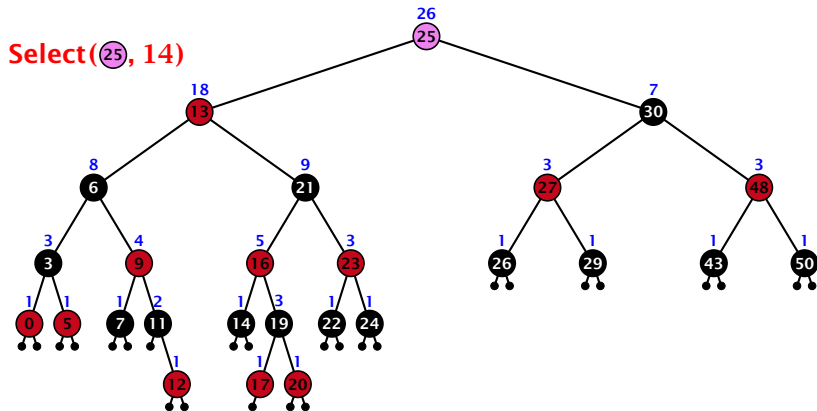
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### Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
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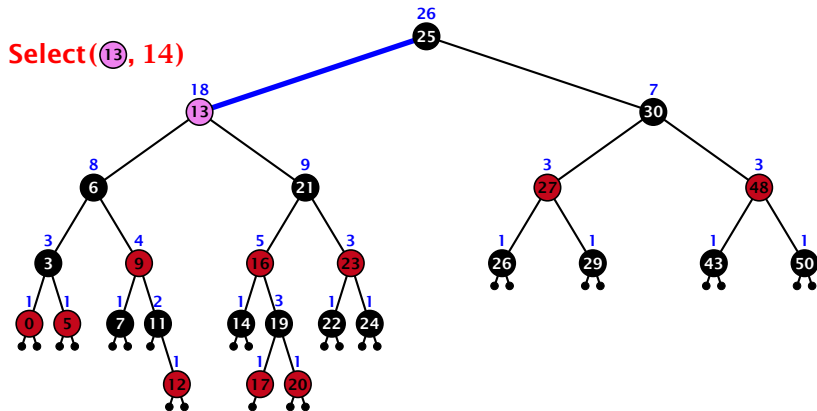
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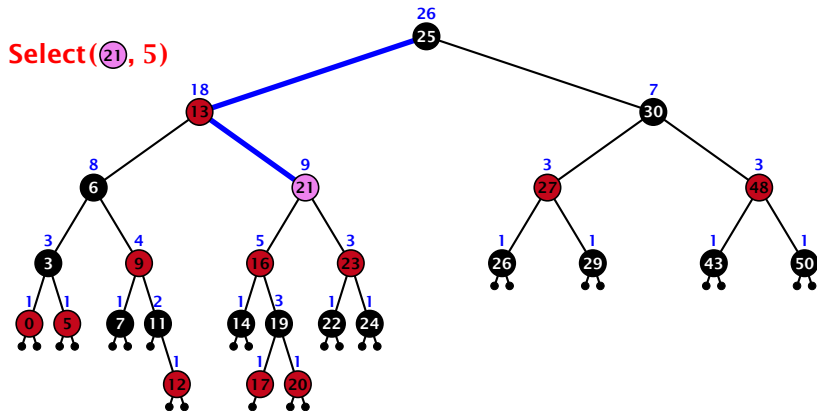
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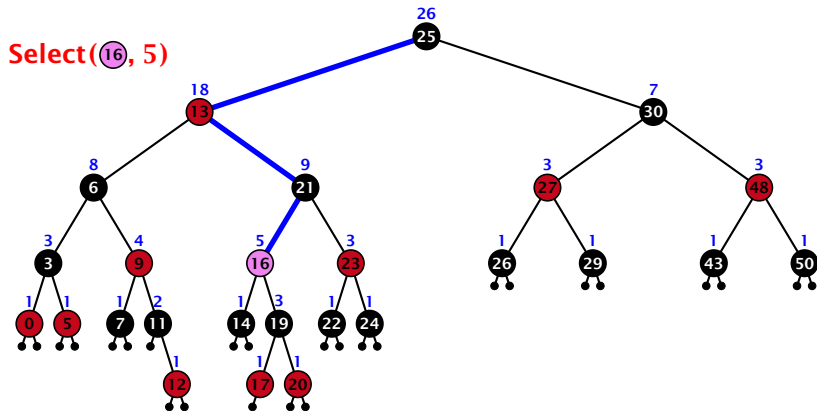


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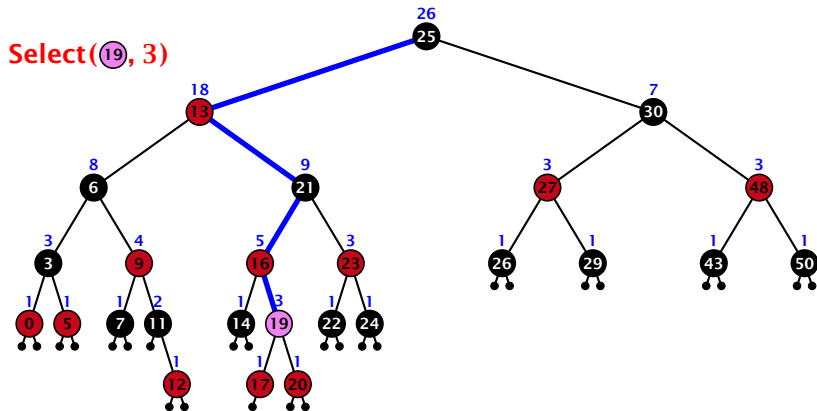
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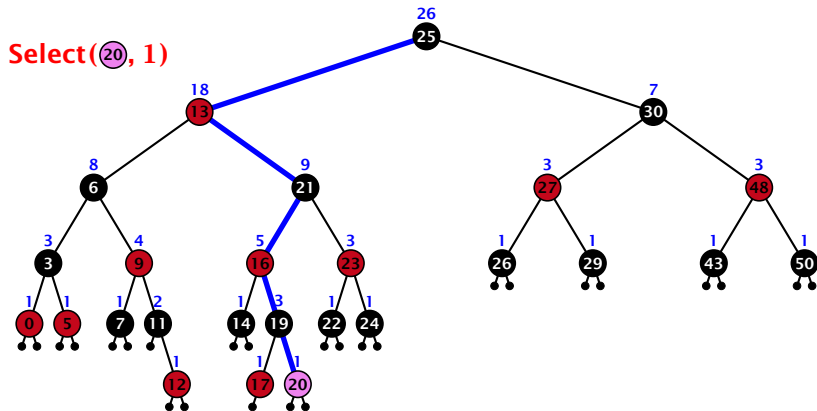
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**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

3. How do we maintain information?

**Search( $k$ ):** Nothing to do.

**Insert( $x$ ):** When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

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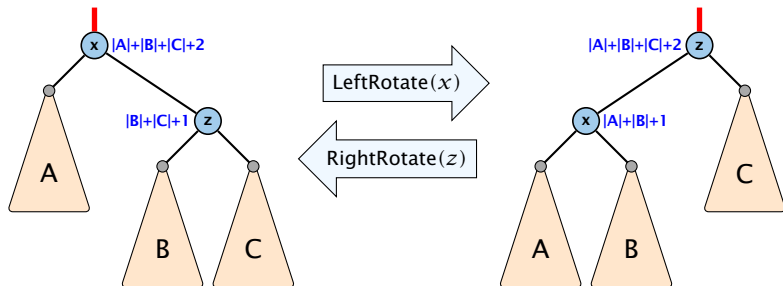
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## Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes  $x$  and  $z$  are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.



## 7.5 ( $a, b$ )-trees

### Definition 7

For  $b \geq 2a - 1$  an  $(a, b)$ -tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex  $v$  has at least  $a$  and at most  $b$  children
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2. every internal non-root vertex  $v$  has at least  $a$  and at most  $b$  children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
5. there is a special dummy leaf node with key-value  $\infty$

## 7.5 (a, b)-trees

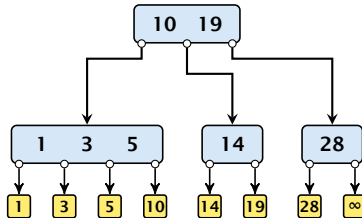
Each internal node  $v$  with  $d(v)$  children stores  $d - 1$  keys  $k_1, \dots, k_{d-1}$ . The  $i$ -th subtree of  $v$  fulfills

$$k_{i-1} < \text{key in } i\text{-th sub-tree} \leq k_i ,$$

where we use  $k_0 = -\infty$  and  $k_d = \infty$ .

## 7.5 (a, b)-trees

### Example 8





## 7.5 ( $a, b$ )-trees

### Variants

- ▶ The dummy leaf element may not exist; it only makes implementation more convenient.
- ▶ Variants in which  $b = 2a$  are commonly referred to as  $B$ -trees.
- ▶ A  $B$ -tree usually refers to the variant in which keys and data are stored at internal nodes.
- ▶ A  $B^+$  tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- ▶ A  $B^*$  tree requires that a node is at least  $2/3$ -full as opposed to  $1/2$ -full (the requirement of a  $B$ -tree).

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## Lemma 9

Let  $T$  be an  $(a, b)$ -tree for  $n > 0$  elements (i.e.,  $n + 1$  leaf nodes) and height  $h$  (number of edges from root to a leaf vertex). Then

1.  $2a^{h-1} \leq n + 1 \leq b^h$
2.  $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

Proof.

The root has degree at least 2 and  $a$  children.

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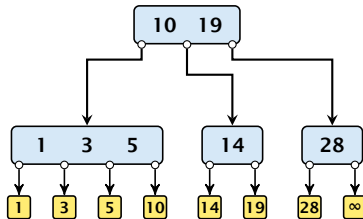
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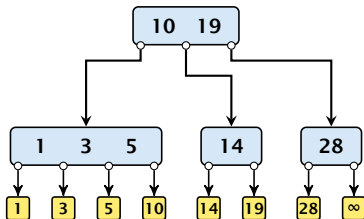


# Search



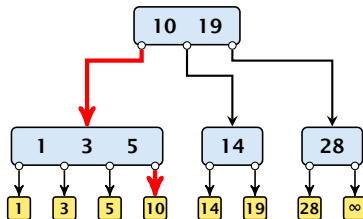
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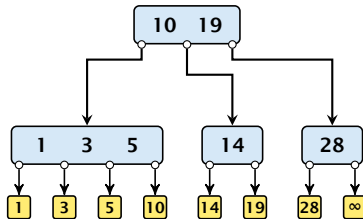
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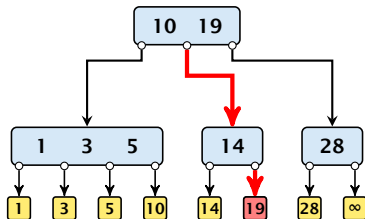
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## Search(19)

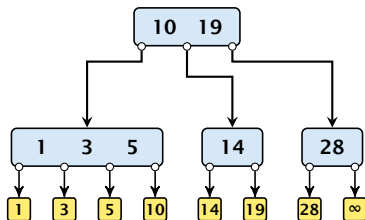


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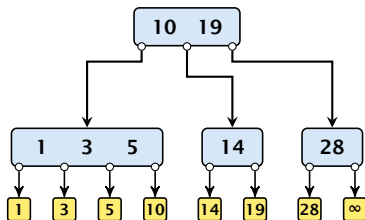
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Time:  $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$ , if the individual nodes are organized as linear lists.

# Insert

Insert element  $x$ :

- ▶ Follow the path as if searching for  $\text{key}[x]$ .
- ▶ If this search ends in leaf  $\ell$ , insert  $x$  before this leaf.
- ▶ For this add  $\text{key}[x]$  to the key-list of the last internal node  $v$  on the path.
- ▶ If after the insert  $v$  contains  $b$  nodes, do  $\text{Rebalance}(v)$ .

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Rebalance( $v$ ):

- ▶ Let  $k_i$ ,  $i = 1, \dots, b$  denote the keys stored in  $v$ .
- ▶ Let  $j := \lfloor \frac{b+1}{2} \rfloor$  be the middle element.
- ▶ Create two nodes  $v_1$ , and  $v_2$ .  $v_1$  gets all keys  $k_1, \dots, k_{j-1}$  and  $v_2$  gets keys  $k_{j+1}, \dots, k_b$ .
- ▶ Both nodes get at least  $\lfloor \frac{b-1}{2} \rfloor$  keys, and have therefore degree at least  $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$  since  $b \geq 2a - 1$ .
- ▶ They get at most  $\lceil \frac{b-1}{2} \rceil$  keys, and have therefore degree at most  $\lceil \frac{b-1}{2} \rceil + 1 \leq b$  (since  $b \geq 2$ ).
- ▶ The key  $k_j$  is promoted to the parent of  $v$ . The current pointer to  $v$  is altered to point to  $v_1$ , and a new pointer (to the right of  $k_j$ ) in the parent is added to point to  $v_2$ .
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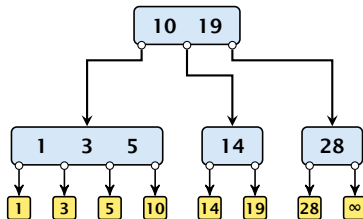
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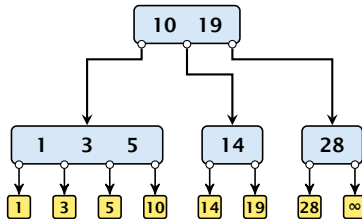
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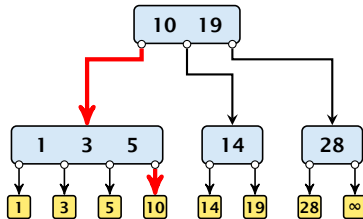
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## Insert(8)



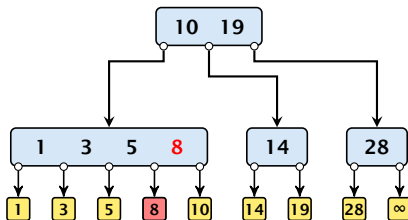
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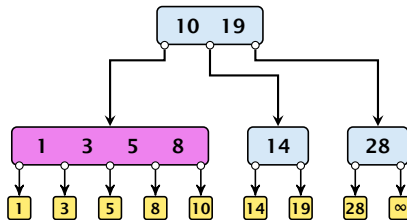
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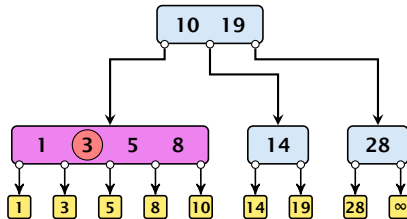
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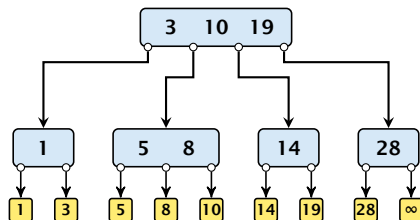


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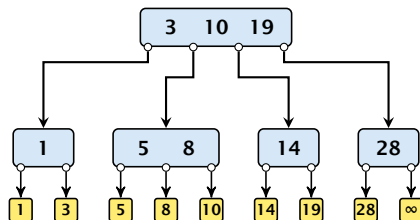


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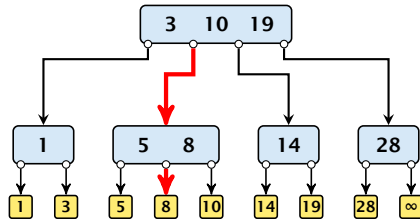
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## Insert(6)



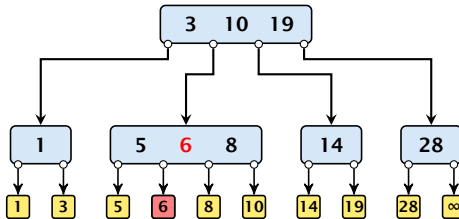
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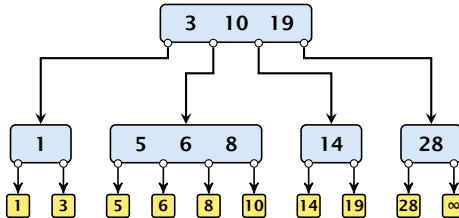
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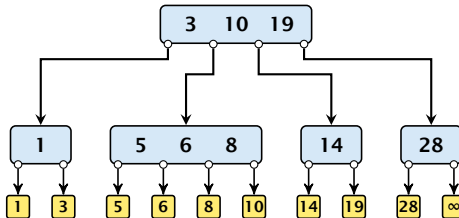
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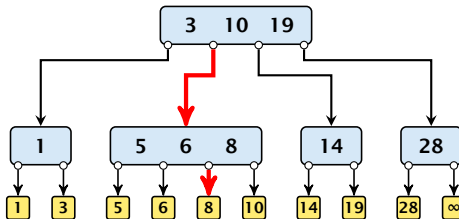
## Insert(7)





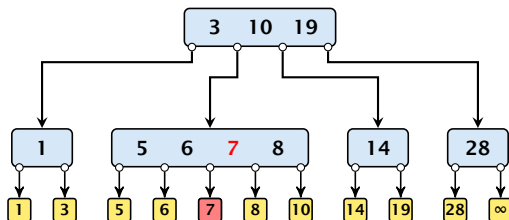
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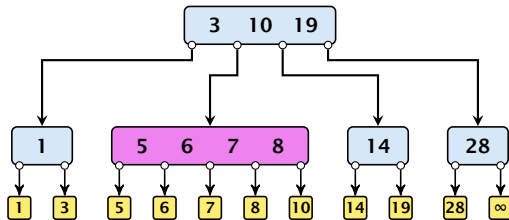
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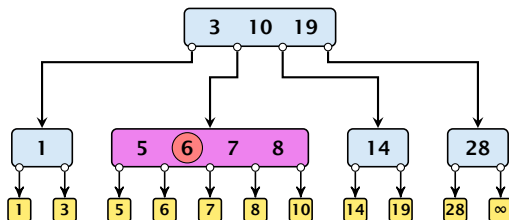
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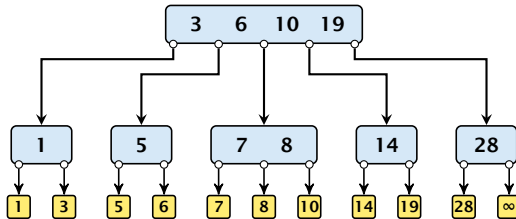
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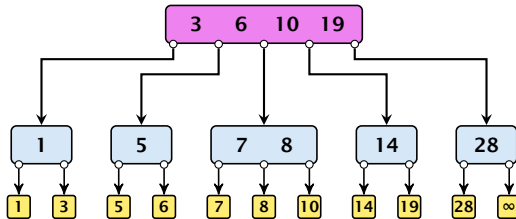
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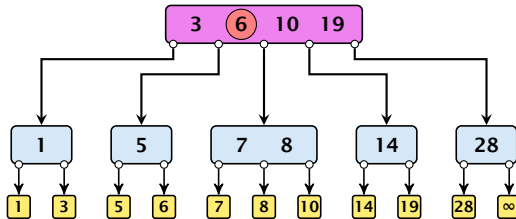
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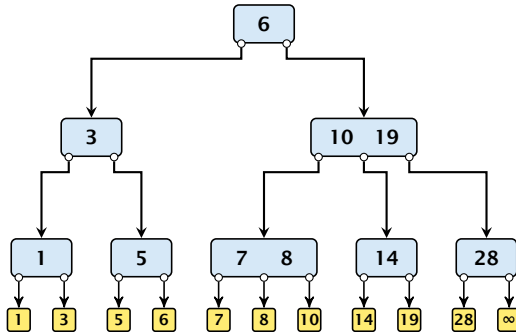
# Insert

## Insert(7)



# Insert

Insert(7)





# Delete

Delete element  $x$  (pointer to leaf vertex):

- ▶ Let  $v$  denote the parent of  $x$ . If  $\text{key}[x]$  is contained in  $v$ , remove the key from  $v$ , and delete the leaf vertex.
- ▶ Otherwise delete the key of the predecessor of  $x$  from  $v$ ; delete the leaf vertex; and replace the occurrence of  $\text{key}[x]$  in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- ▶ If now the number of keys in  $v$  is below  $a - 1$  perform  $\text{Rebalance}'(v)$ .

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Rebalance' ( $v$ ):

- ▶ If there is a neighbour of  $v$  that has at least  $a$  keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- ▶ If not: merge  $v$  with one of its neighbours.
- ▶ The merged node contains at most  $(a - 2) + (a - 1) + 1$  keys, and has therefore at most  $2a - 1 \leq b$  successors.
- ▶ Then rebalance the parent.
- ▶ During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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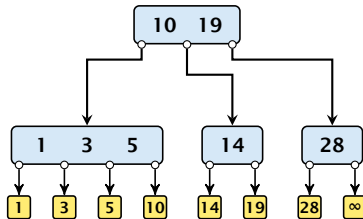
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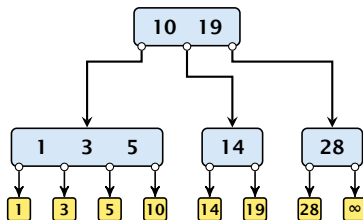


# Delete



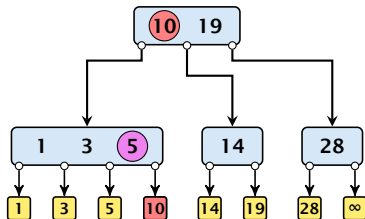
# Delete

Delete(10)



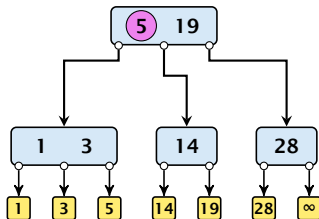
# Delete

Delete(10)

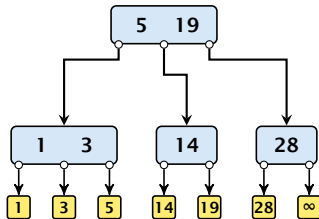


# Delete

Delete(10)

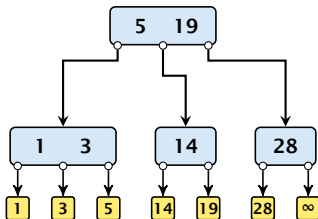


# Delete



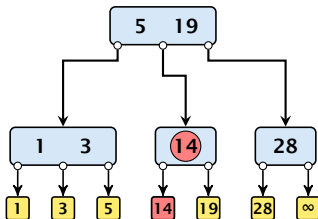
# Delete

Delete(14)



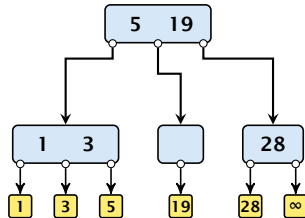
# Delete

Delete(14)



# Delete

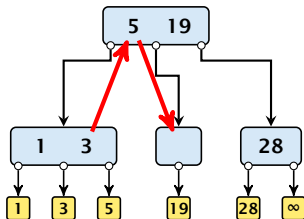
Delete(14)





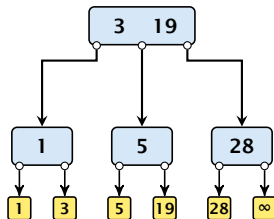
# Delete

Delete(14)

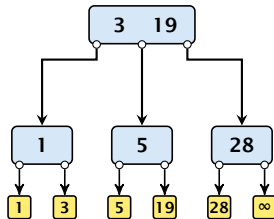


# Delete

## Delete(14)

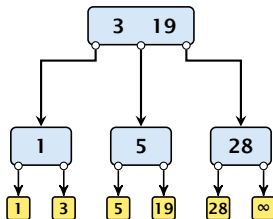


# Delete



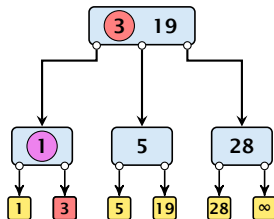
# Delete

## Delete(3)



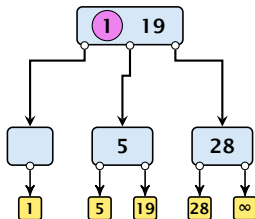
# Delete

## Delete(3)



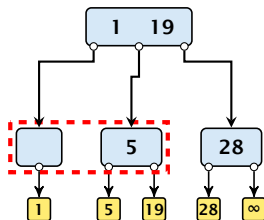
# Delete

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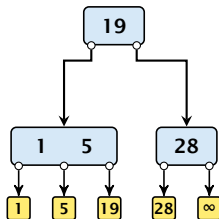
# Delete

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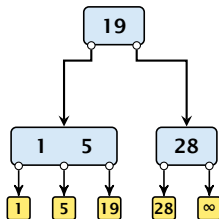
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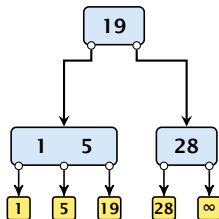


# Delete



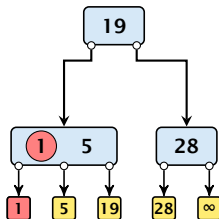
# Delete

## Delete(1)



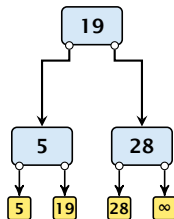
# Delete

## Delete(1)

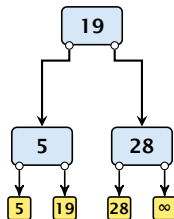


# Delete

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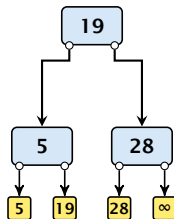


# Delete



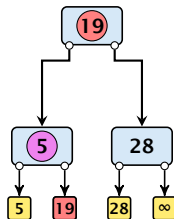
# Delete

## Delete(19)



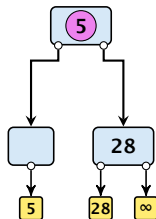
# Delete

Delete(19)



# Delete

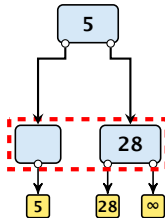
## Delete(19)





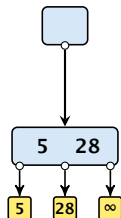
# Delete

## Delete(19)



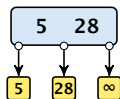
# Delete

## Delete(19)



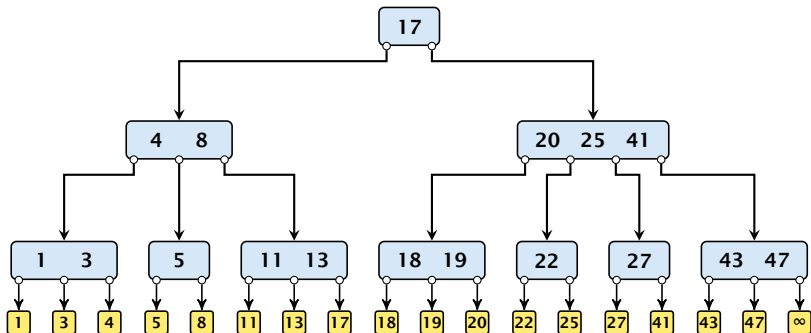
# Delete

## Delete(19)



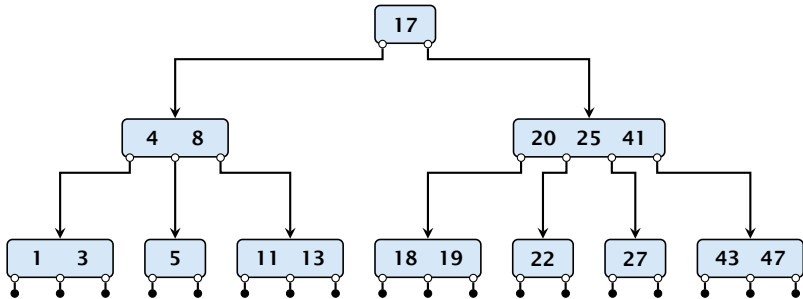
## (2, 4)-trees and red black trees

There is a close relation between red-black trees and (2,4)-trees:



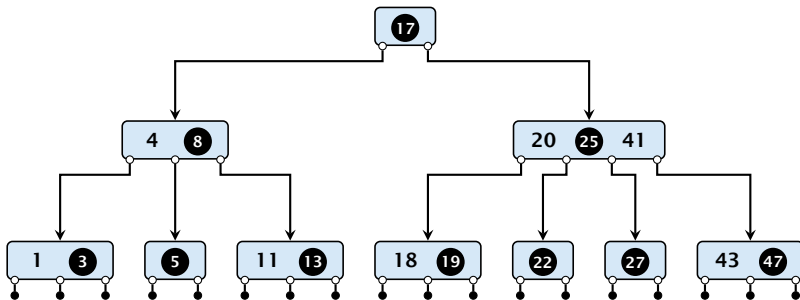
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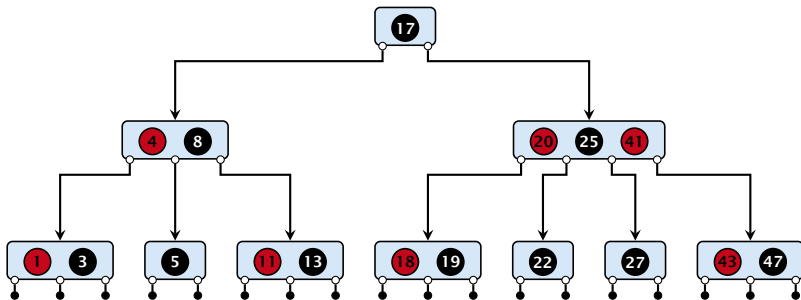
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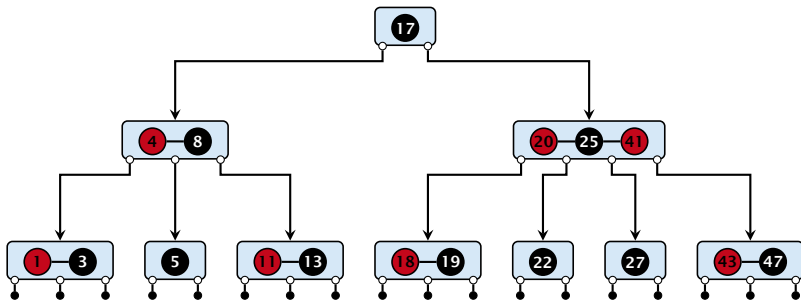
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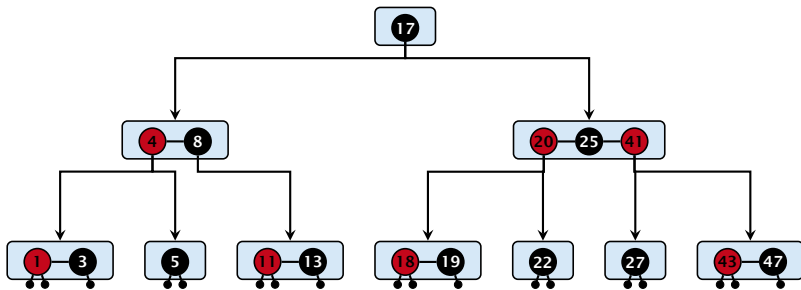
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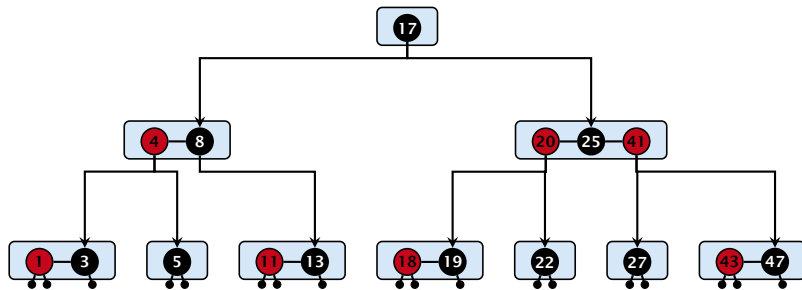
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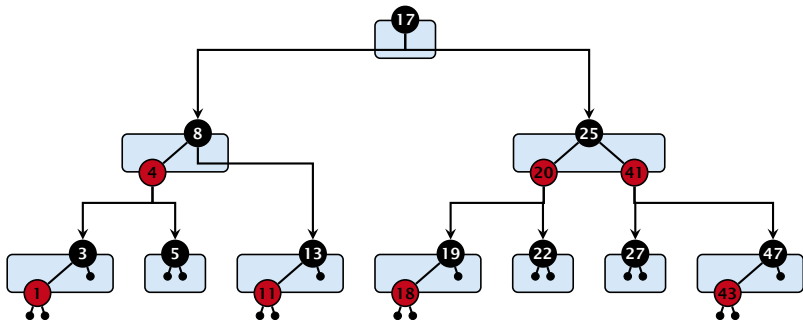
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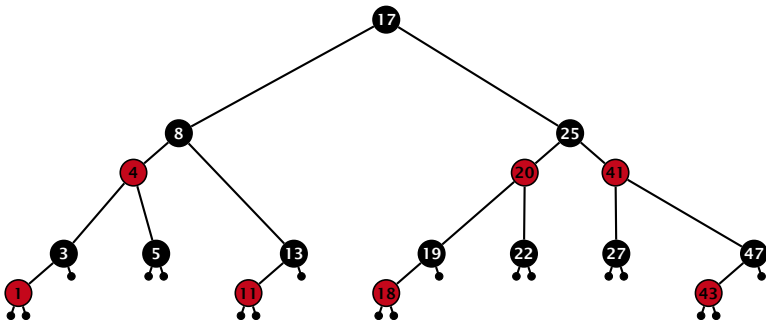
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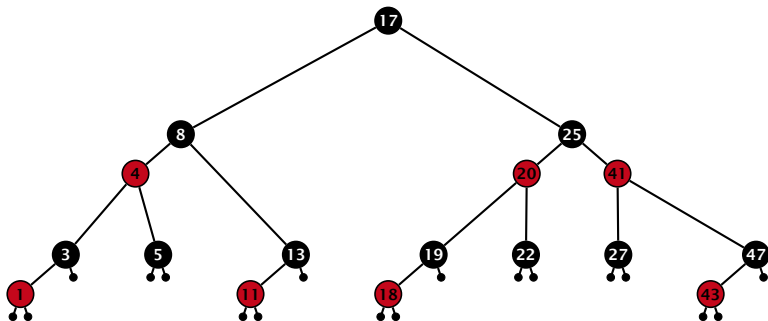
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Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same (2, 4)-tree.

## 7.6 Skip Lists

### Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search  $\Theta(n)$
- ▶ time for insert  $\Theta(n)$  (dominated by searching the item)
- ▶ time for delete  $\Theta(1)$  if we are given a handle to the object, otw.  $\Theta(n)$



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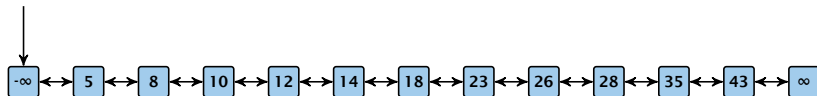
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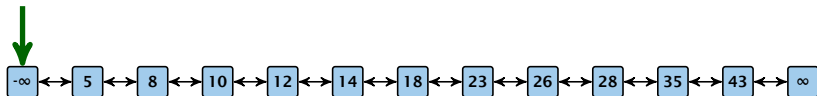




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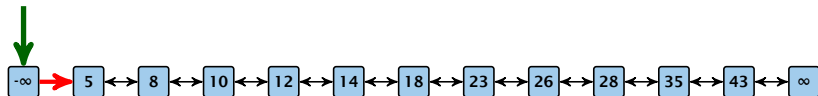
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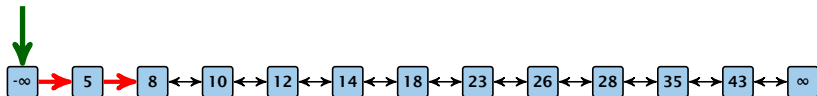
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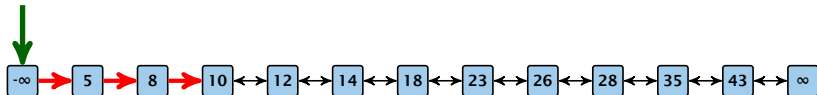
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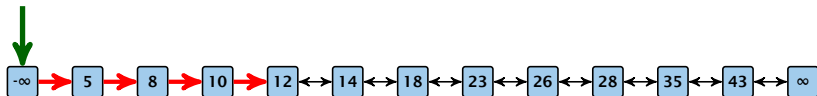
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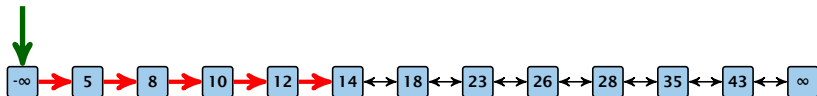
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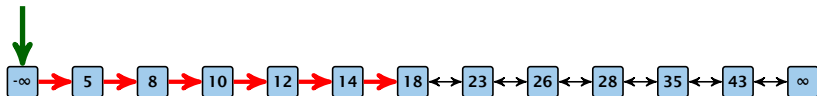
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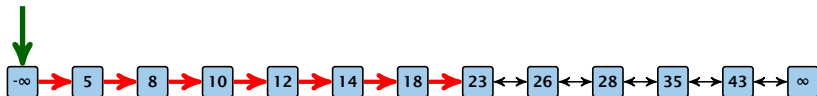
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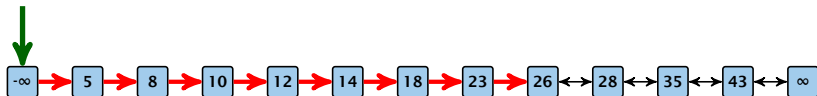




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## 7.6 Skip Lists

How can we improve the search-operation?

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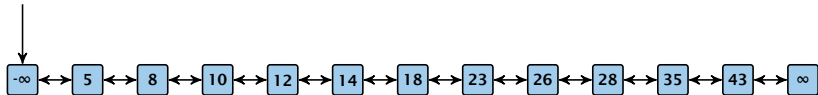
How can we improve the search-operation?

**Add an express lane:**

## 7.6 Skip Lists

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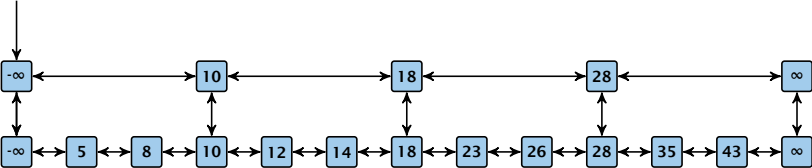
**Add an express lane:**



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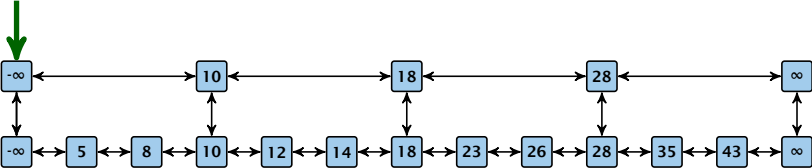
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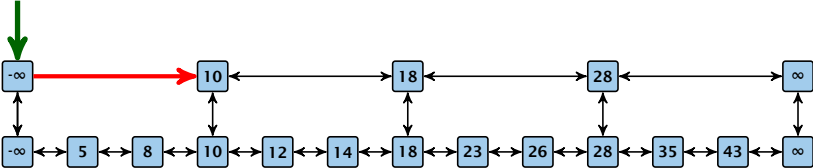
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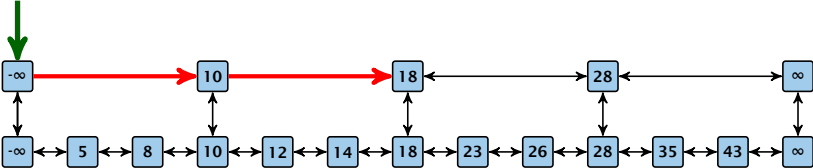
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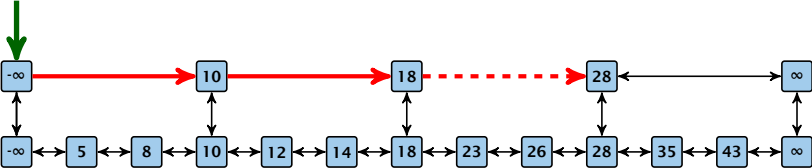




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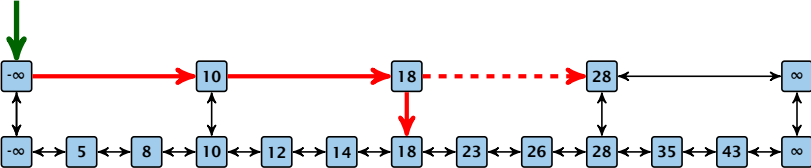
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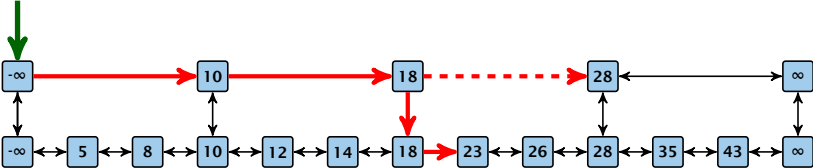
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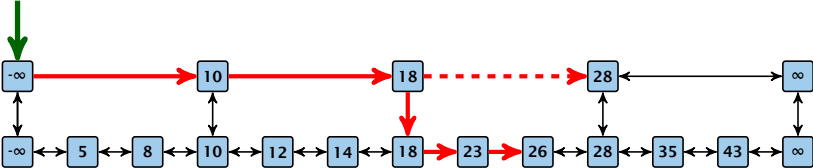
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# 7.6 Skip Lists

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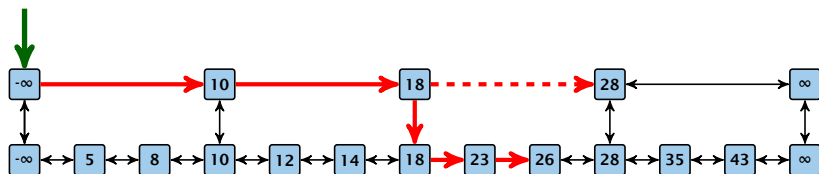
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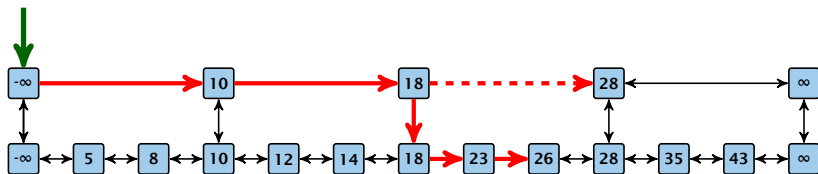


Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

## 7.6 Skip Lists

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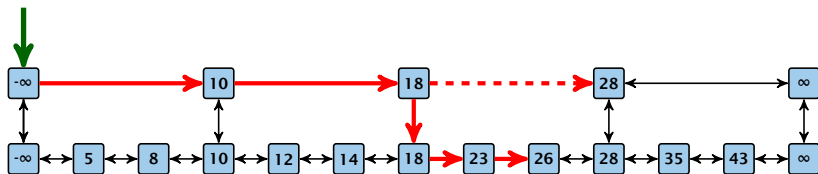
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Worst case search time:  $|L_1| + \frac{|L_0|}{|L_1|}$  (ignoring additive constants)

Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

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Add more express lanes. Lane  $L_i$  contains roughly every  $\frac{L_{i-1}}{L_i}$ -th item from list  $L_{i-1}$ .



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- ▶ Find the largest item in list  $L_k$  that is smaller than  $x$ . At most  $|L_k| + 2$  steps.

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- ▶ Find the largest item in list  $L_{k-2}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-2}|}{4} \rceil + 2$  steps.

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- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.

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Choose ratios between list-lengths evenly, i.e.,  $\frac{|L_{i-1}|}{|L_i|} = r$ , and, hence,  $L_k \approx r^{-k}n$ .

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Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

## 7.6 Skip Lists

### How to do insert and delete?

How can we do insert and delete efficiently? The worst number of elements to traverse to insert or delete may require a lot of reorganization.

Use randomization instead!

## 7.6 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

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### Insert:

- ▶ A search operation gives you the insert position for element  $x$  in every list.
- ▶ Flip a coin until it shows head, and record the number  $t \in \{1, 2, \dots\}$  of trials needed.
- ▶ Insert  $x$  into lists  $L_0, \dots, L_{t-1}$ .

### Delete:

- ▶ You get all predecessors via backward pointers.
- ▶ Delete  $x$  in all lists it actually appears in.

The time for both operations is dominated by the search time.

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### Delete:

Find all predecessor and successor pointers.

Remove all nodes which appear in it.

The time for both operations is dominated by the search time.

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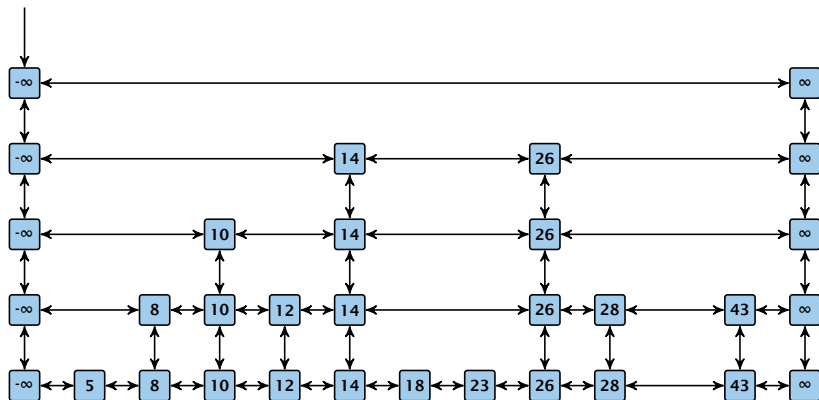
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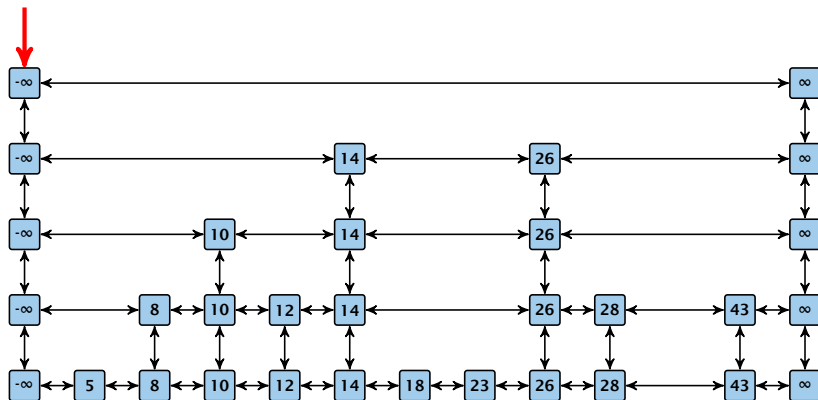
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Insert (35):



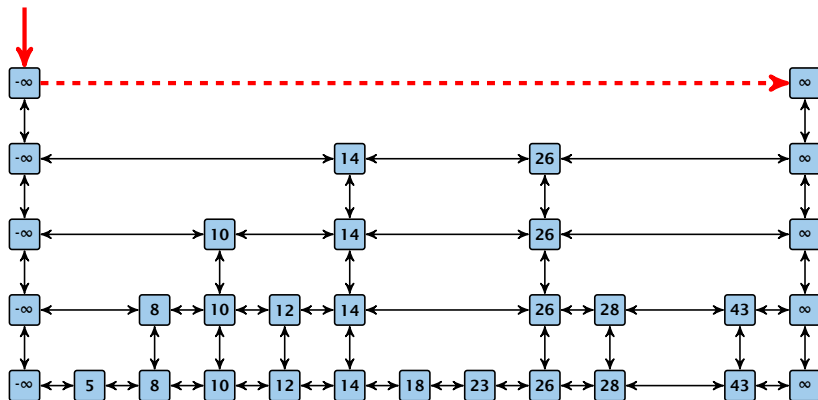
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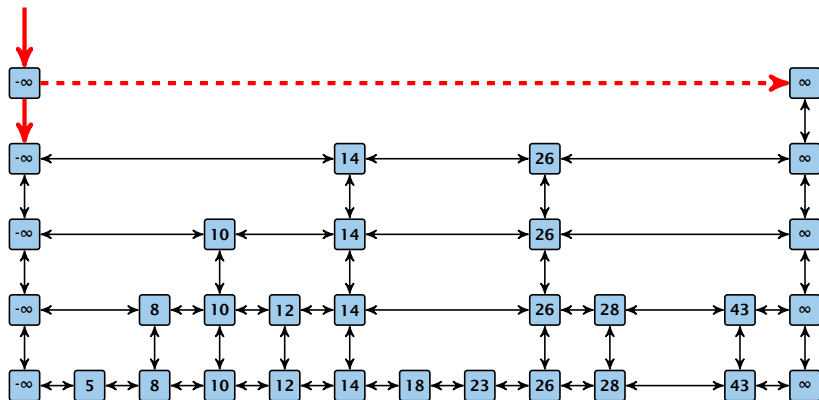
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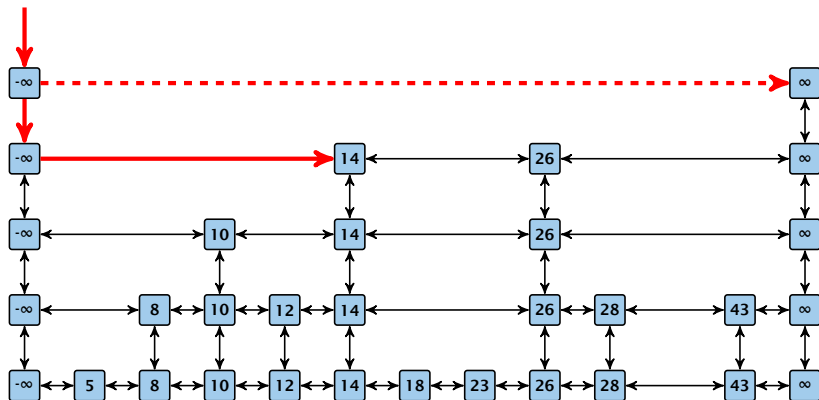
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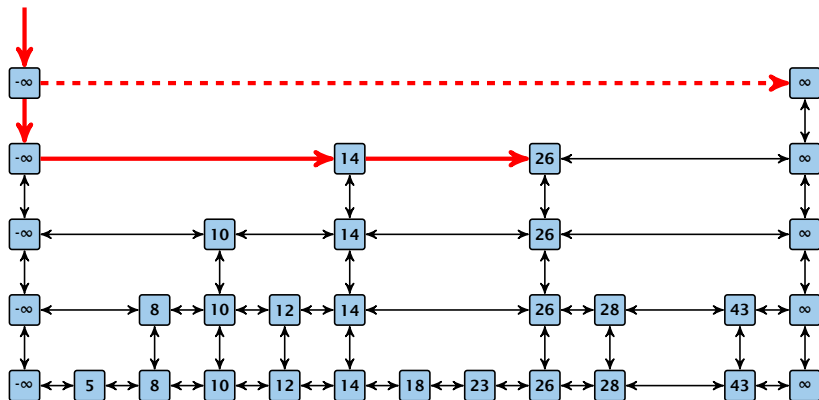
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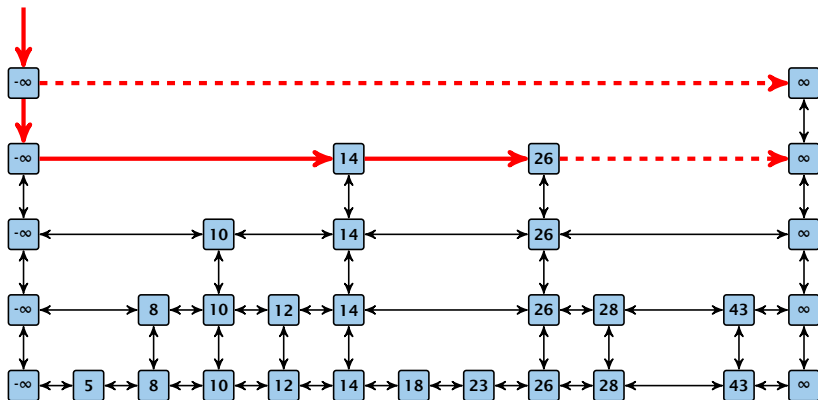
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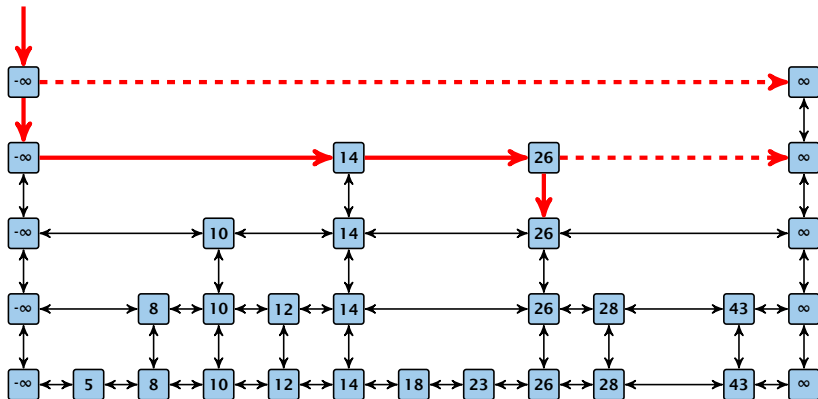
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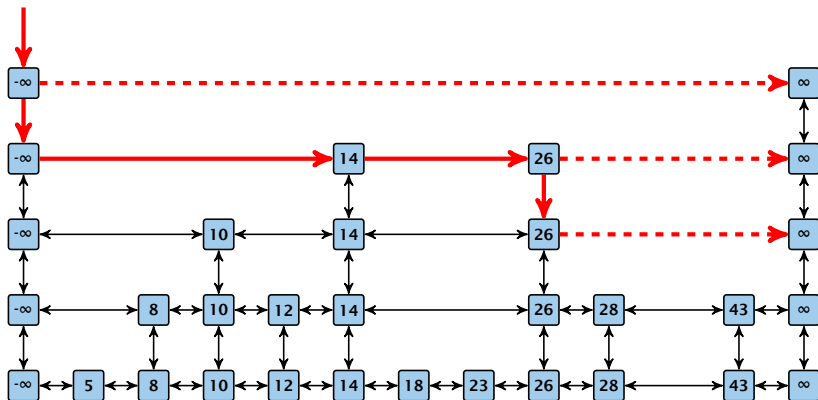
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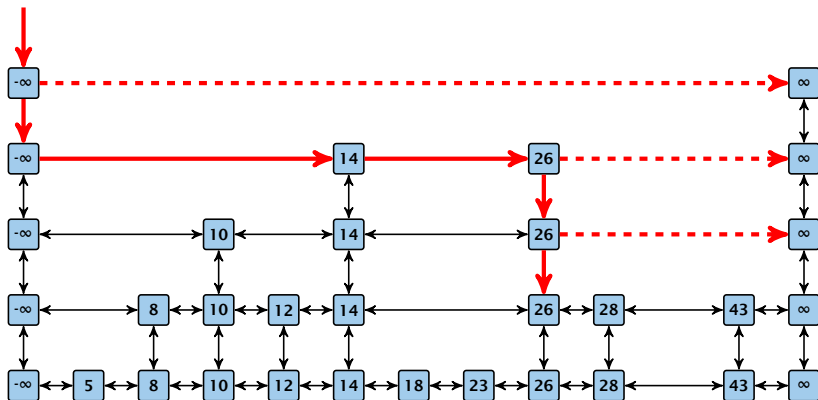
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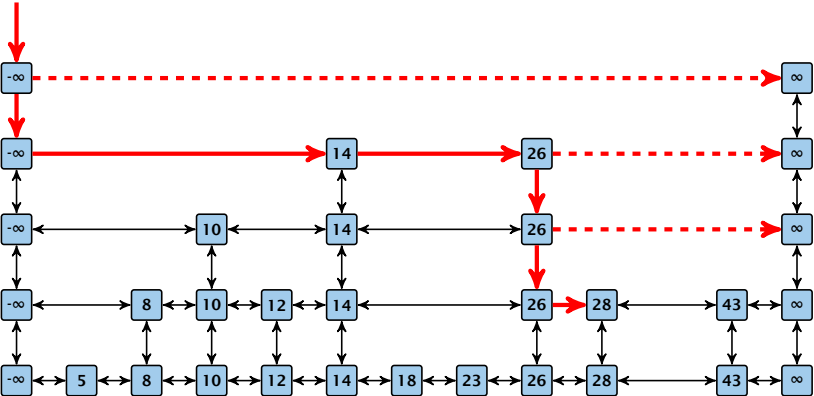
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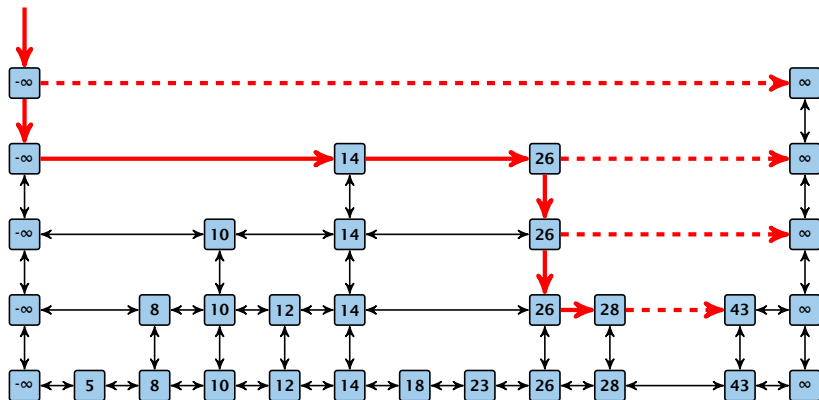
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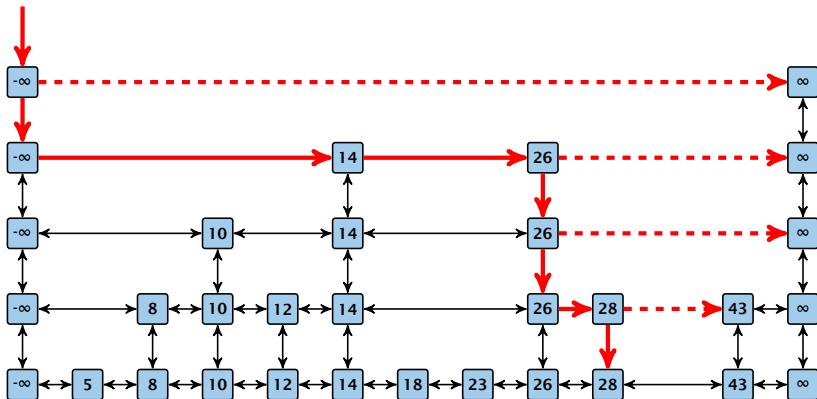
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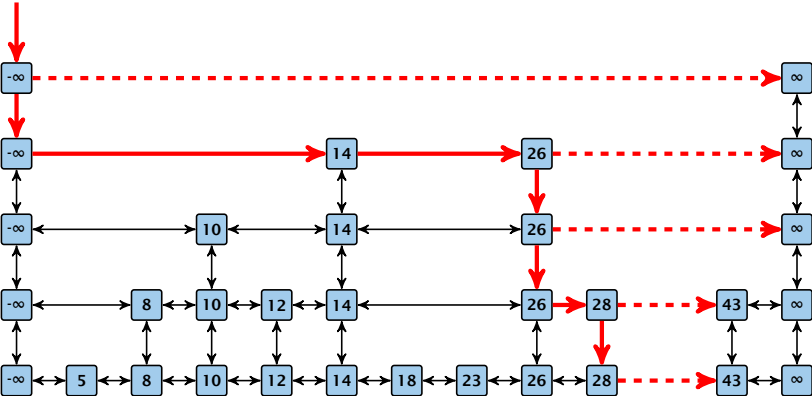
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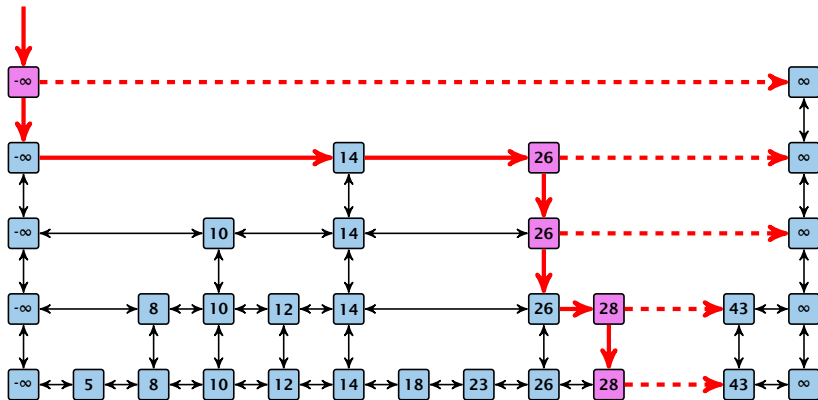
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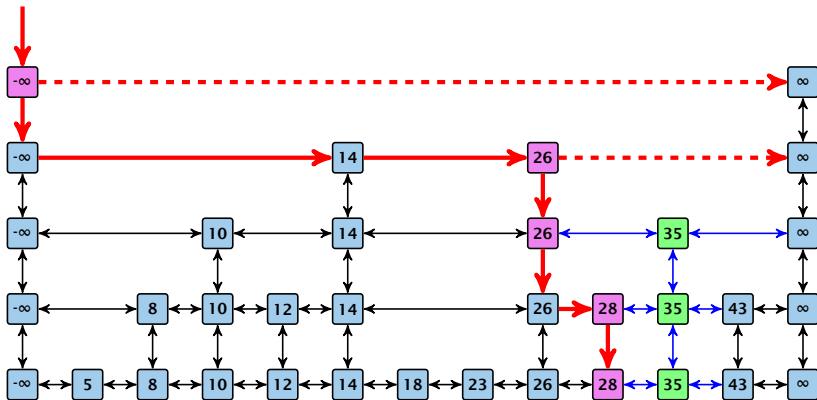
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# High Probability

## Definition 10 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

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# High Probability

Suppose there are **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

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This means  $\Pr[E_1 \wedge \dots \wedge E_\ell]$  holds with high probability.



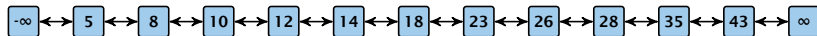
## 7.6 Skip Lists

### Lemma 11

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

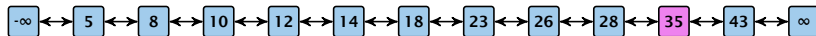
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Backward analysis:



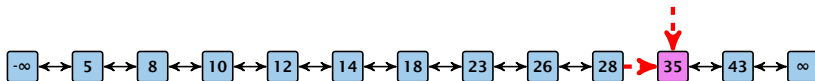
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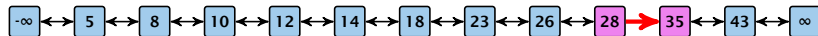
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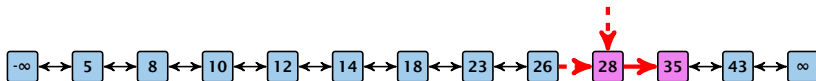
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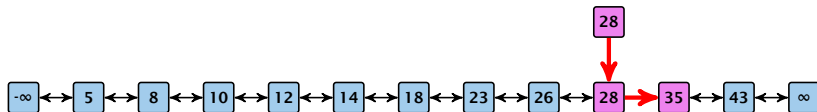
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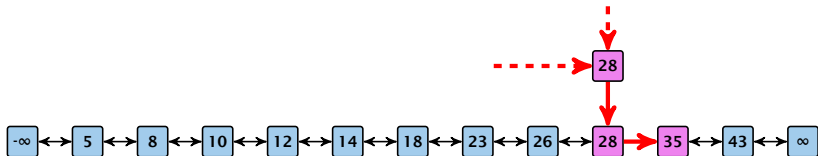
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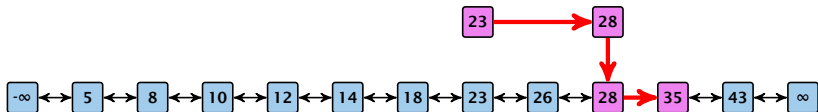
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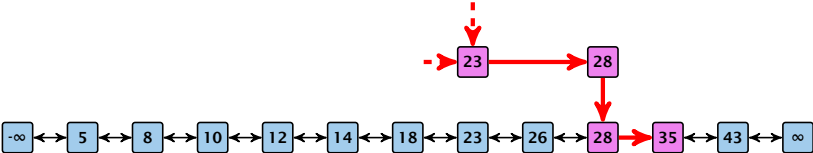
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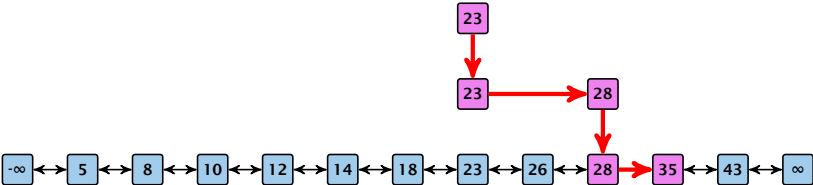
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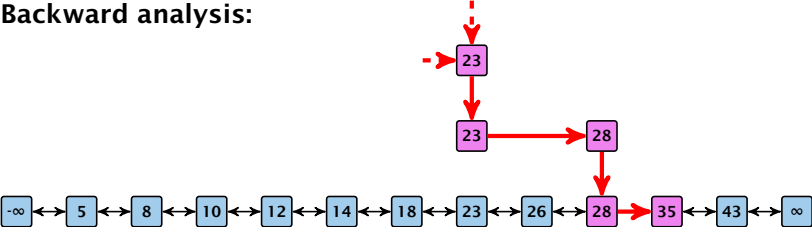
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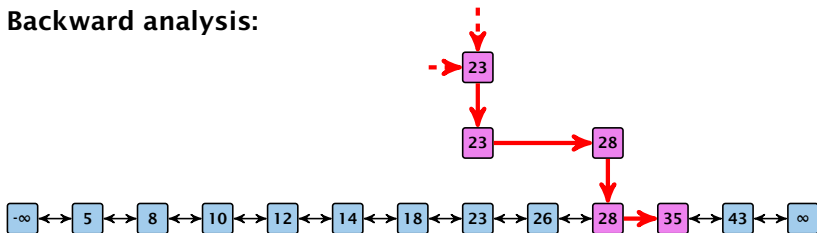
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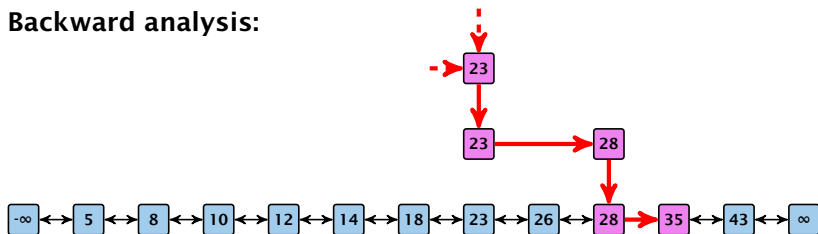
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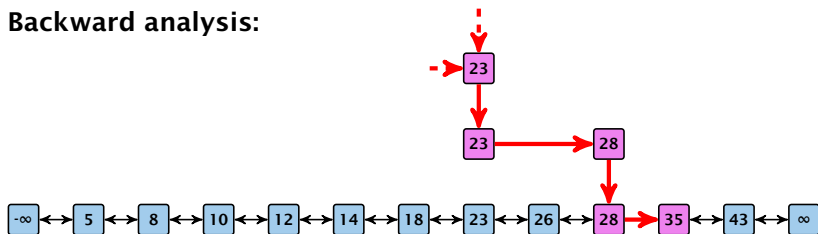
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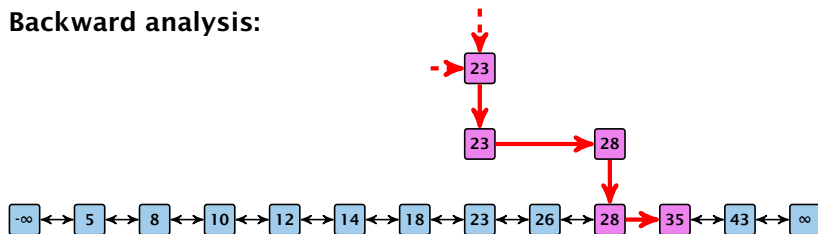
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At each point the path goes up with probability  $1/2$  and left with probability  $1/2$ .

We show that w.h.p.:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.



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In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

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This means, the search requires at most  $z$  steps, w. h. p.

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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

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- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n-1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n-1]$ .

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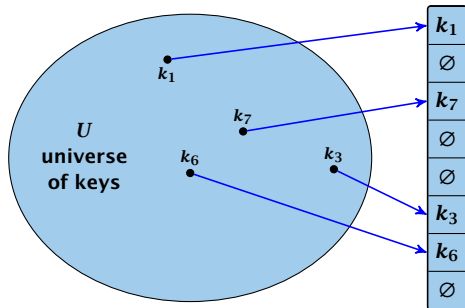
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- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n-1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n-1]$ .

### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

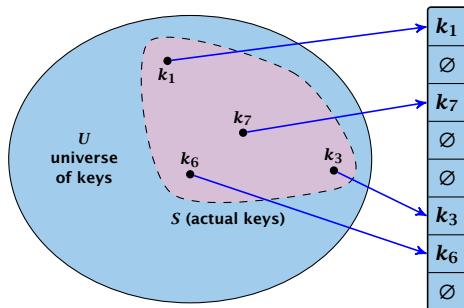
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

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# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

## Lemma 12

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}}.$$

## Uniform hashing:

Choose a hash function uniformly at random from all functions  $f: U \rightarrow [0, \dots, n-1]$ .

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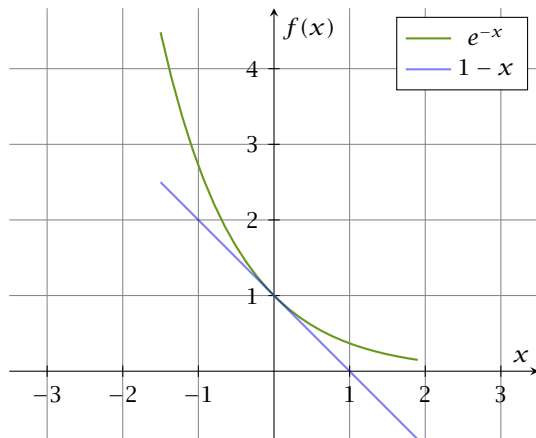
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Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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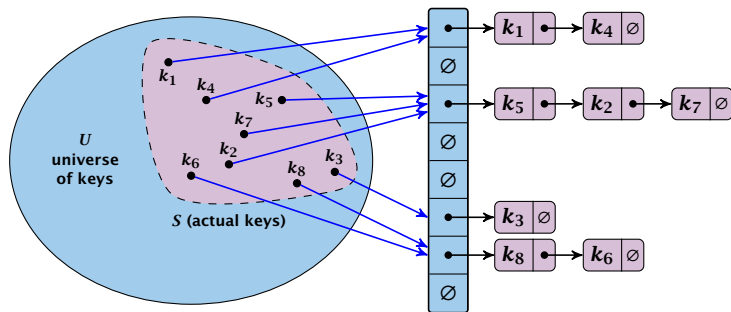
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# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.



# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;
- ▶ We parameterize the complexity results in terms of  $\alpha := \frac{m}{n}$ , the so-called **fill factor** of the hash-table.

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$$A^- = 1 + \alpha .$$

# Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

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Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$ .

**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

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# Open Addressing

Choices for  $h(k, j)$ :

▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

▶ Quadratic probing:

$$h(k, i) = h(k) + c_1 i + c_2 i^2 \pmod n.$$

▶ Double hashing:

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (teilerfremd); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (teilerfremd); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

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## Lemma 13

*Let  $L$  be the method of linear probing for resolving collisions:*

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

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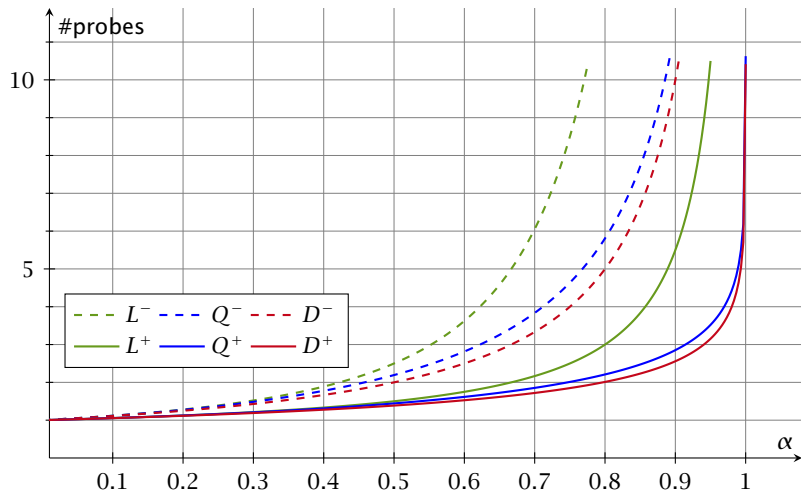
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# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .



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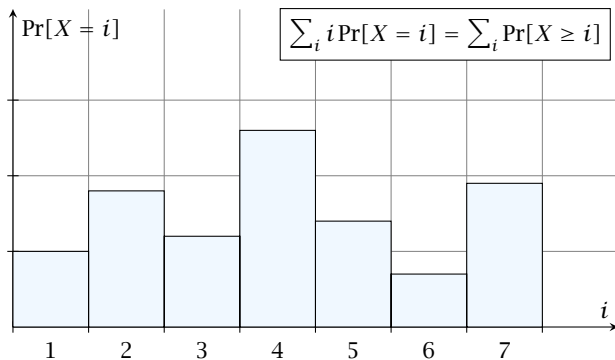
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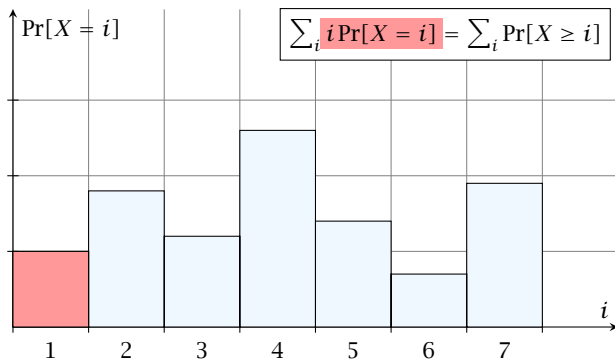
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

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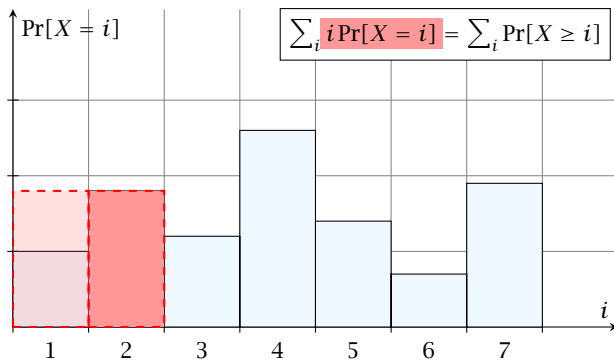
$i = 1$





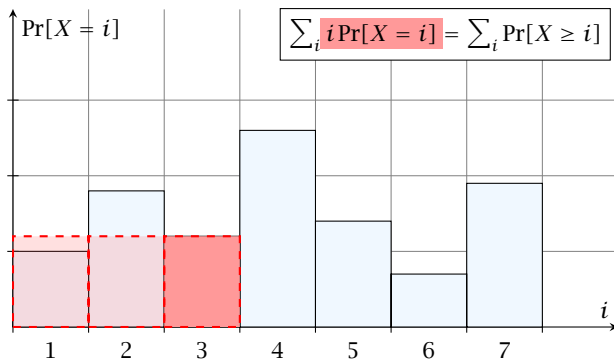
# Analysis of Idealized Open Address Hashing

$i = 2$



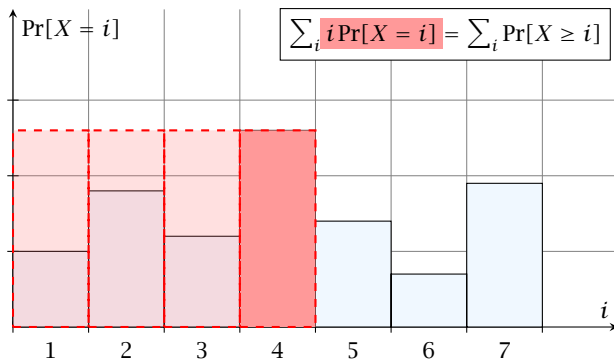
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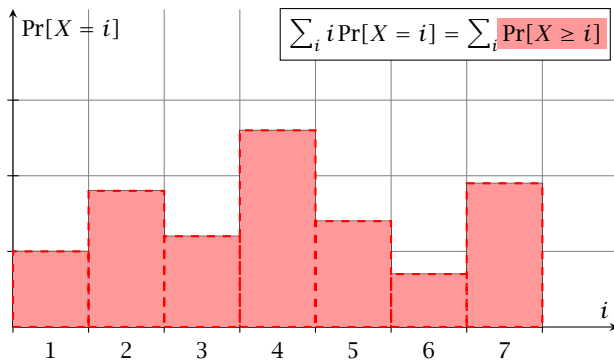
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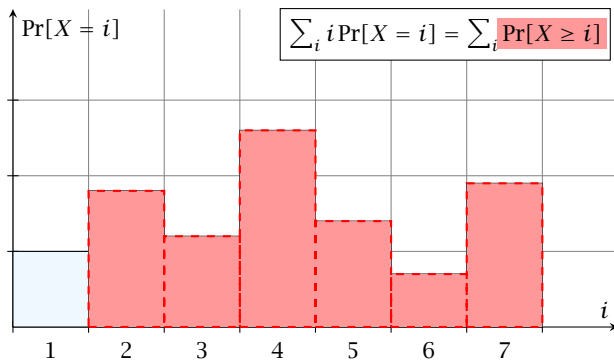
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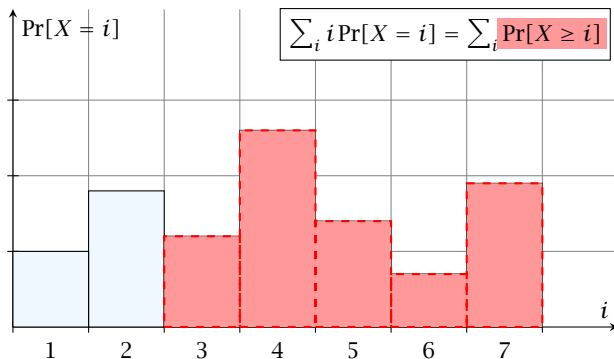
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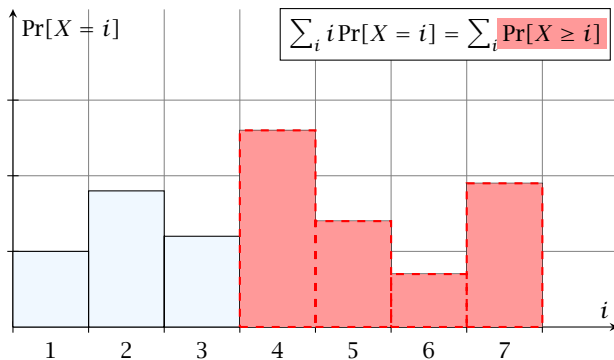
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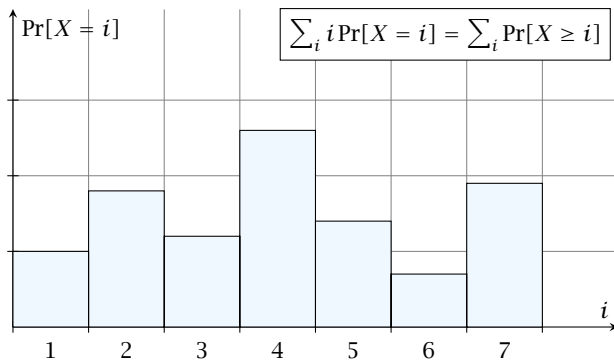


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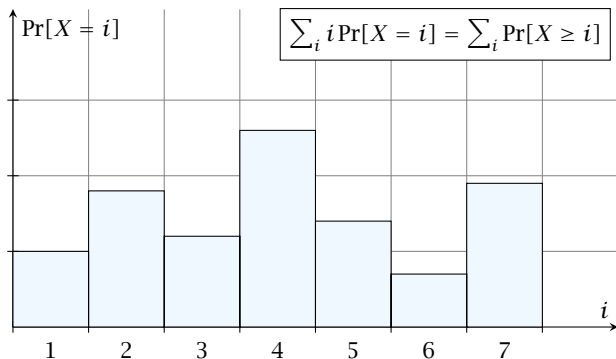


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The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

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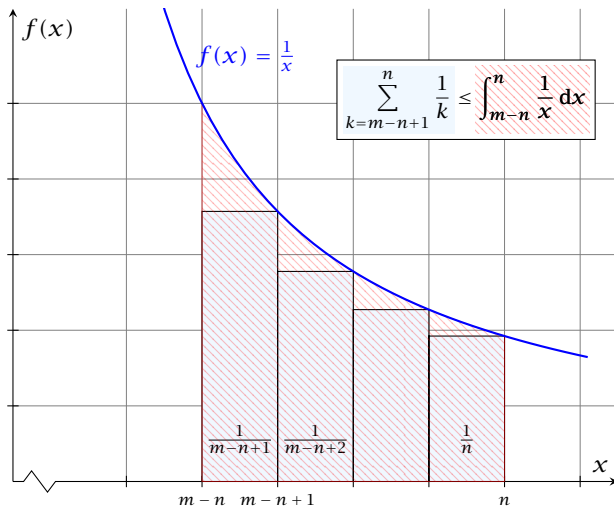
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# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted-marker**.
  - ▶ Deleted markers are ignored by the probe sequence, but the element can be inserted there.
  - ▶ Deleted markers can be removed by rehashing the table. This amortizes the cost for the deletions.
- ▶ The table could fill up with deleted-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
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$p$  is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

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# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

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## Definition 16

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

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A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

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## Definition 18

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

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A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

Let  $U := \{0, \dots, p-1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p-1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p-1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

## Lemma 20

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

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If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

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- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

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$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

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This happens with probability at most  $\frac{1}{n}$ .



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It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 21

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

# Universal Hashing

For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

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Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lfloor \frac{q}{n} \rfloor - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

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Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lfloor \frac{q}{n} \rfloor - 1\}\}}_{=: B_i}$$

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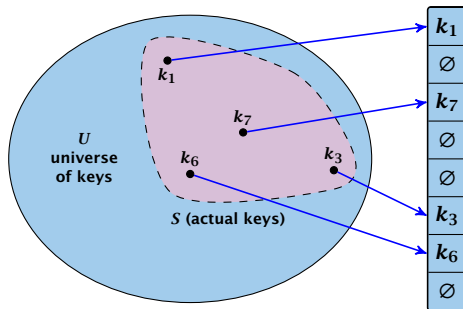
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This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose  $n = m^2$  the **expected number** of collisions is strictly less than  $\frac{1}{2}$ .

Can we get an upper bound on the **probability of having collisions**?

The probability of having 1 or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

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We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from  $S$  to  $m$  buckets.

Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

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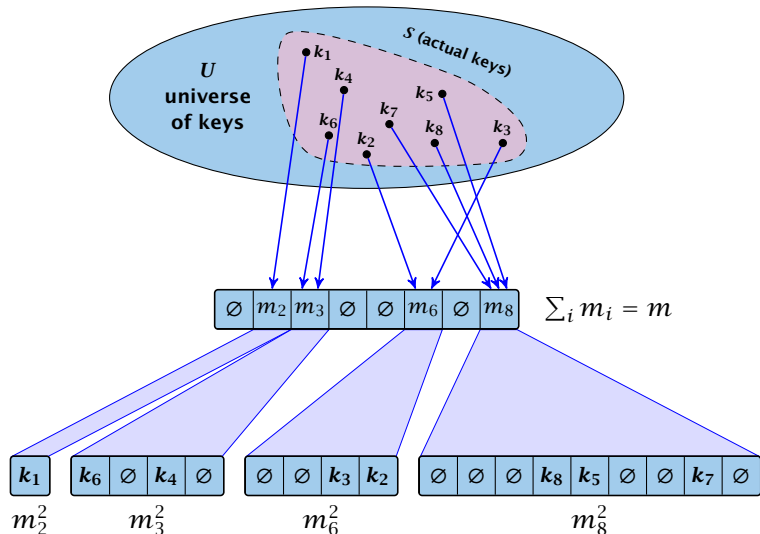
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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

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We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!



# Cuckoo Hashing

## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

Two hash-tables  $T_1$  and  $T_2$  and two hash functions  $h_1$  and  $h_2$ , with hash functions  $h_1$  and  $h_2$ .

An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .

Insertion and deletion takes constant time if the above constraints are met.

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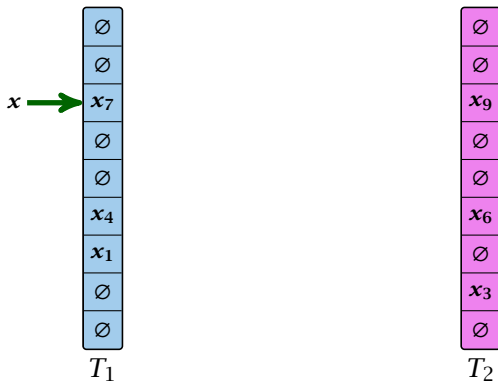
$T_1$



$T_2$

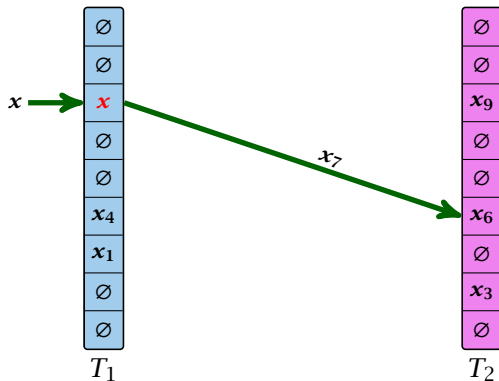
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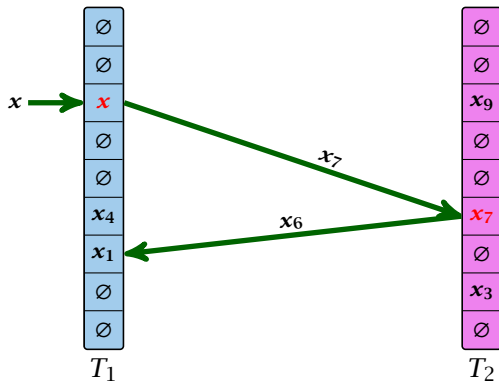
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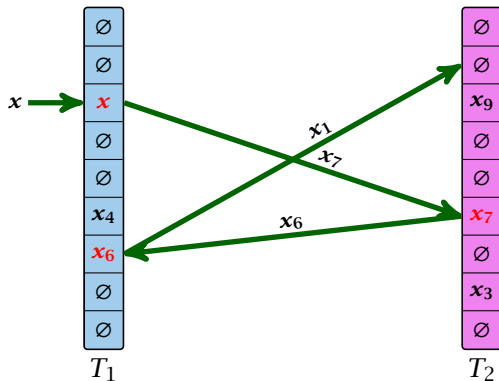
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## Algorithm 13 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after  $\text{maxsteps}$  steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

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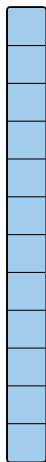
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# Cuckoo Hashing: Insert

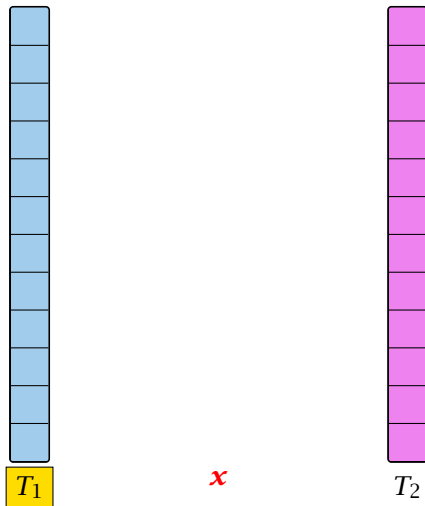


$T_1$

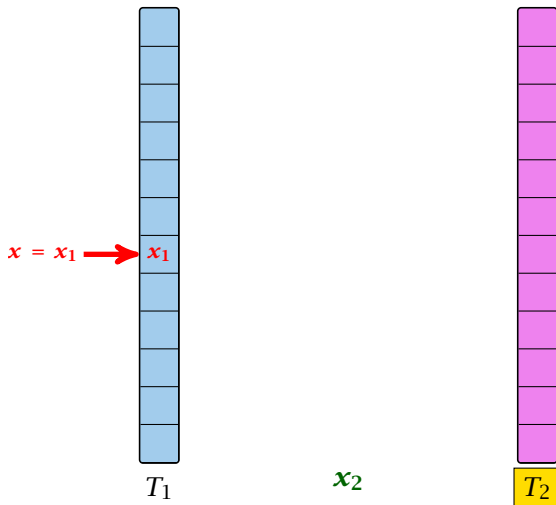


$T_2$

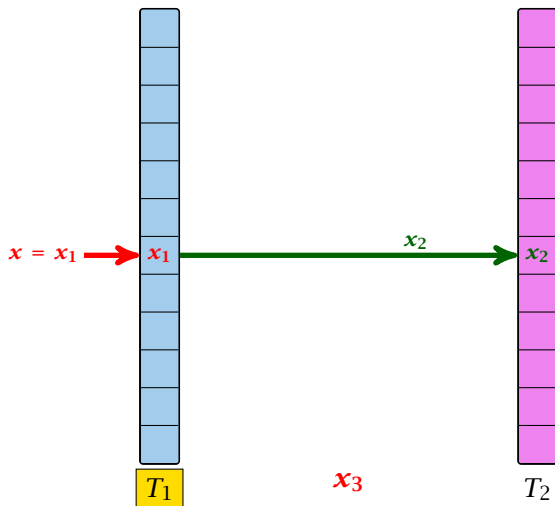
# Cuckoo Hashing: Insert



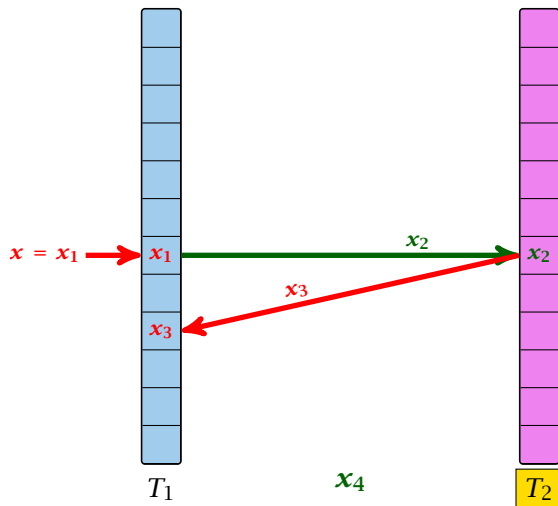
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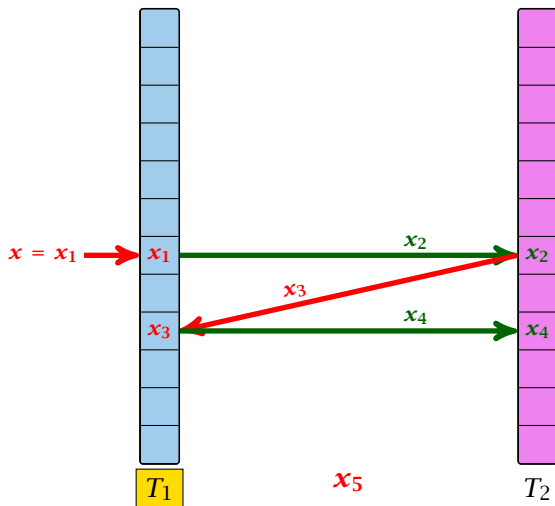
# Cuckoo Hashing: Insert



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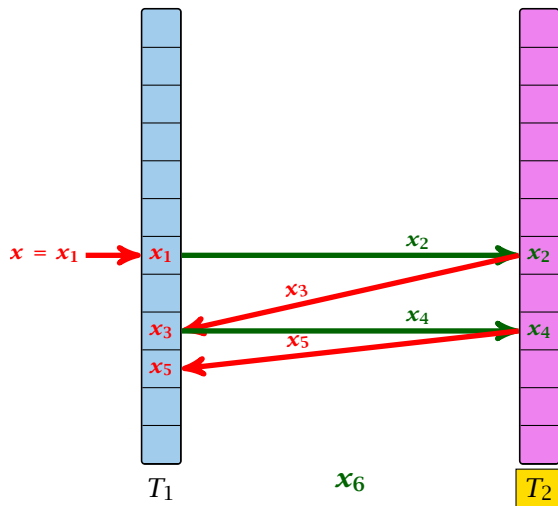


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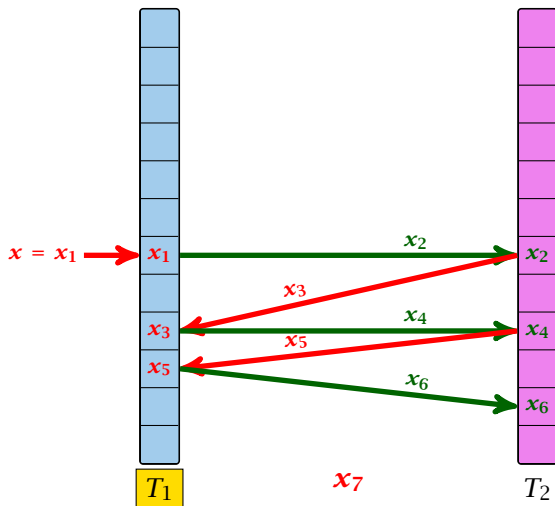




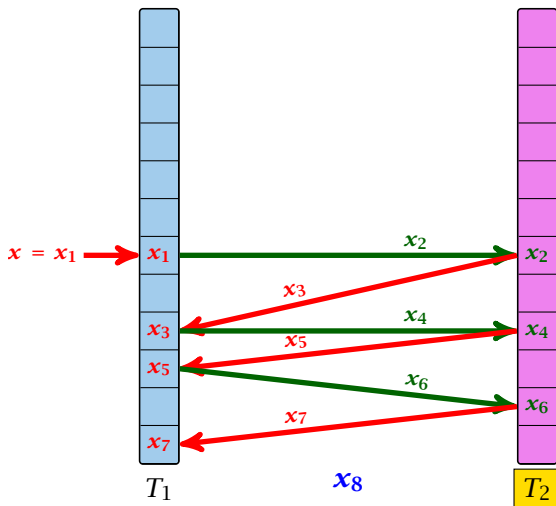
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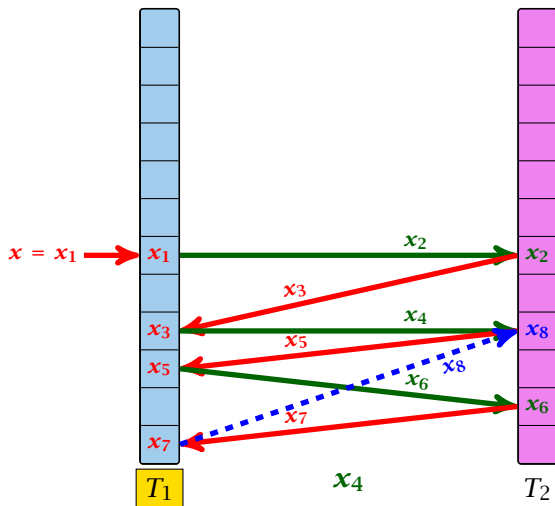
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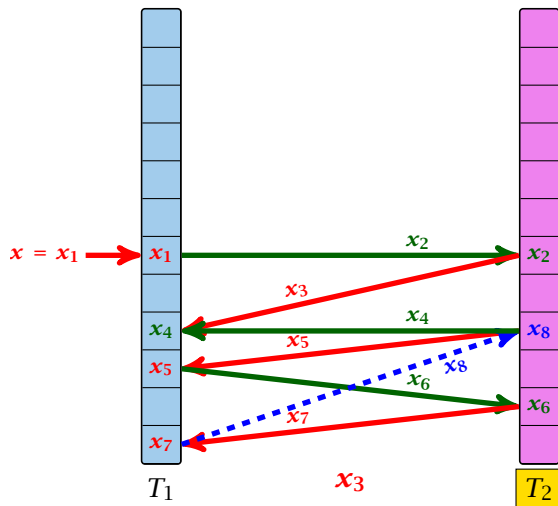
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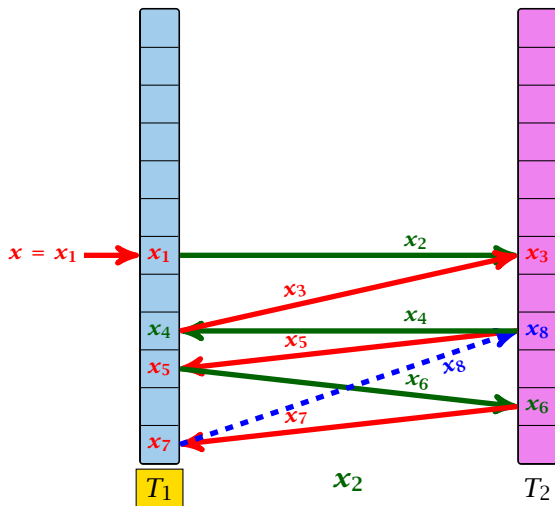
# Cuckoo Hashing: Insert



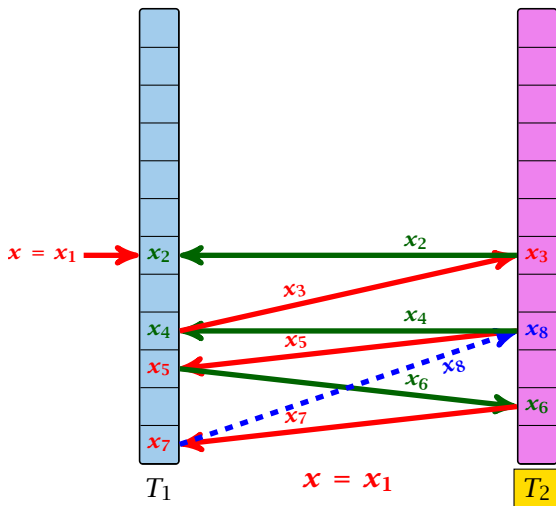
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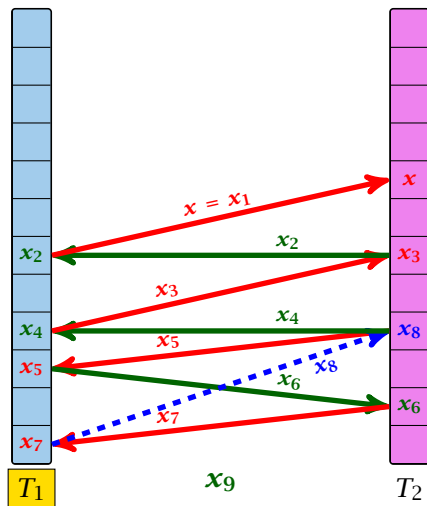
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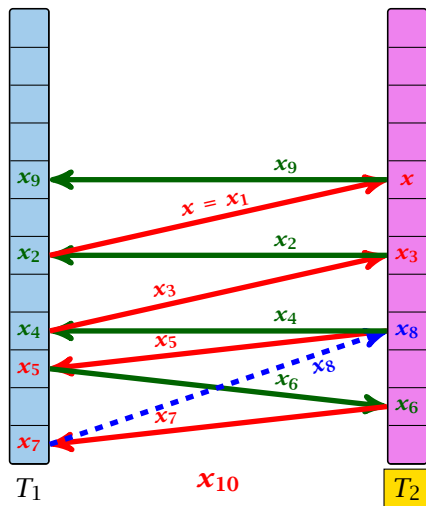


# Cuckoo Hashing: Insert

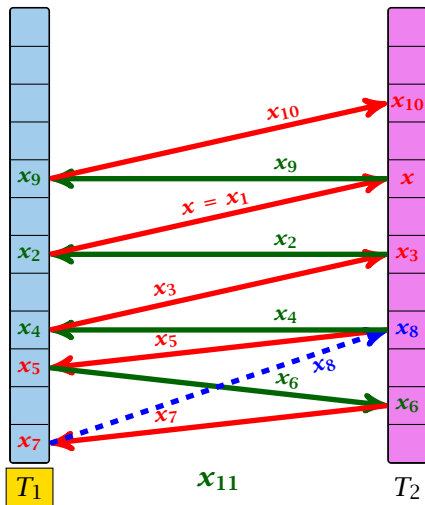




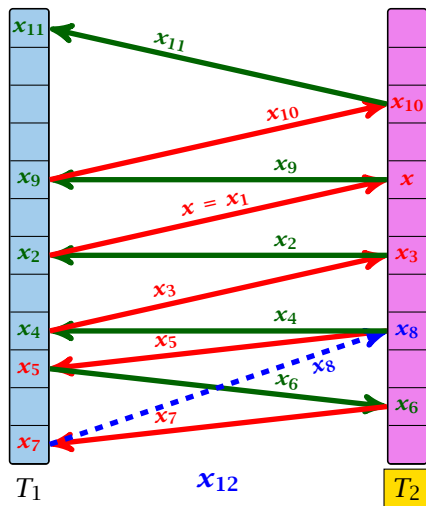
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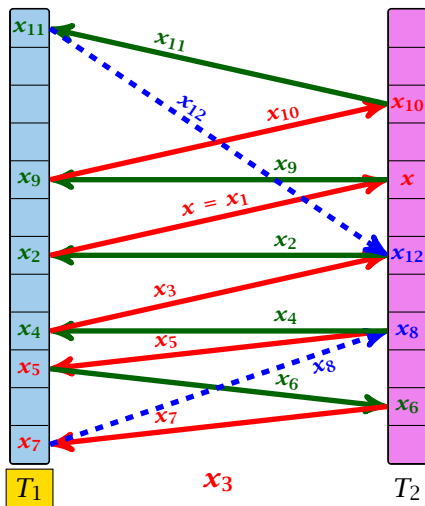
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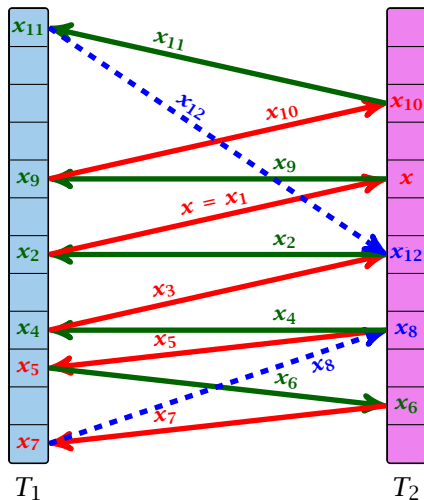
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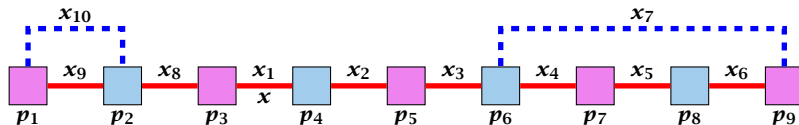
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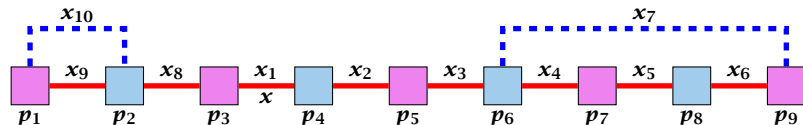


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A cycle-structure of size  $s$  is defined by

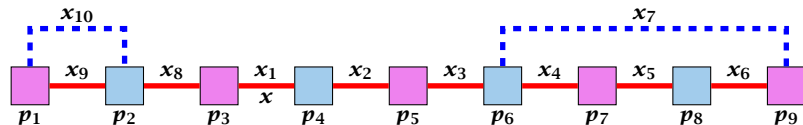
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A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
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- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

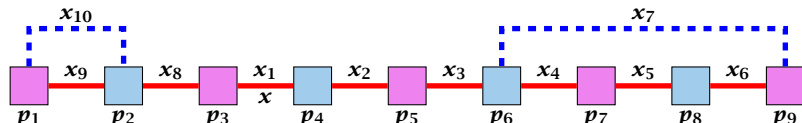


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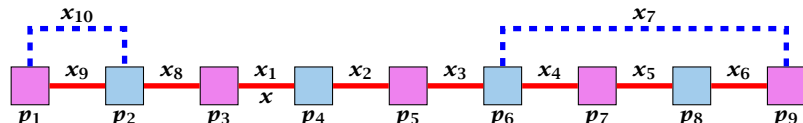
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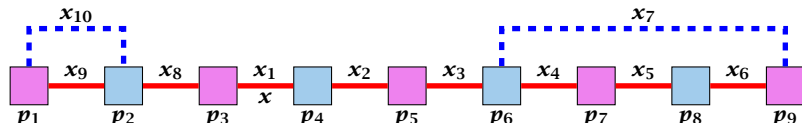
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A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

**Observation:**

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .

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# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

These events are independent.

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The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

What is the probability that **there exists** an active cycle structure of size  $s$ ?

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What is the probability that **there exists** an active cycle structure of size  $s$ ?

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

There are  $s$  ways to pick the  $s$  keys that form the cycle, and  $s$  ways to pick the forward and backward links.

There are at most  $s$  possibilities to choose where to place

the keys in the buckets. There are  $n$  choices for each of the

$s-1$  keys, and  $m$  choices for each of the  $s-1$  buckets.

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.
- ▶ There are at most  $s$  possibilities to choose where to place key  $x$ .
- ▶ There are  $m^{s-1}$  possibilities to choose the keys apart from  $x$ .
- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

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The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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Here we used the fact that  $(1 + \epsilon)m \leq n$ .

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Hence,

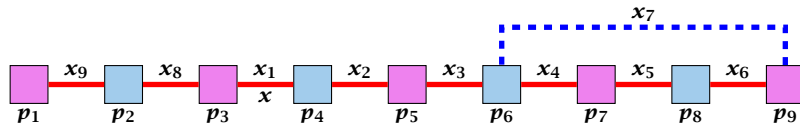
$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.



# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

## Lemma 22

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of **distinct** keys.*

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## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

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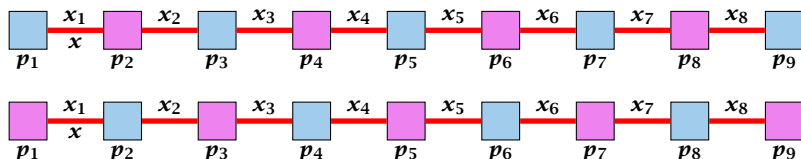
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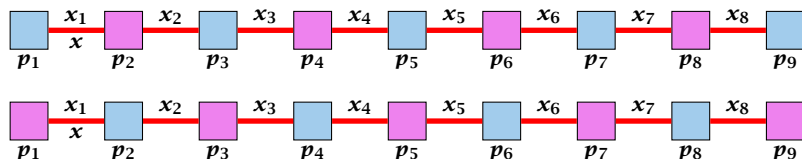
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A path-structure of size  $s$  is defined by

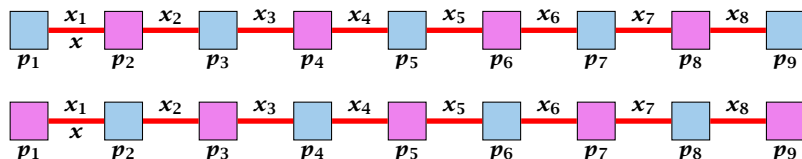
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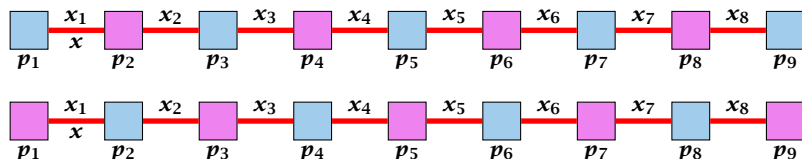


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## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

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This gives  $\text{maxsteps} = \Theta(\log m)$ .

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $q = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching maxsteps without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

The expected number of unsuccessful rehashes is

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The expected cost for all rehashes is

$$E \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

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Since  $\text{maxsteps}$  is  $\Theta(\log m)$  the largest size of a path-structure or cycle-structure contains just  $\Theta(\log m)$  different keys.

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# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (table-expand).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (table-shrink).
- ▶ Note that right after a change in table-size we have  $m = \alpha n/2$ . In order for a table-expand to occur at least  $\alpha n/2$  insertions are required. Similar, for a table-shrink at least  $\alpha n/4$  deletions must occur.
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# Cuckoo Hashing

## Lemma 23

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+c)}$ .



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# Dijkstra's Shortest Path Algorithm

## Algorithm 14 Shortest-Path( $G = (V, E, d), s \in V$ )

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
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# Prim's Minimum Spanning Tree Algorithm

**Algorithm 15** Prim-MST( $G = (V, E, d), s \in V$ )

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
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# Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
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How good a running time can we obtain?

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<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	$n$	$n \log n$	$n \log n$	$n$
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
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delete-min	$\log n$	$\log n$	$\log n$	$\log n$
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Note that most applications use `build()` only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an **amortized** guarantee.

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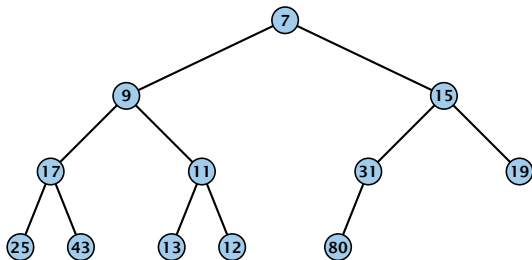
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Using Binary Heaps, Prim and Dijkstra run in time  $\mathcal{O}((|V| + |E|) \log |V|)$ .

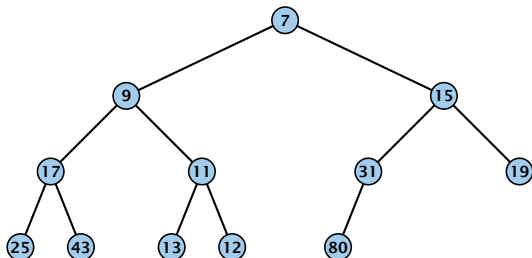
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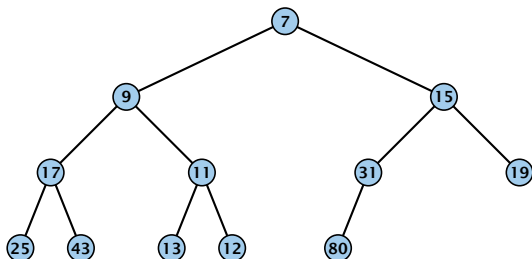
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- ▶ **Heap property:** A node's key is not larger than the key of one of its children.





# Binary Heaps

## Operations:

- ▶ `minimum()`: return the root-element. Time  $\mathcal{O}(1)$ .
- ▶ `is-empty()`: check whether root-pointer is null. Time  $\mathcal{O}(1)$ .

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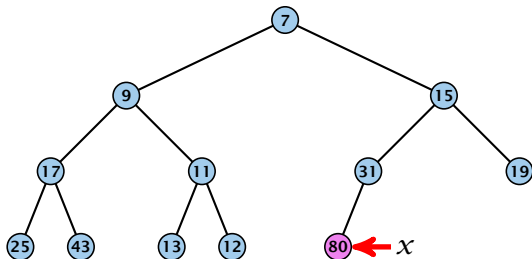
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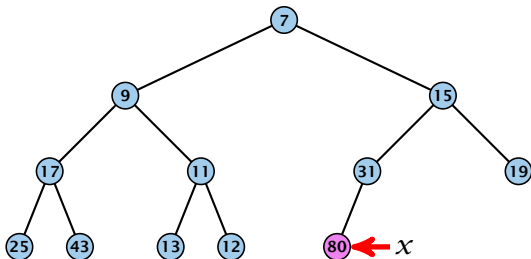
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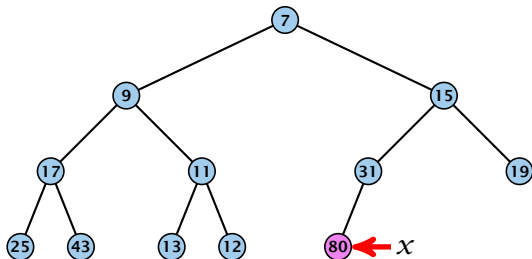
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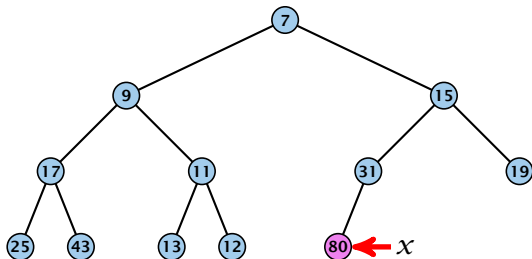
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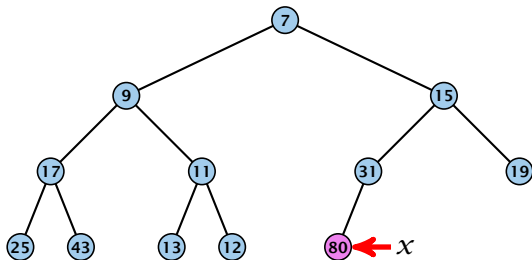
Maintain a pointer to the **last element**  $x$ .

- ▶ We can compute the successor of  $x$  (last element when an element is inserted) in time  $\mathcal{O}(\log n)$ .

go up until the last edge used was left edge

if right edge used, go to left child

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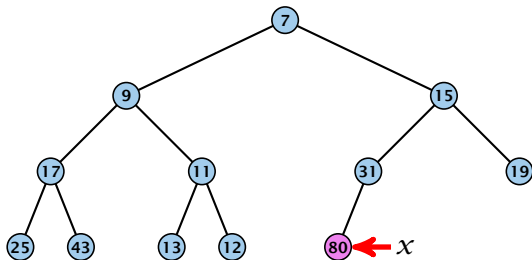
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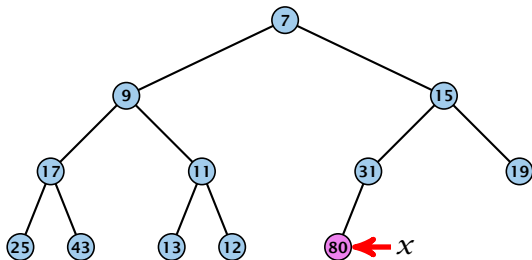
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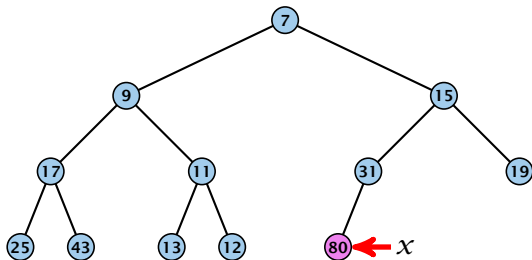
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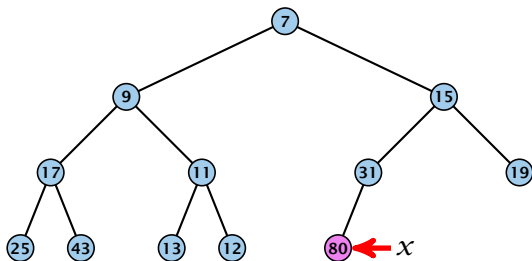
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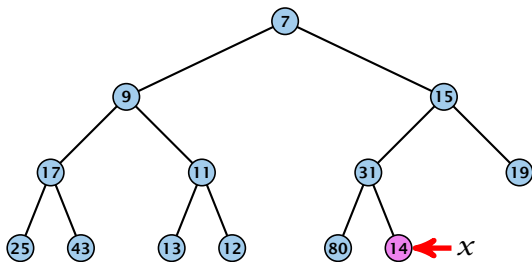
1. Insert element at successor of  $x$ .
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Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

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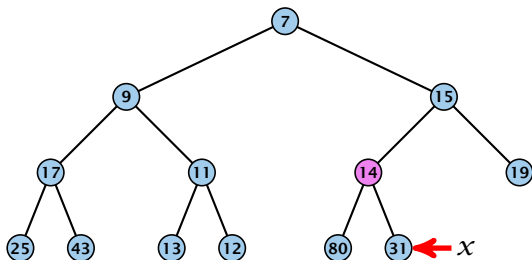
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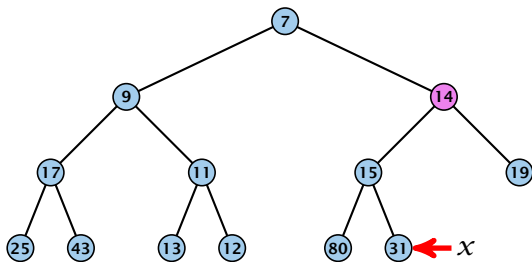
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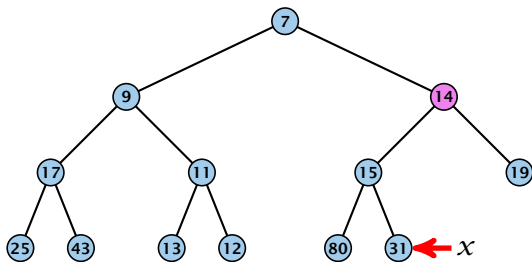
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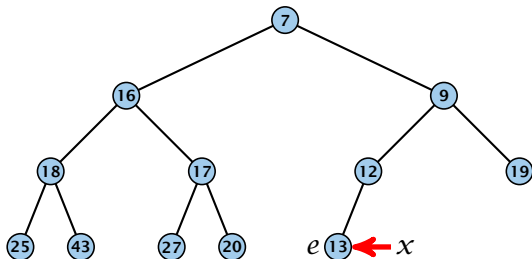


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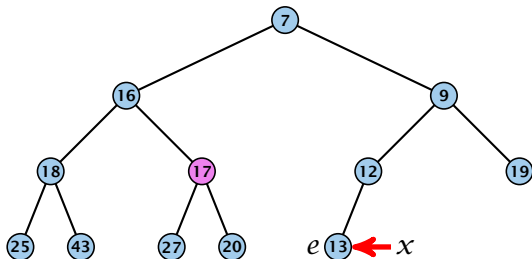
1. Exchange the element to be deleted with the element  $e$  pointed to by  $x$ .
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At its new position  $e$  may either travel up or down in the tree (but not both directions).

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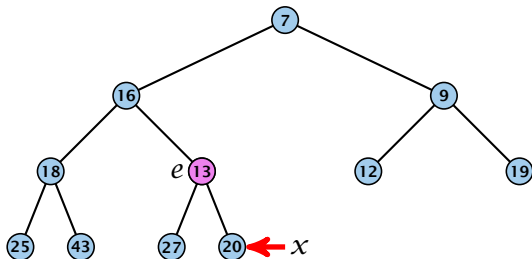
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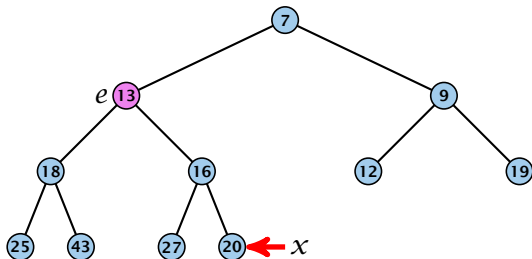
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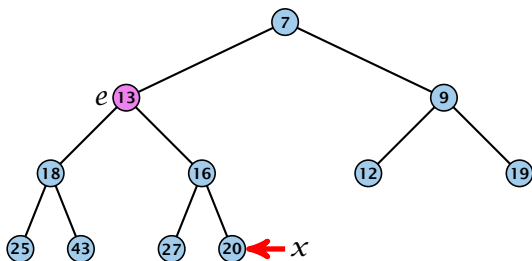
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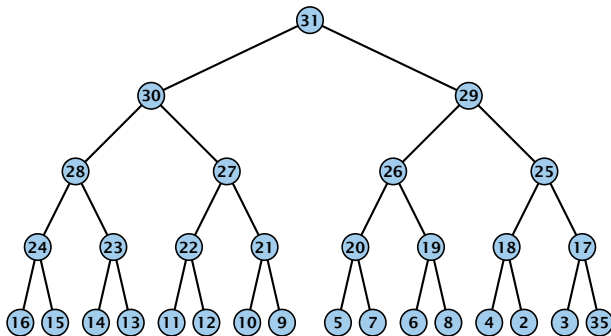
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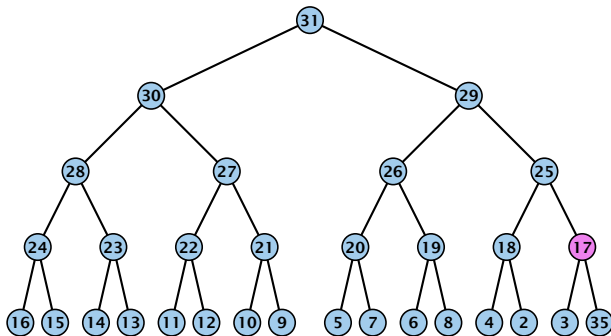
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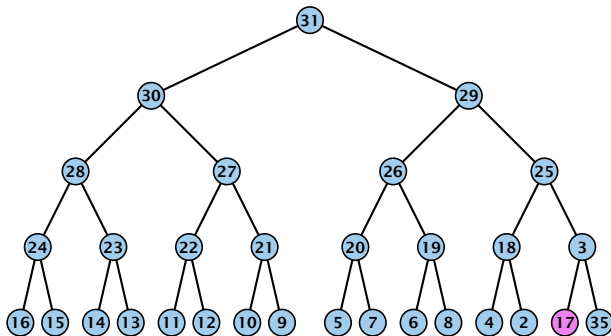


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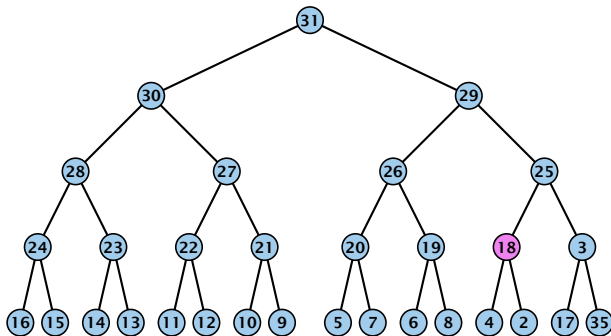
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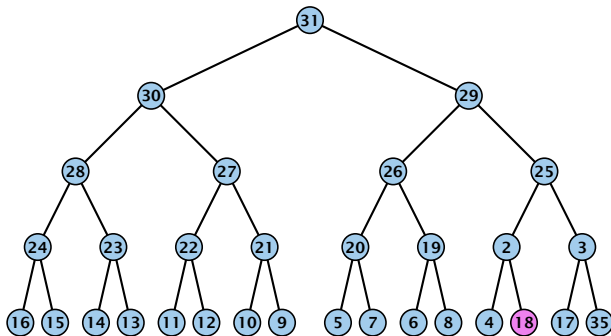
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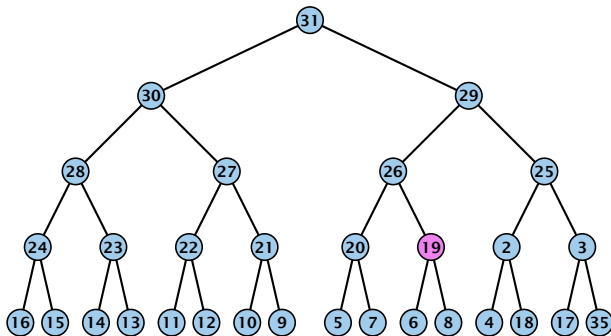
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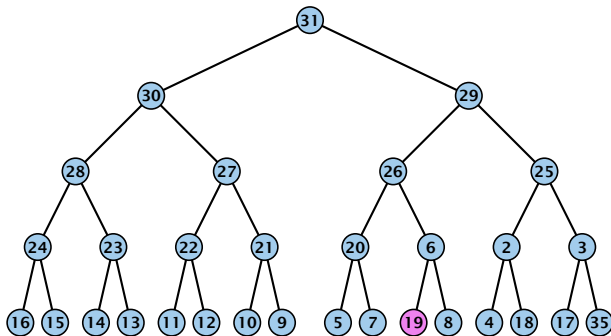
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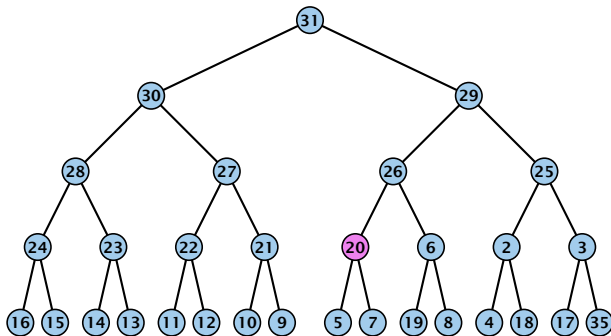
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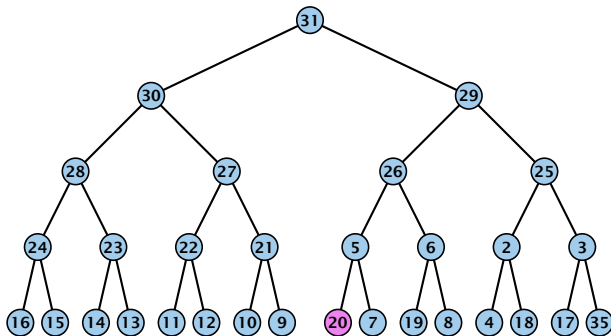
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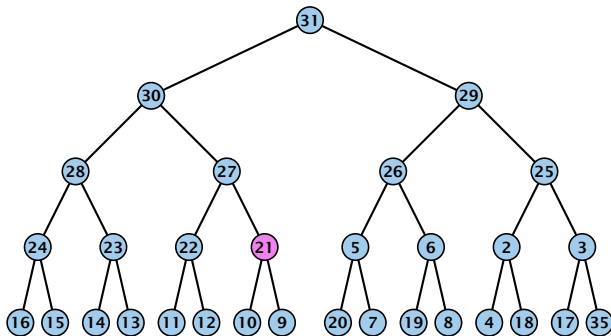
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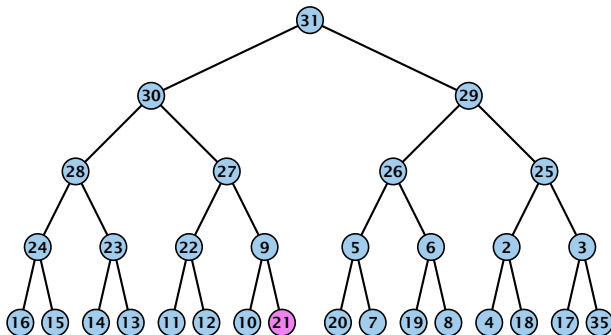


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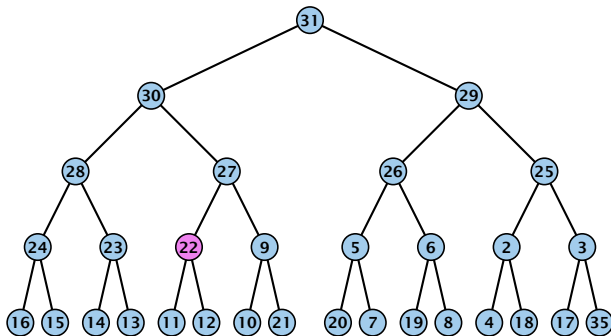
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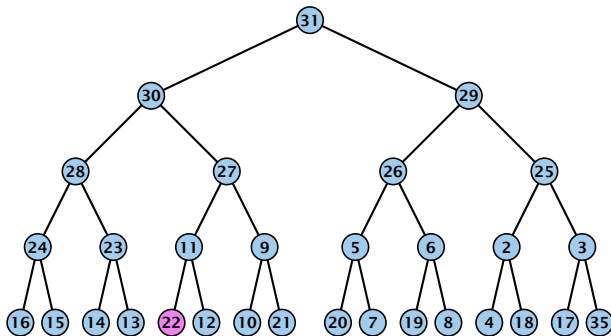
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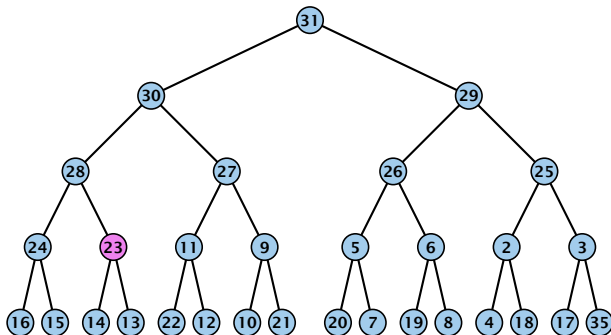
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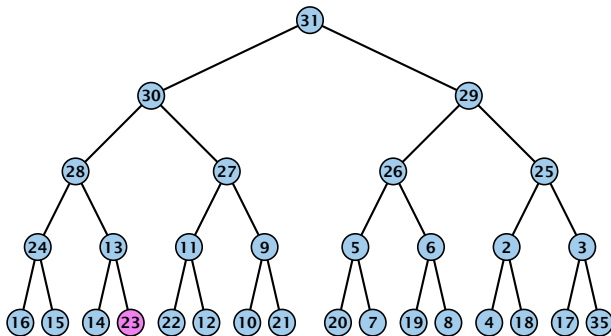
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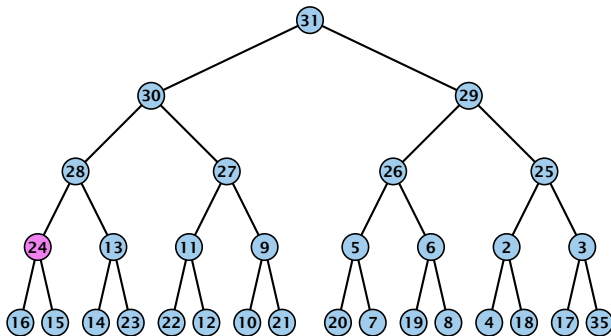
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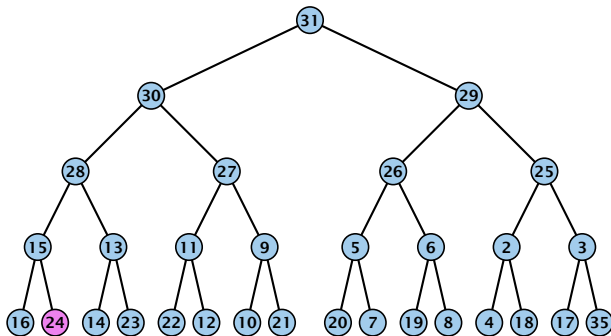
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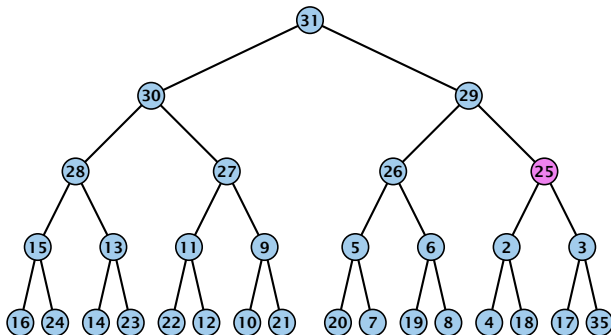
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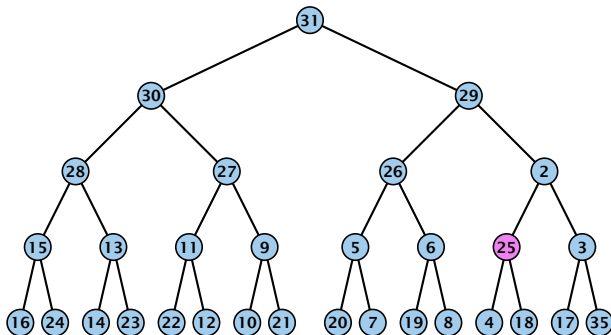


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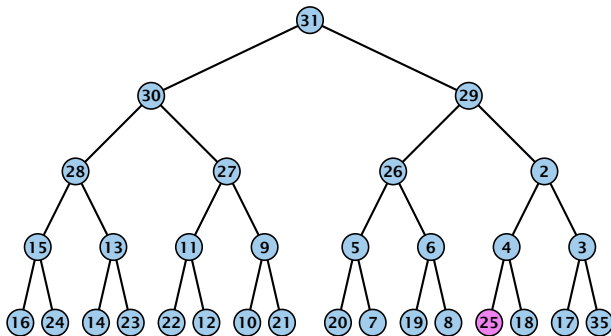
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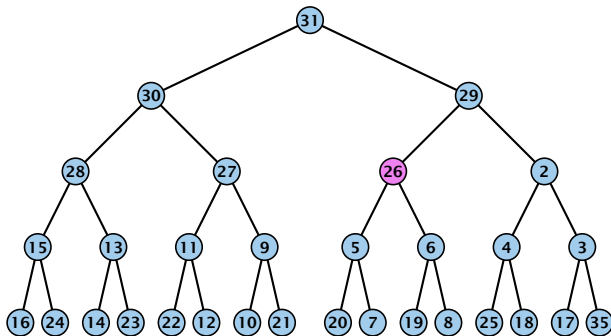
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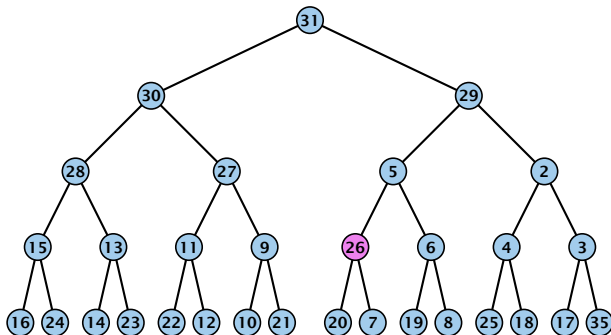
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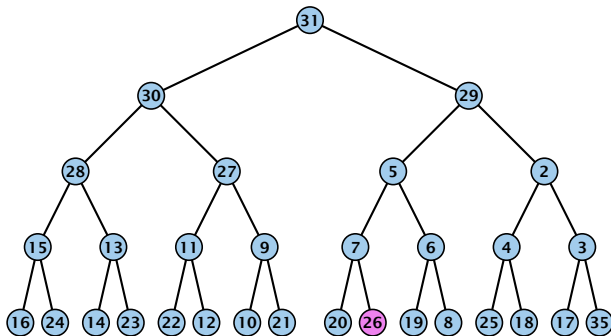
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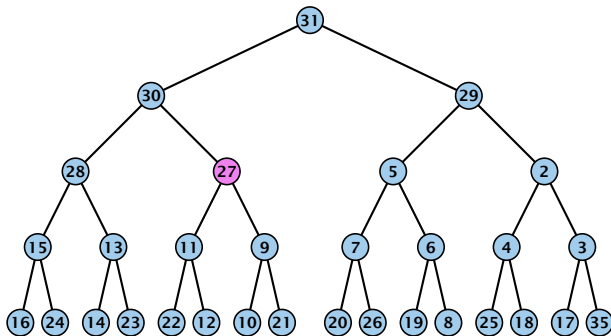
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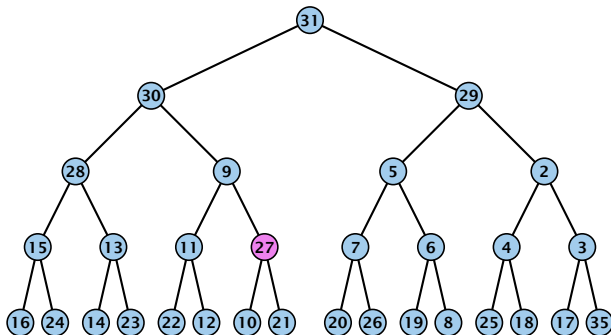
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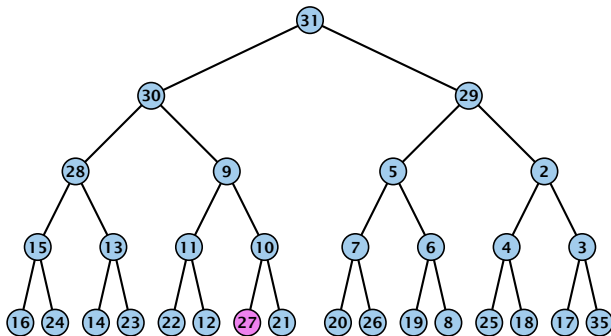
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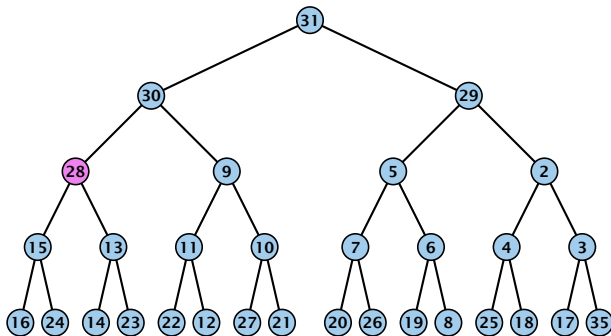


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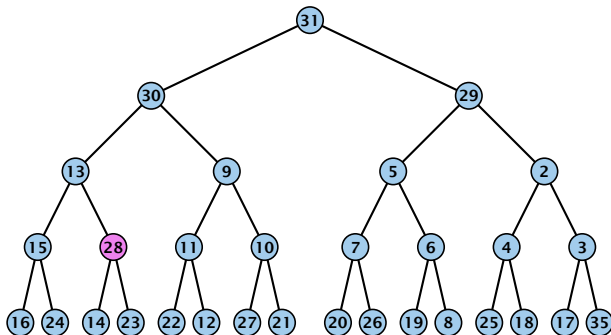
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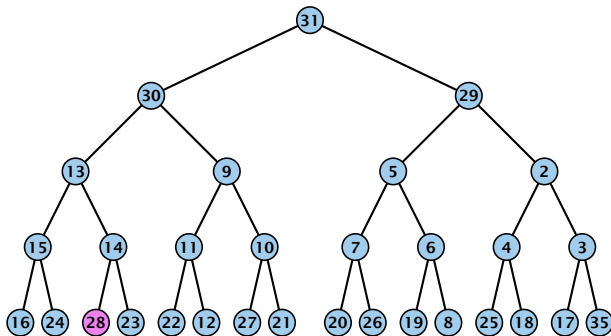
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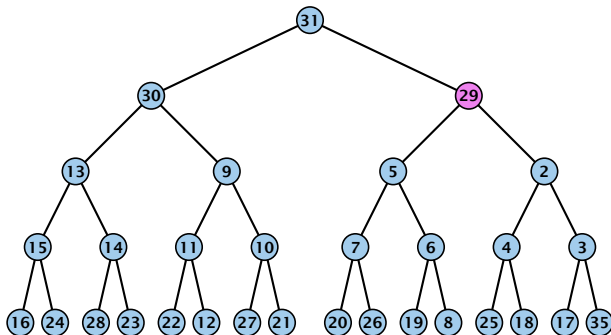
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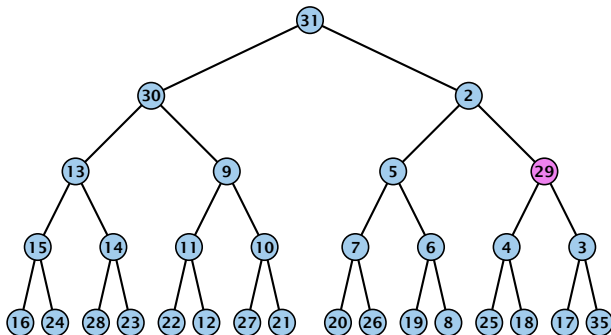
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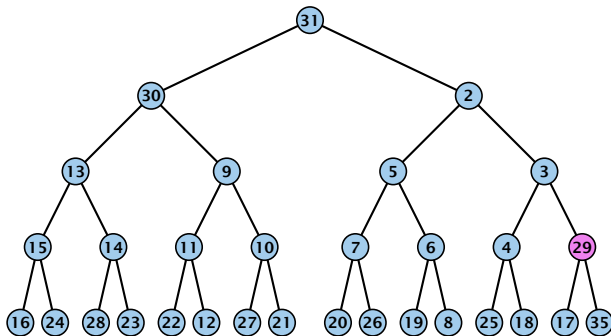
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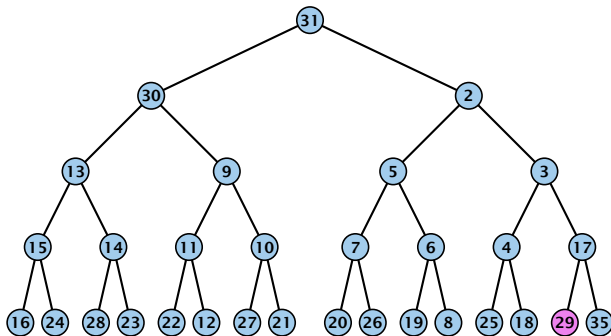
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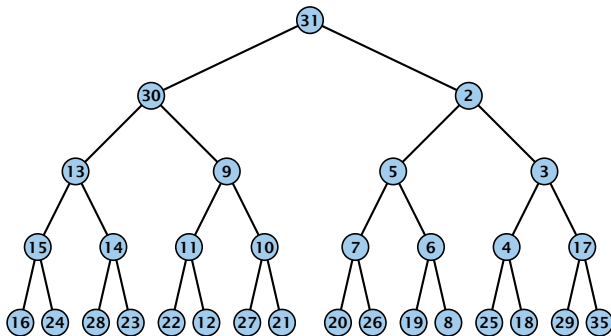
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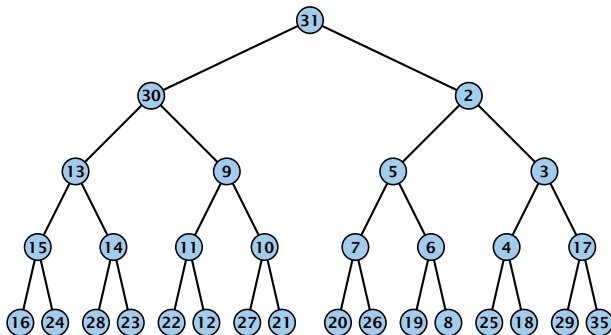


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- ▶ **delete( $h$ )**: Swap with  $x$  and bubble up or sift-down. Time  $\mathcal{O}(\log n)$ .
- ▶ **build( $x_1, \dots, x_n$ )**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time  $\mathcal{O}(n)$ .

# Binary Heaps

The standard implementation of binary heaps is via arrays. Let  $A[0, \dots, n-1]$  be an array

- ▶ The parent of  $i$ -th element is at position  $\lfloor \frac{i-1}{2} \rfloor$ .
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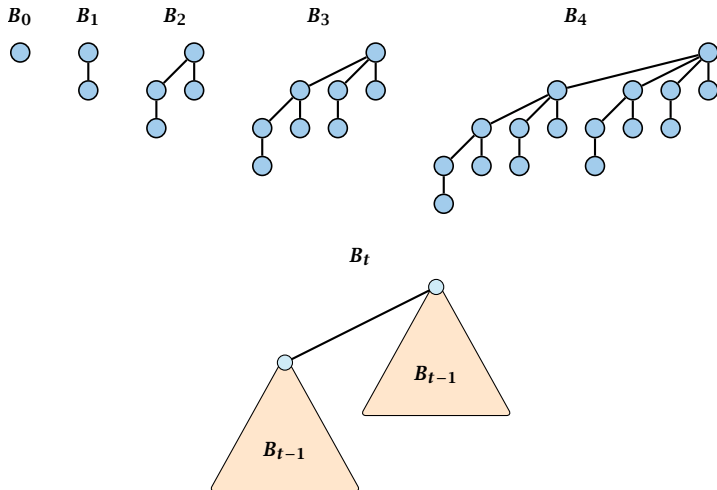
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## 8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	$n$	$n \log n$	$n \log n$	$n$
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	$n$	$n \log n$	<b><math>\log n</math></b>	1

# Binomial Trees





## Properties of Binomial Trees

- ▶  $B_k$  has  $2^k$  nodes.
- ▶  $B_k$  has height  $k$ .
- ▶ The root of  $B_k$  has degree  $k$ .
- ▶  $B_k$  has  $\binom{k}{\ell}$  nodes on level  $\ell$ .
- ▶ Deleting the root of  $B_k$  gives trees  $B_0, B_1, \dots, B_{k-1}$ .

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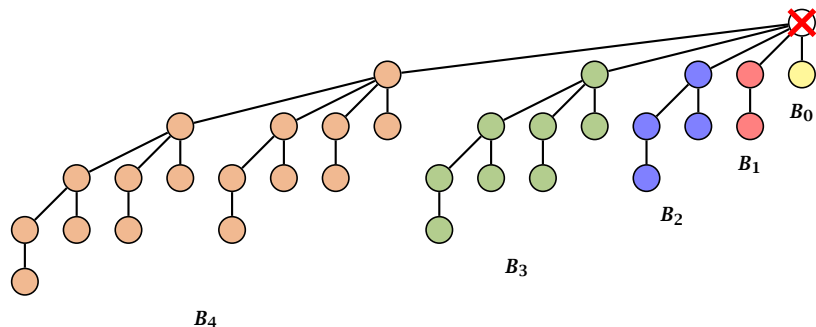
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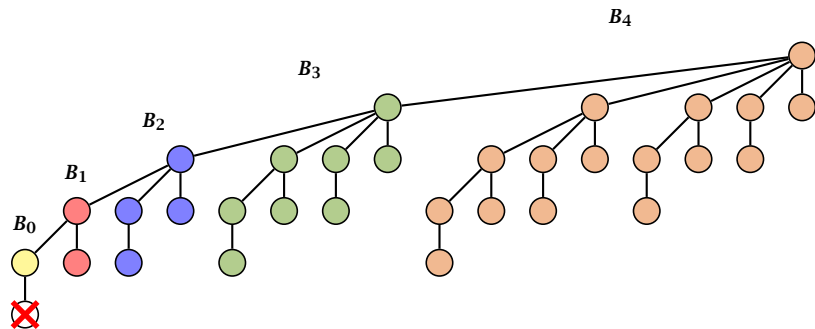
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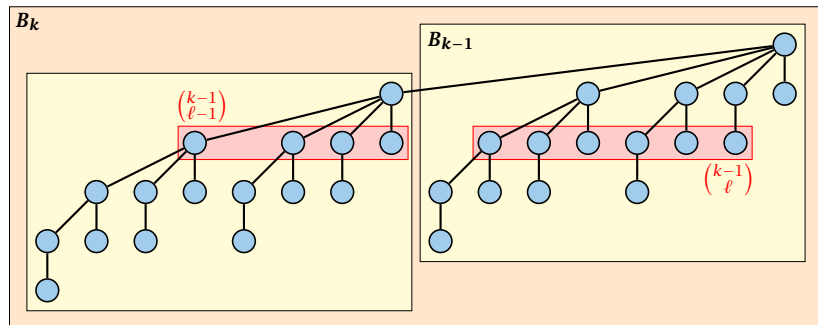
Deleting the root of  $B_5$  leaves sub-trees  $B_4$ ,  $B_3$ ,  $B_2$ ,  $B_1$ , and  $B_0$ .

# Binomial Trees



Deleting the leaf furthest from the root (in  $B_5$ ) leaves a path that connects the roots of sub-trees  $B_4$ ,  $B_3$ ,  $B_2$ ,  $B_1$ , and  $B_0$ .

# Binomial Trees

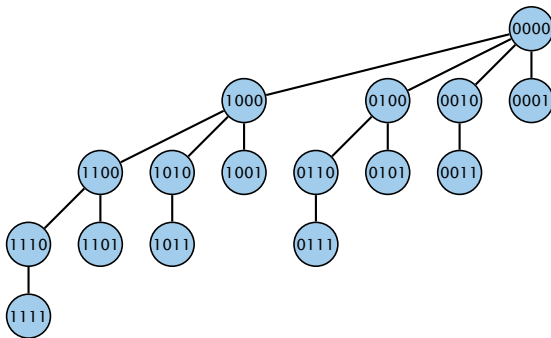


The number of nodes on level  $\ell$  in tree  $B_k$  is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$



# Binomial Trees

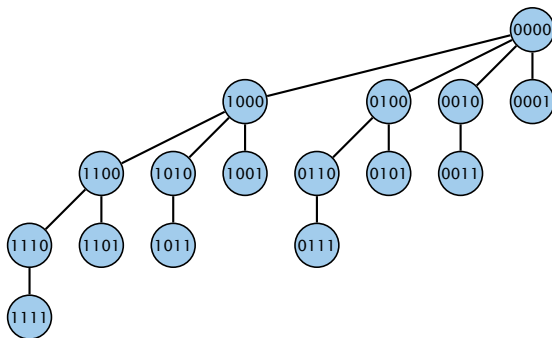


The binomial tree  $B_k$  is a sub-graph of the hypercube  $H_k$ .

The parent of a node with label  $b_n, \dots, b_1, b_0$  is obtained by setting the least significant 1-bit to 0.

The  $\ell$ -th level contains nodes that have  $\ell$  1's in their label.

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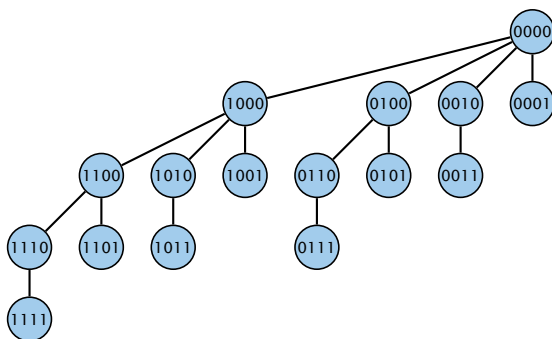


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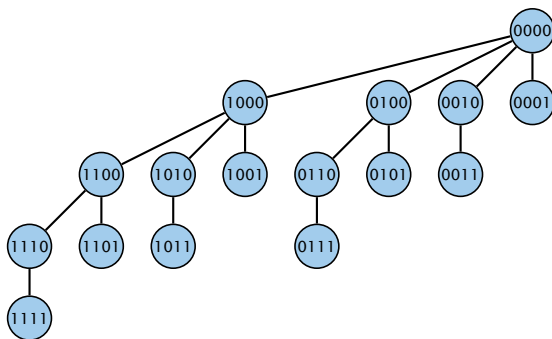


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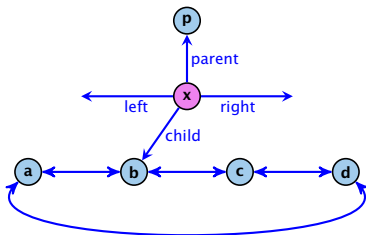
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## 8.2 Binomial Heaps

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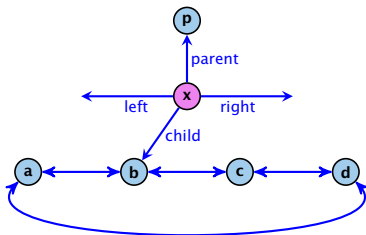
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
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- ▶ Pointers  $x.\text{left}$  and  $x.\text{right}$  point to the left and right sibling of  $x$  (if  $x$  does not have siblings then  $x.\text{left} = x.\text{right} = x$ ).



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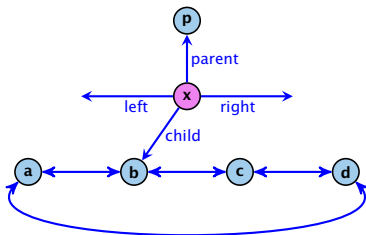
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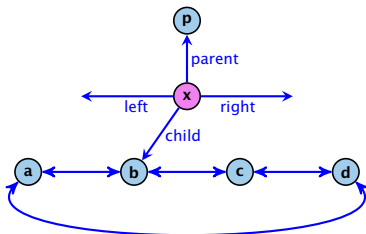
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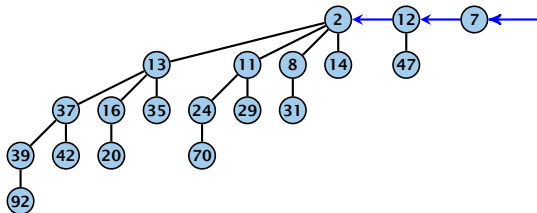




## 8.2 Binomial Heaps

- ▶ Given a pointer to a node  $x$  we can splice out the sub-tree rooted at  $x$  in constant time.
- ▶ We can add a child-tree  $T$  to a node  $x$  in constant time if we are given a pointer to  $x$  and a pointer to the root of  $T$ .

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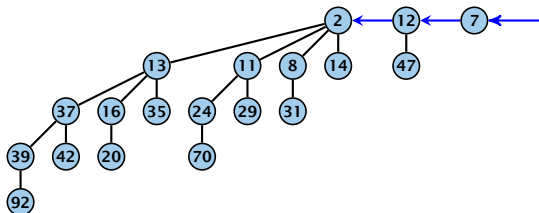


In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees  $B_0$ ,  $B_1$ , and  $B_4$ .

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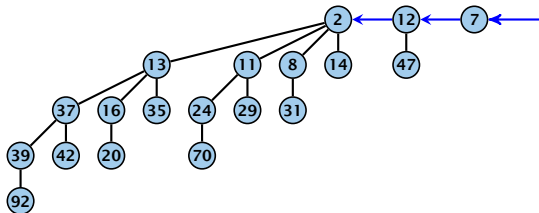


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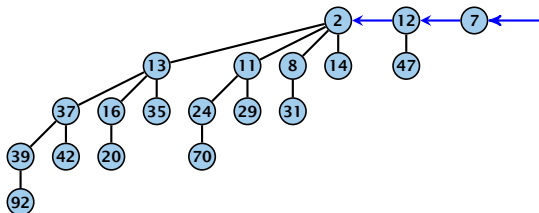


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Given the number  $n$  of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

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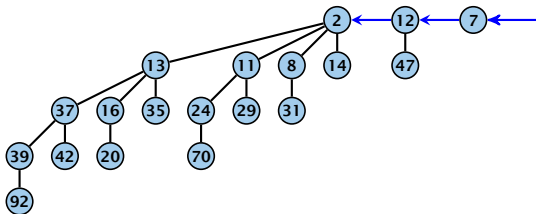
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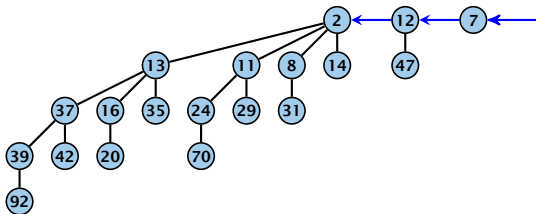
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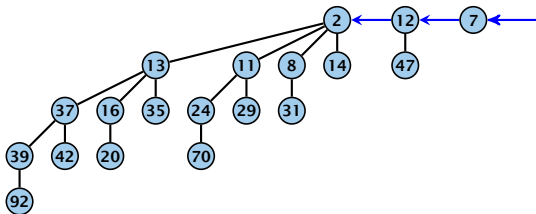
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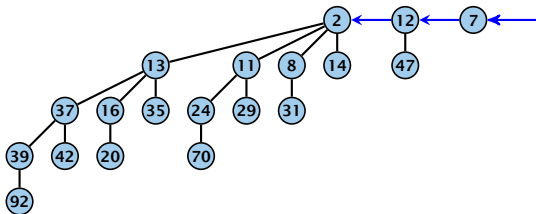
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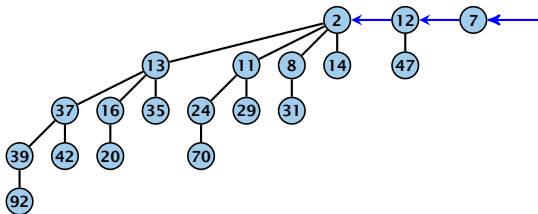
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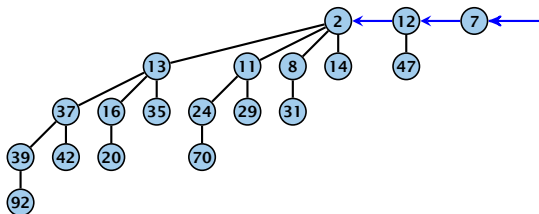
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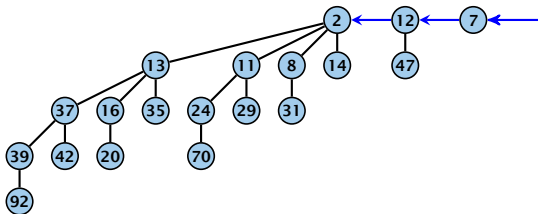
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# Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For many trees this technique is analogous to binary addition.



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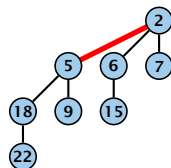
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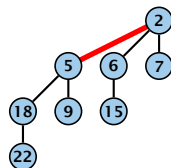
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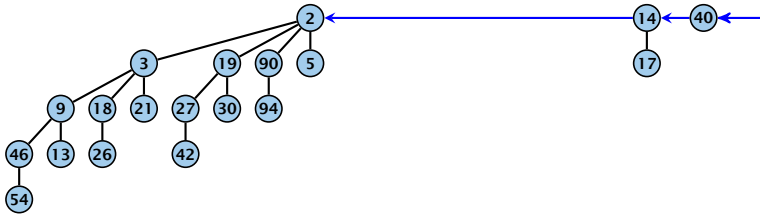
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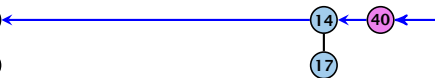
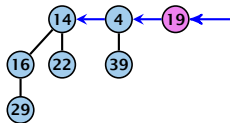
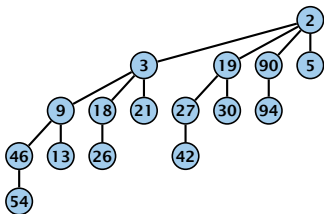
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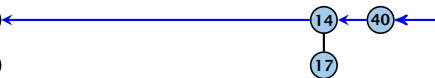
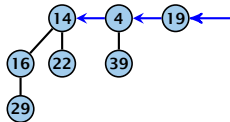
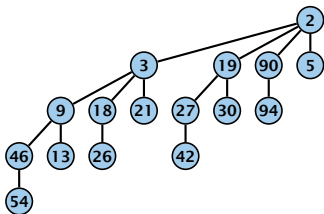
Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.

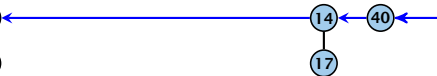
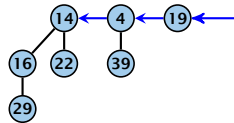
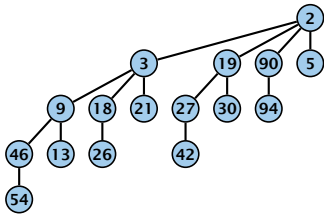


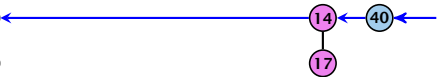
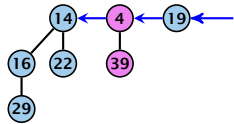
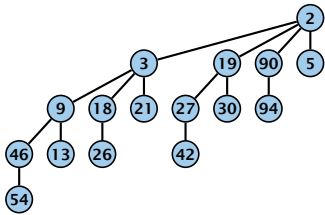


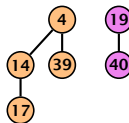
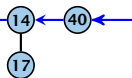
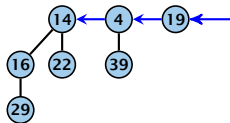
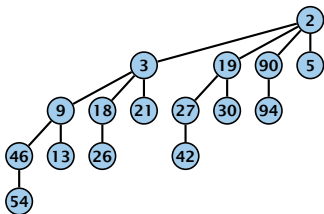


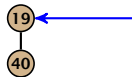
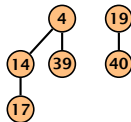
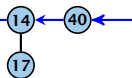
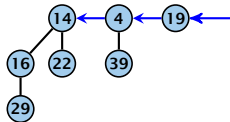
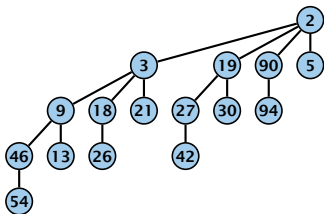


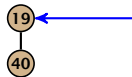
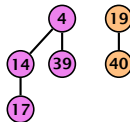
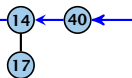
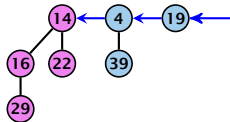
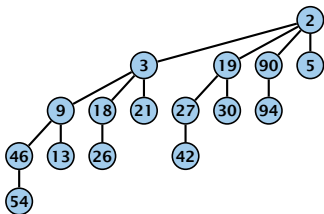


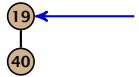
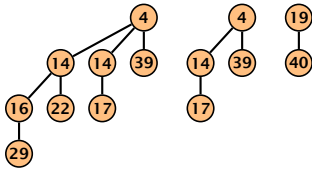
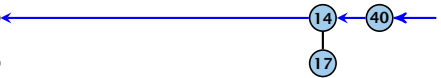
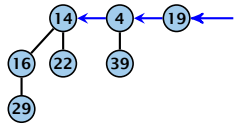
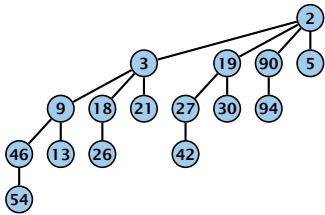


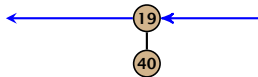
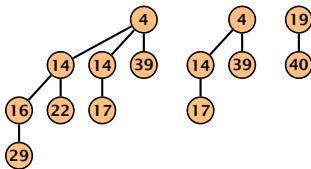
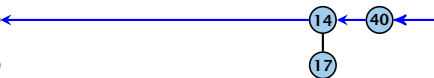
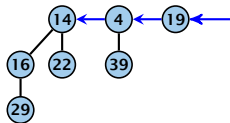
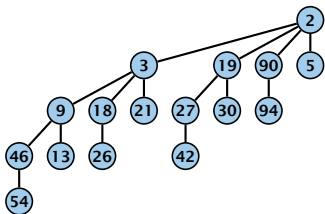


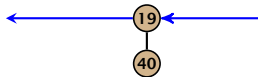
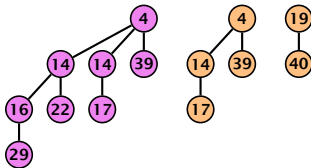
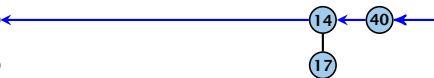
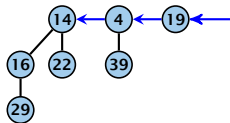
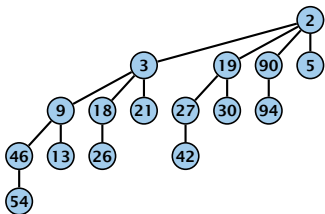






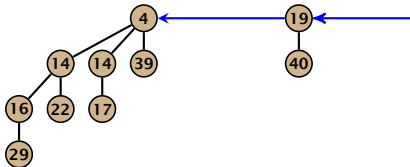
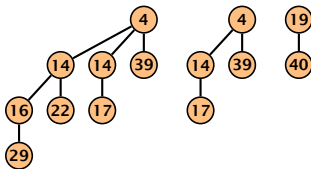
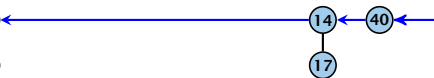
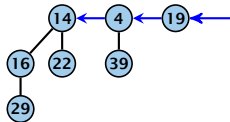
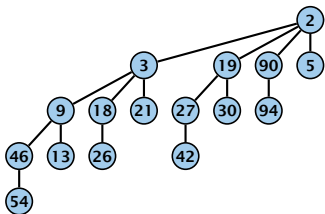




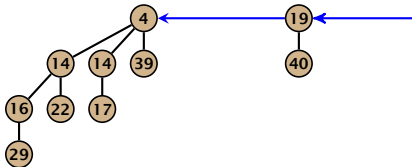
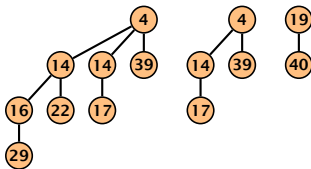
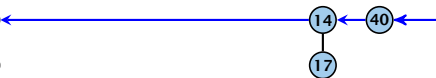
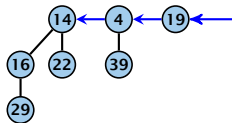
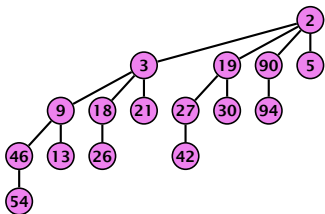




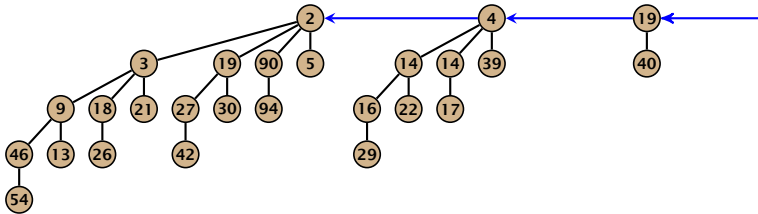
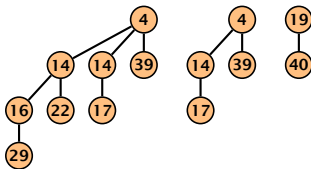
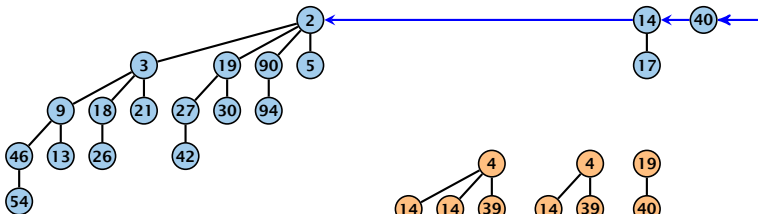
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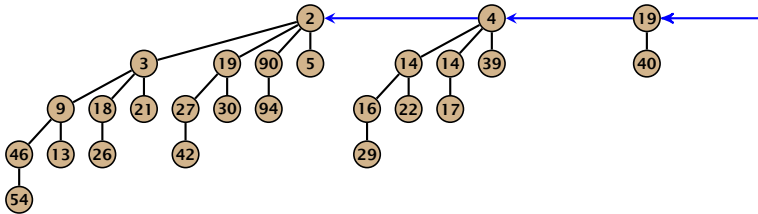
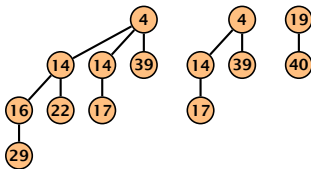
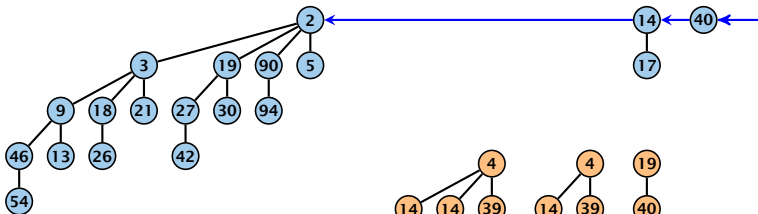
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## 8.2 Binomial Heaps

**$S_1$ . merge( $S_2$ ):**

- ▶ Analogous to binary addition.
- ▶ Time is proportional to the number of trees in both heaps.
- ▶ Time:  $\mathcal{O}(\log n)$ .

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**`S.insert(x)`:**

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### **S. minimum():**

- ▶ Find the minimum key-value among all roots.
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## 8.2 Binomial Heaps

### **S. delete-min():**

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### **S. decrease-key(handle $h$ ):**

- ▶ Decrease the key of the element pointed to by  $h$ .
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time:  $\mathcal{O}(\log n)$  since the trees have height  $\mathcal{O}(\log n)$ .

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### ***S*. delete(handle *h*):**

- ▶ Execute *S*. decrease-key(*h*,  $-\infty$ ).
- ▶ Execute *S*. delete-min().
- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

### **$S$ . delete(handle $h$ ):**

- ▶ Execute  $S$ . decrease-key( $h, -\infty$ ).
- ▶ Execute  $S$ . delete-min().
- ▶ Time:  $\mathcal{O}(\log n)$ .

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## 8.2 Binomial Heaps

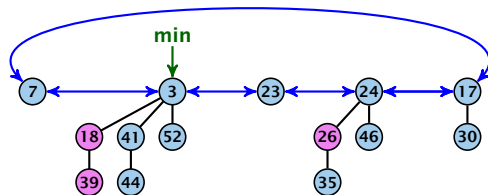
**$S$ . delete(handle  $h$ ):**

- ▶ Execute  $S$ . decrease-key( $h, -\infty$ ).
- ▶ Execute  $S$ . delete-min().
- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



## 8.3 Fibonacci Heaps

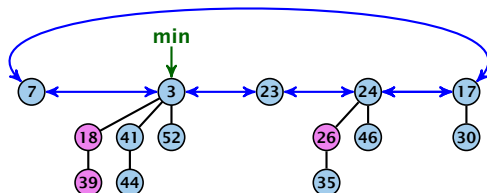
### Additional implementation details:

- ▶ Every node  $x$  stores its degree in a field  $x.degree$ . Note that this can be updated in constant time when adding a child to  $x$ .
- ▶ Every node stores a boolean value  $x.marked$  that specifies whether  $x$  is **marked** or not.

## 8.3 Fibonacci Heaps

### The potential function:

- ▶  $t(S)$  denotes the number of trees in the heap.
- ▶  $m(S)$  denotes the number of marked nodes.
- ▶ We use the potential function  $\Phi(S) = t(S) + 2m(S)$ .



The potential is  $\Phi(S) = 5 + 2 \cdot 3 = 11$ .

## 8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use  $c$  to denote the amount of work that a unit of potential can pay for.

## 8.3 Fibonacci Heaps

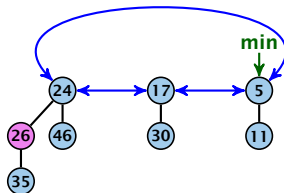
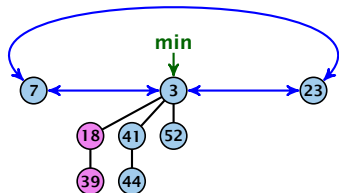
### S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Amortized cost  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### $S$ . merge( $S'$ )

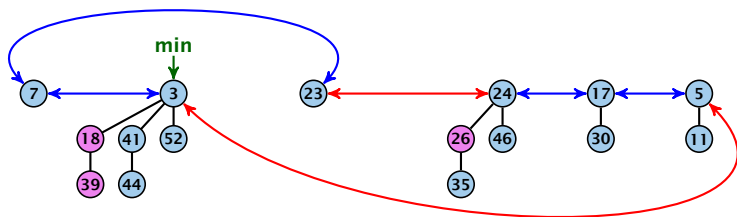
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



## 8.3 Fibonacci Heaps

### S. merge( $S'$ )

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



### Running time:

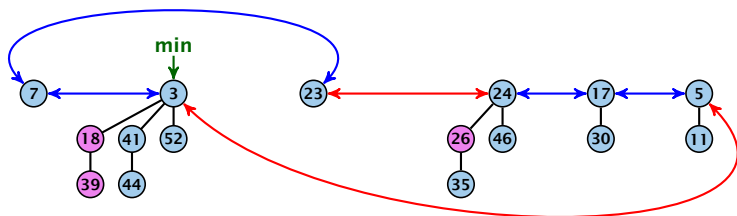
- ▶ Actual cost  $\mathcal{O}(1)$ .



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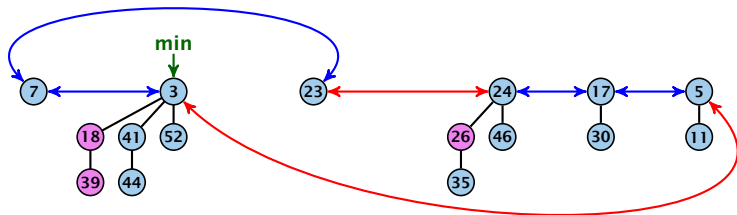
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## 8.3 Fibonacci Heaps

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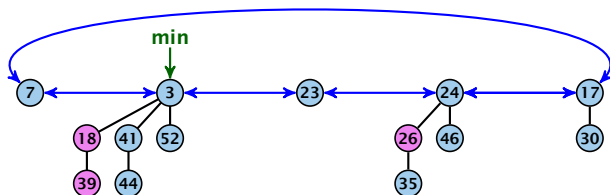
### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Hence, amortized cost is  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### S. insert( $x$ )

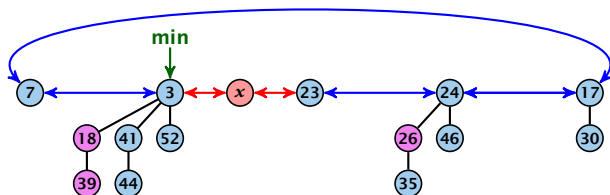
- ▶ Create a new tree containing  $x$ .
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- ▶ Update min-pointer, if necessary.



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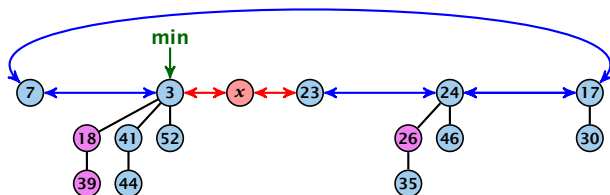
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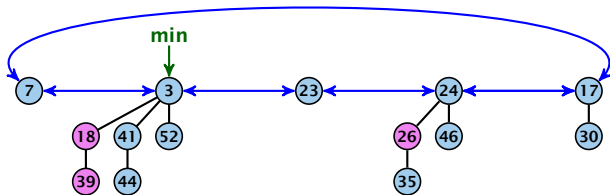


### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ Change in potential is  $+1$ .
- ▶ Amortized cost is  $c + \mathcal{O}(1) = \mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

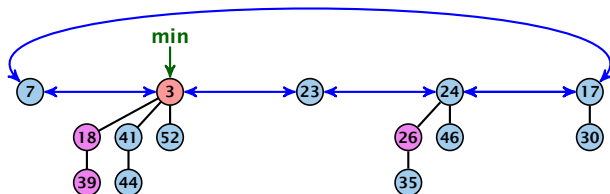
S. delete-min( $x$ )



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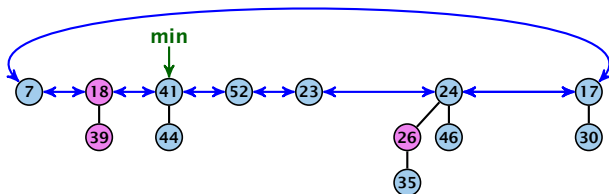
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## 8.3 Fibonacci Heaps

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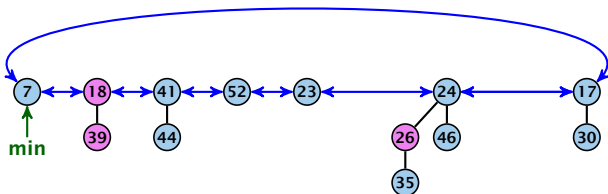




## 8.3 Fibonacci Heaps

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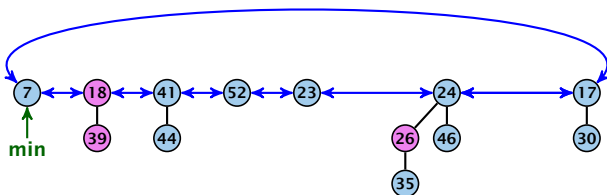
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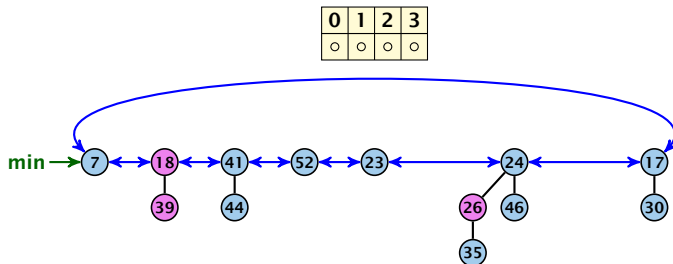
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- ▶ Consolidate root-list so that no roots have the same degree. Time  $t \cdot \mathcal{O}(1)$  (see next slide).

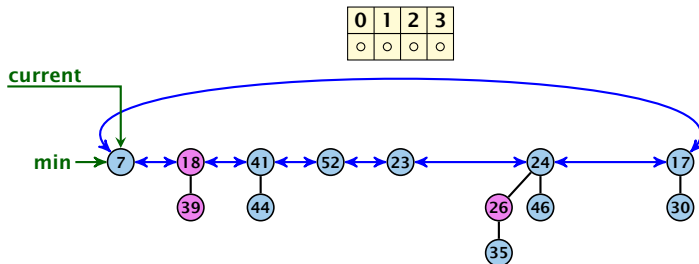
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Consolidate:



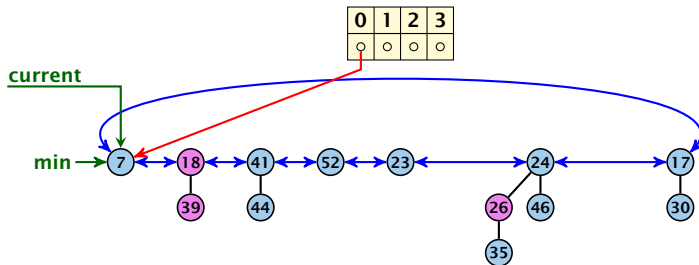
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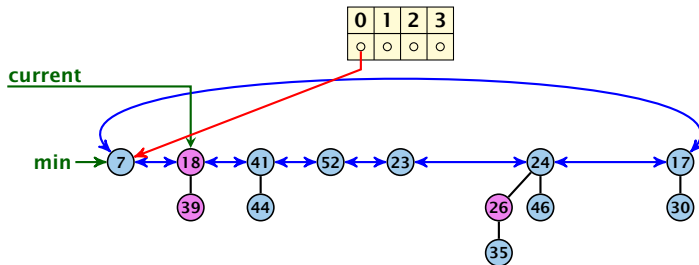
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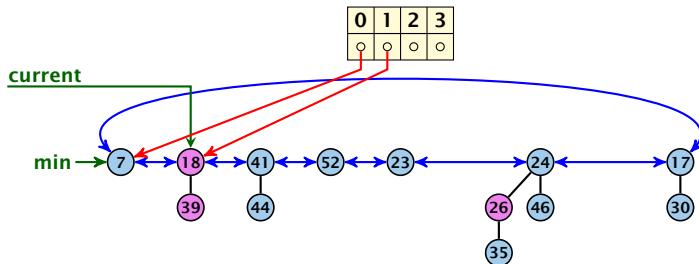
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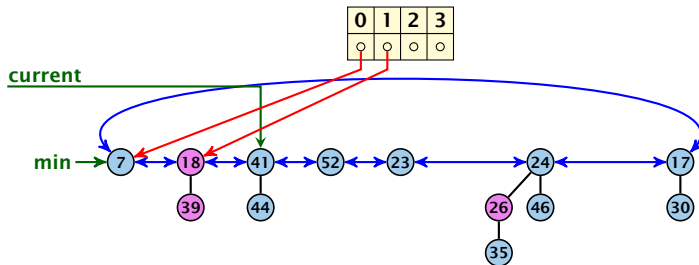
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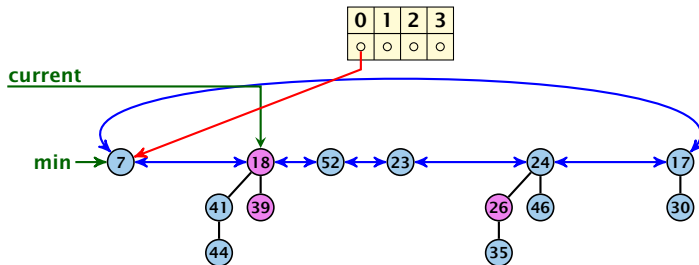
Consolidate:





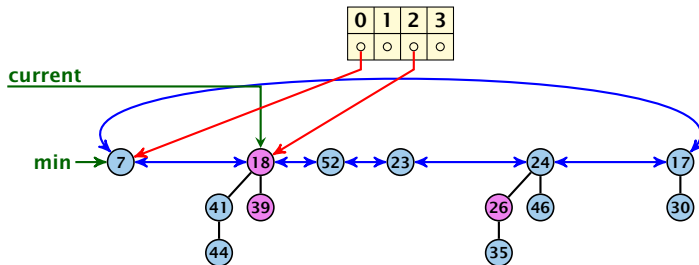
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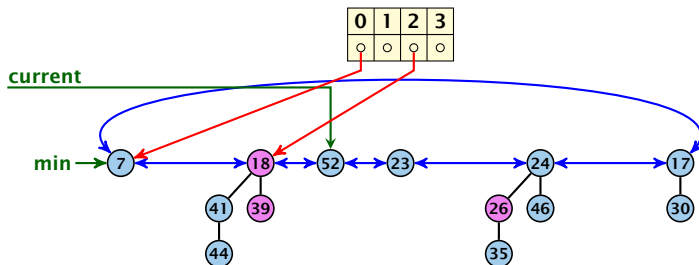
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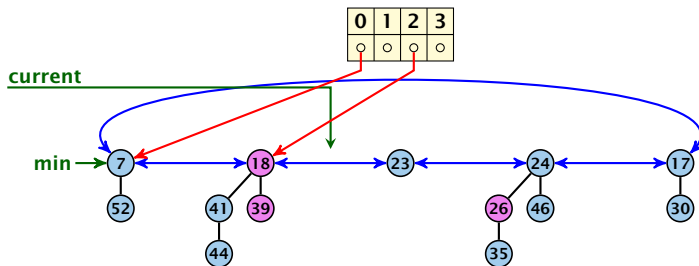
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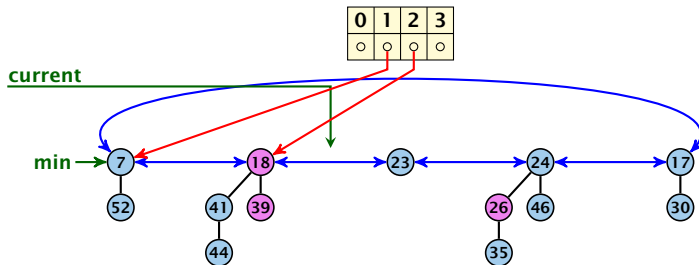
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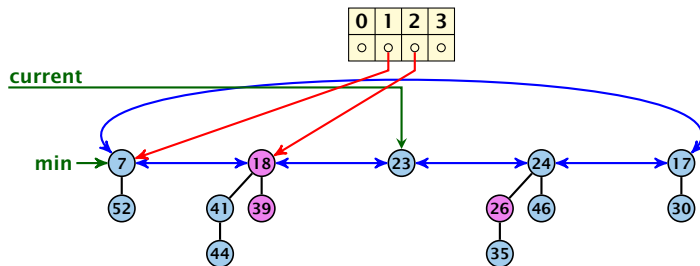
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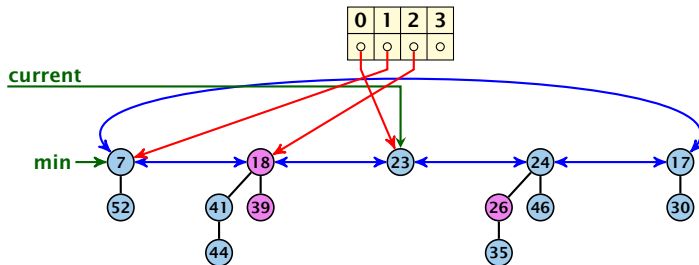
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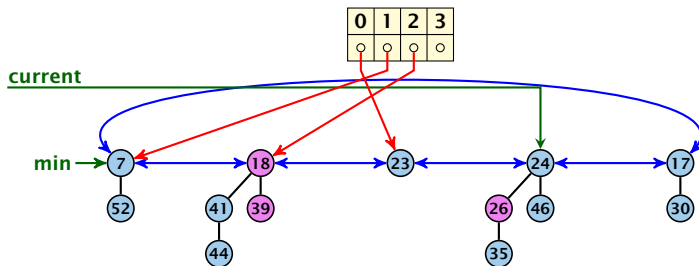
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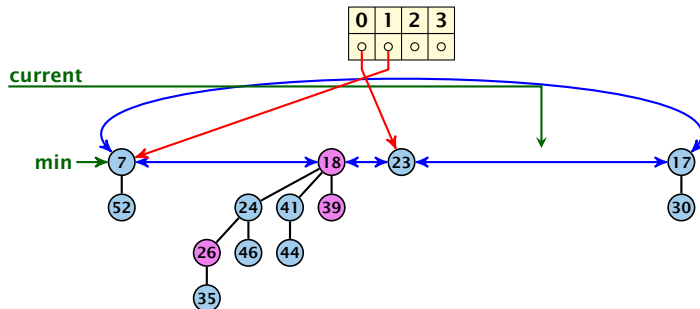
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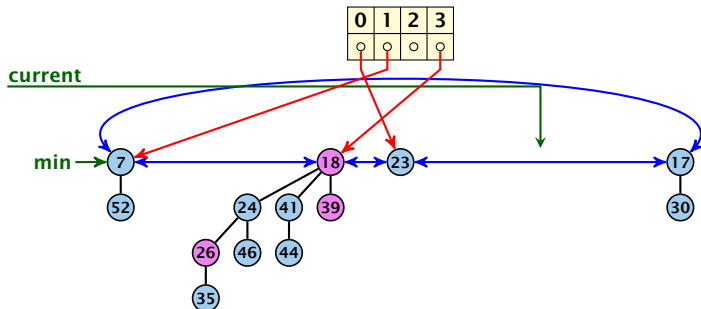
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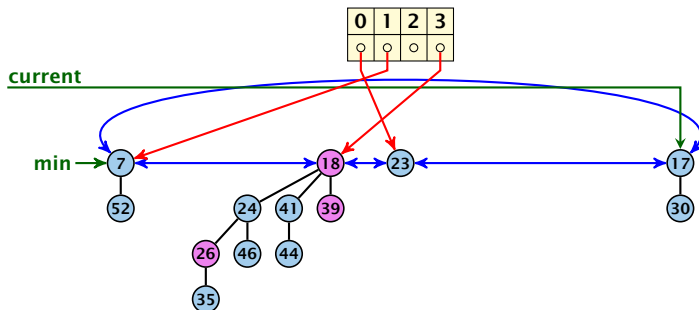
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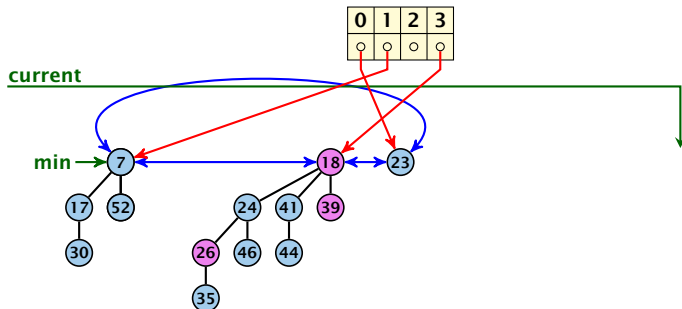
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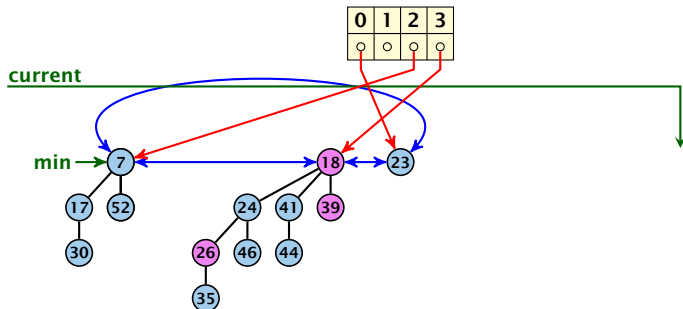
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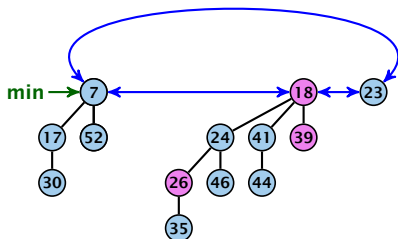
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for  $c \geq c_1$  .

## 8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

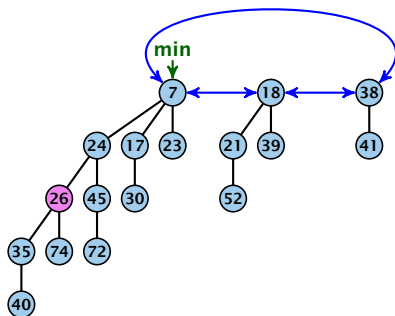
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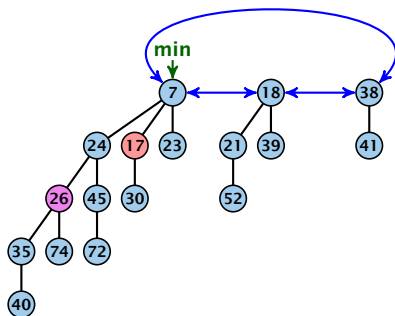
## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by  $h$ . Nothing else to do.

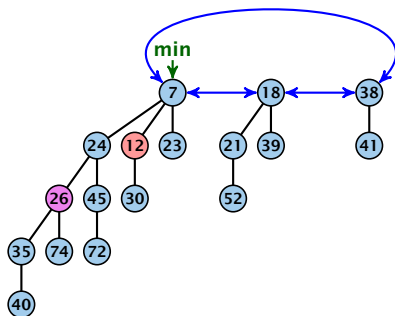
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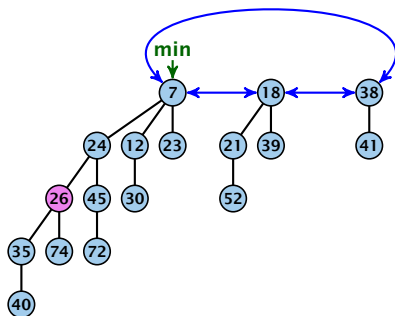
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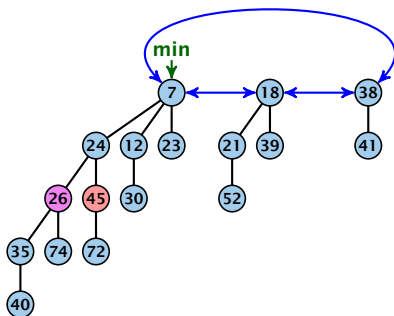
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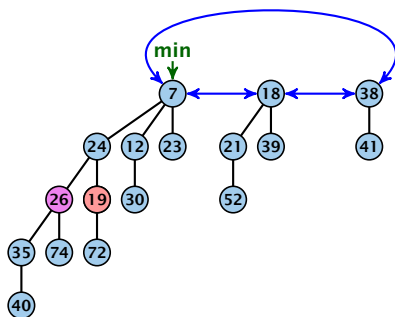


### Case 2: heap-property is violated, but parent is not marked

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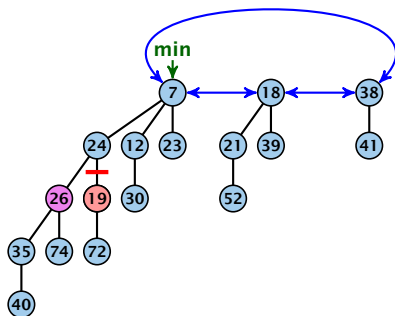
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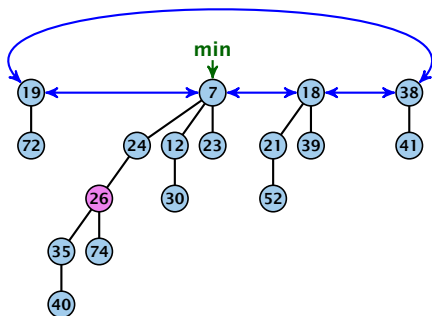
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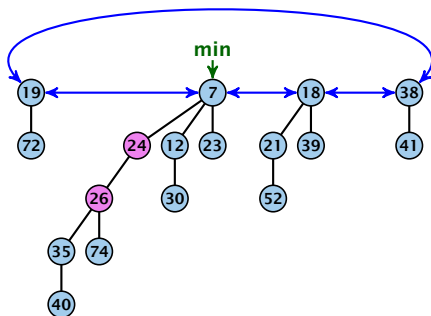
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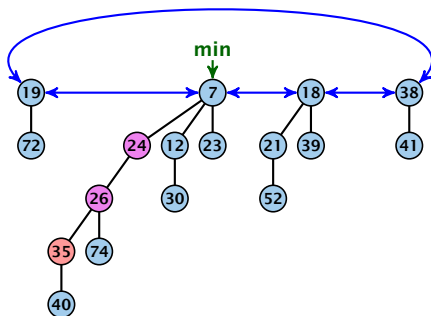
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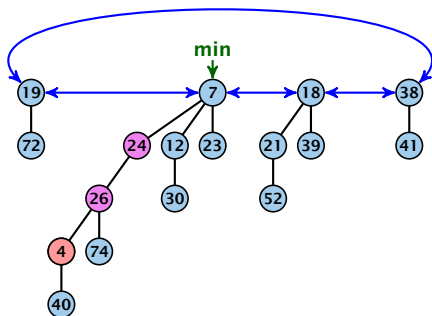
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### Case 3: heap-property is violated, and parent is marked

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- ▶ Continue cutting the parent until you arrive at an unmarked node.

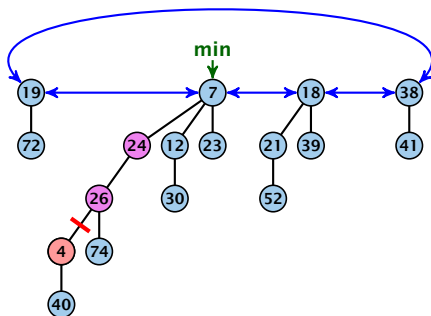
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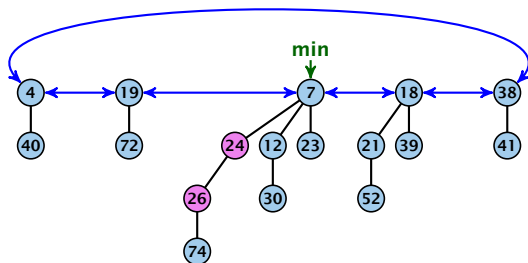
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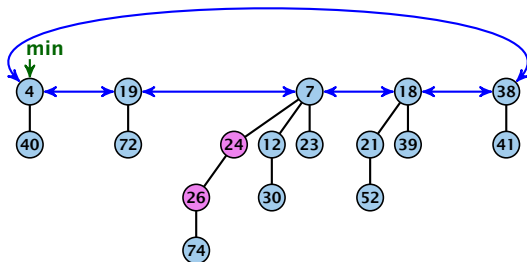


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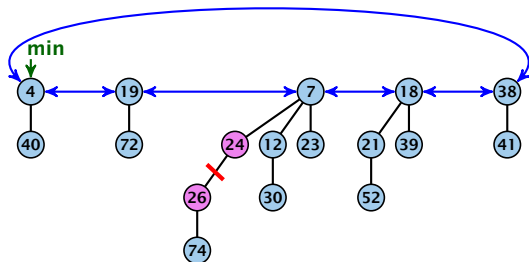
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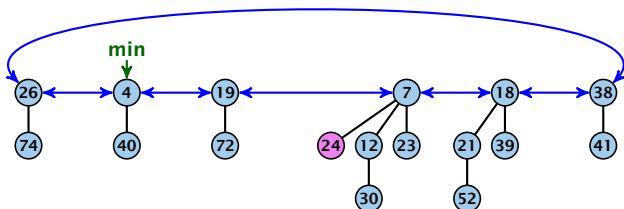
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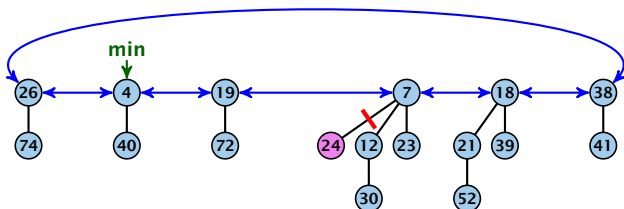
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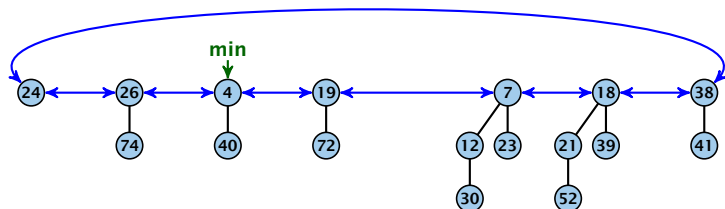
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- ▶ Cut the parent edge of  $x$ , and make  $x$  into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

# Fibonacci Heaps: decrease-key(handle $h, v$ )

## Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of  $\ell$  cuts.
- ▶ Hence, cost is at most  $c_2 \cdot (\ell + 1)$ , for some constant  $c_2$ .

## Amortized cost:

- ▶ Every time we cut, we mark a node. As every cut creates one new root, the number of roots increases by one. As every root has the first cut mark, the number of roots is at least the number of cut marks. As every cut marks a node, the last cut may mark a node.

- ▶ Amortized cost is at most  $c_2$ .

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### Amortized cost:

- ▶ Every cut creates one new root.
- ▶ Every root has at most  $\ell$  children.
- ▶ Every root marks a node, the last cut may mark a node.
- ▶ Amortized cost is  $O(\log n)$ .



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## Amortized cost:

- ▶ Every cut creates one new root.
- ▶ Every root has at most  $\ell$  children.
- ▶ Hence, the number of roots is at most  $\ell + 1$ .
- ▶ Hence, the number of nodes is at most  $(\ell + 1)^2$ .
- ▶ Hence, the cost of a decrease-key is at most  $c_2 \cdot (\ell + 1)$ .

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## Amortized cost:

- ▶  $t' = t + \ell$ , as every cut creates one new root.
- ▶  $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$ , since all but the first cut unmarks a node; the last cut may mark a node.
- ▶  $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
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$$\text{if } c \geq c_2.$$

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# Delete node

***H. delete(x):***

- ▶ decrease value of  $x$  to  $-\infty$ .
- ▶ delete-min.

**Amortized cost:  $\mathcal{O}(D_n)$**

- ▶  $\mathcal{O}(1)$  for decrease-key.
- ▶  $\mathcal{O}(D_n)$  for delete-min.

## 8.3 Fibonacci Heaps

### Lemma 24

Let  $x$  be a node with degree  $k$  and let  $y_1, \dots, y_k$  denote the children of  $x$  in the order that they were linked to  $x$ . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

## 8.3 Fibonacci Heaps

### Proof

- ▶ When  $y_i$  was linked to  $x$ , at least  $y_1, \dots, y_{i-1}$  were already linked to  $x$ .
- ▶ Hence, at this time  $\text{degree}(x) \geq i - 1$ , and therefore also  $\text{degree}(y_i) \geq i - 1$  as the algorithm links nodes of equal degree only.
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## 8.3 Fibonacci Heaps

### Definition 25

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

### Facts:

1.  $F_k \geq \phi^k$ .
2. For  $k \geq 2$ :  $F_k = 2 + \sum_{i=0}^{k-2} F_i$ .

The above facts can be easily proved by induction. From this it follows that  $s_k \geq F_k \geq \phi^k$ , which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$

## 9 Union Find

**Union Find Data Structure  $\mathcal{P}$ :** Maintains a partition of **disjoint** sets over elements.

- ▶  $\mathcal{P}$ . **makeset**( $x$ ): Given an element  $x$ , adds  $x$  to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for  $x$  in the data-structure.
- ▶  $\mathcal{P}$ . **find**( $x$ ): Given a handle for an element  $x$ ; find the set that contains  $x$ . Returns a representative/identifier for this set.
- ▶  $\mathcal{P}$ . **union**( $x, y$ ): Given two elements  $x$ , and  $y$  that are currently in sets  $S_x$  and  $S_y$ , respectively, the function replaces  $S_x$  and  $S_y$  by  $S_x \cup S_y$  and returns an identifier for the new set.



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## 9 Union Find

### Algorithm 16 Kruskal-MST( $G = (V, E), w$ )

```
1:  $A \leftarrow \emptyset$ ;  
2: for all  $v \in V$  do  
3:    $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$   
4: sort edges in non-decreasing order of weight  $w$   
5: for all  $(u, v) \in E$  in non-decreasing order do  
6:   if  $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$  then  
7:      $A \leftarrow A \cup \{(u, v)\}$   
8:      $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$ 
```

# List Implementation

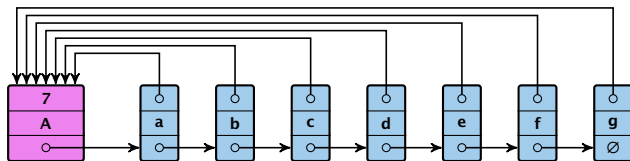
- ▶ The elements of a set are stored in a list; each node has a backward pointer to the head.
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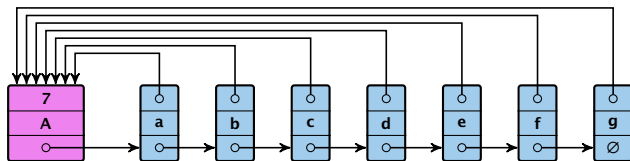


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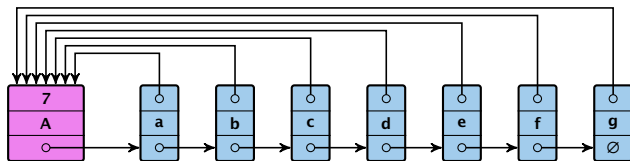
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- ▶ Determine sets  $S_x$  and  $S_y$ .
- ▶ Traverse the smaller list (say  $S_y$ ), and change all backward pointers to the head of list  $S_x$ .
- ▶ Insert list  $S_y$  at the head of  $S_x$ .
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- ▶ Time:  $\min\{|S_x|, |S_y|\}$ .

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# List Implementation

## **union( $x, y$ )**

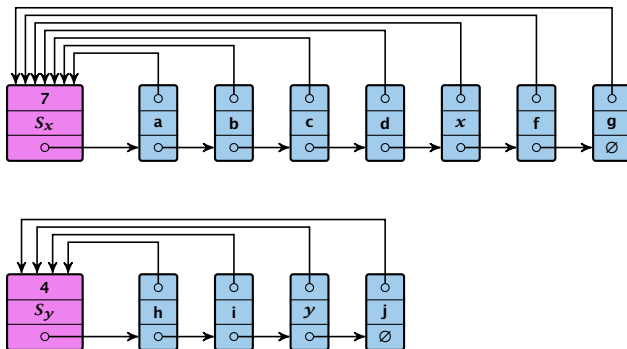
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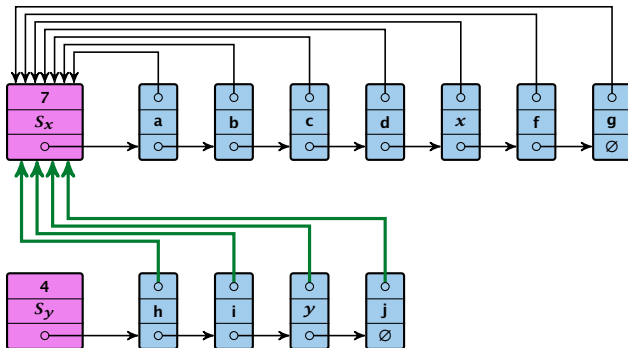
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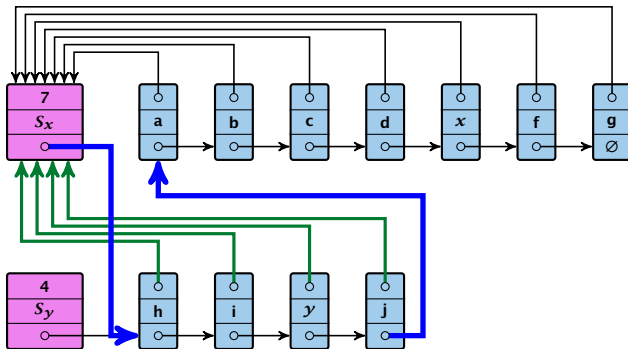




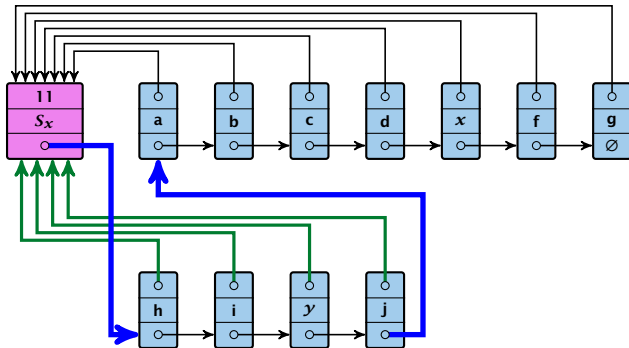
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## Running times:

- ▶  $\text{find}(x)$ : constant
- ▶  $\text{makeset}(x)$ : constant
- ▶  $\text{union}(x, y)$ :  $\mathcal{O}(n)$ , where  $n$  denotes the number of elements contained in the set system.

# List Implementation

## Lemma 26

*The list implementation for the ADT union find fulfills the following amortized time bounds:*

- ▶  $\text{find}(x): \mathcal{O}(1)$ .
- ▶  $\text{makeset}(x): \mathcal{O}(\log n)$ .
- ▶  $\text{union}(x, y): \mathcal{O}(1)$ .

# The Accounting Method for Amortized Time Bounds

- ▶ There is a bank account for every element in the data structure.
- ▶ Initially the balance on all accounts is zero.
- ▶ Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
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- ▶ For an operation whose actual cost exceeds the amortized cost we charge the **excess** to the elements involved.
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**union( $x, y$ ):**

Let  $x$  and  $y$  be two disjoint sets. The cost to convert  $x$  and  $y$  into a single set is

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## Lemma 27

*An element is charged at most  $\lfloor \log_2 n \rfloor$  times, where  $n$  is the total number of elements in the set system.*

### Proof.

Whenever an element  $x$  is charged the number of elements in  $x$ 's set doubles. This can happen at most  $\lfloor \log n \rfloor$  times.  $\square$

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- ▶ Maintain nodes of a set in a tree.
- ▶ The root of the tree is the label of the set.
- ▶ Only pointer to parent exists; we cannot list all elements of a given set.
- ▶ Example:

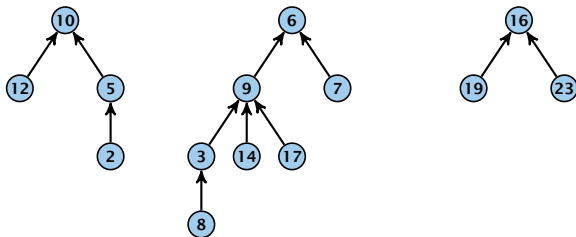


Set system  $\{2, 5, 10, 12\}$ ,  $\{3, 6, 7, 8, 9, 14, 17\}$ ,  $\{16, 19, 23\}$ .



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# Implementation via Trees

## **makeset( $x$ )**

- ▶ Create a singleton tree. Return pointer to the root.
- ▶ Time:  $\mathcal{O}(1)$ .

## **find( $x$ )**

Start at element  $x$  in the tree, and repeatedly update  $x$  to be its parent until it reaches the root.

The root is the element  $x$  such that  $\text{parent}(x) = x$ .

Time complexity:  $\mathcal{O}(h)$ , where  $h$  is the height of the tree.

Amortized time complexity:  $\mathcal{O}(\alpha(n))$ , where  $\alpha$  is the inverse Ackermann function.

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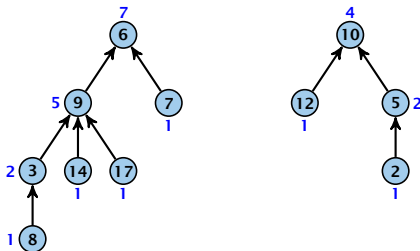
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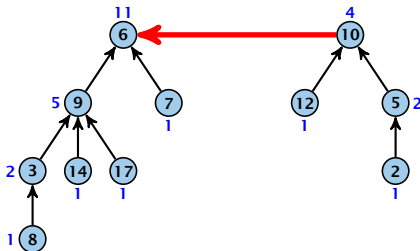


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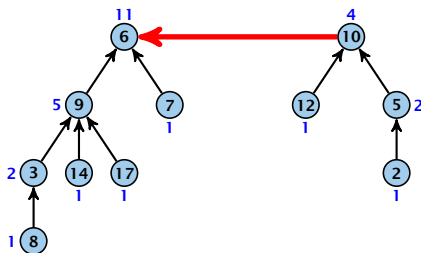


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- ▶ Time: constant for  $\text{link}(a, b)$  plus two find-operations.

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# Path Compression

**find( $x$ ):**

- ▶ Go upward until you find the root.
- ▶ Re-attach all visited nodes as children of the root.
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Time complexity:  $O(\alpha(n))$  (inverse Ackermann function)

Space complexity:  $O(n)$

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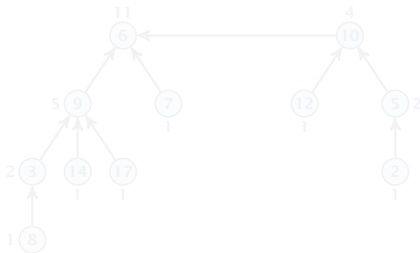
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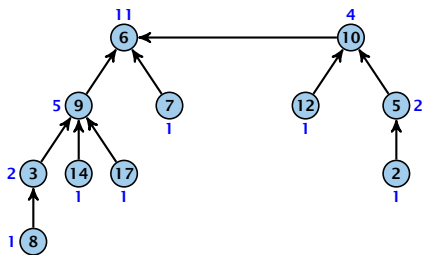
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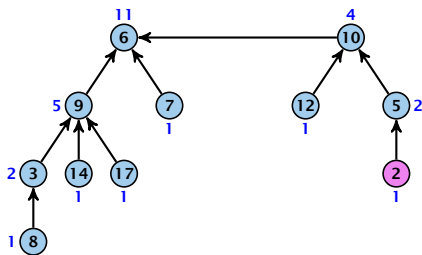


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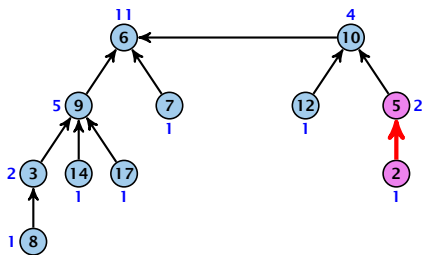


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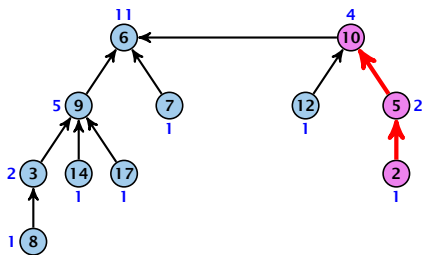


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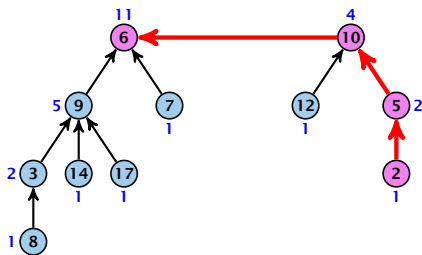
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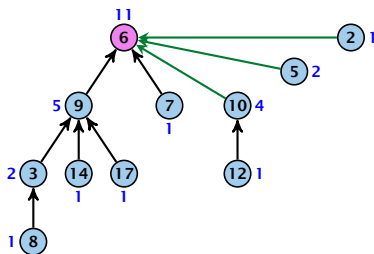


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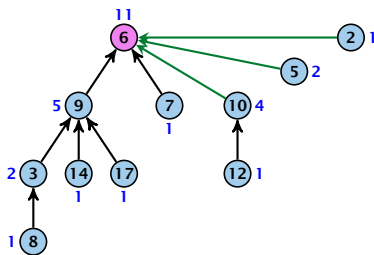


- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

# Path Compression

**find(x):**

- ▶ Go upward until you find the root.
- ▶ Re-attach all visited nodes as children of the root.
- ▶ Speeds up successive find-operations.



- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

# Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time  $\mathcal{O}(\log n)$ .

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# Amortized Analysis

## Definitions:

$n(x)$  = the number of nodes that were in the sds-tree created at  $x$  when  $x$  became the child of another node (i.e. the number of nodes if  $x$  is the root).

$rank(x)$  = the same as the size of  $x$ 's subtree in the case that there are no find-operations.

## Lemma 29

*The rank of a parent must be strictly larger than the rank of a child.*

# Amortized Analysis

## Definitions:

- ▶  $\text{size}(v) :=$  the number of nodes that were in the sub-tree rooted at  $v$  when  $v$  became the child of another node (or the number of nodes if  $v$  is the root).

Note that this is the same as the size of  $v$ 's subtree in the case that there are no find-operations.

- ▶  $\text{rank}(v) = \lfloor \log(\text{size}(v)) \rfloor$ .
- ▶  $\Rightarrow \text{size}(v) \geq 2^{\text{rank}(v)}$ .

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*The rank of a parent must be strictly larger than the rank of a child.*

# Amortized Analysis

## Lemma 30

There are at most  $n/2^s$  nodes of rank  $s$ .

Proof.

Let's say a node  $x$  has rank  $s$ . Then it has at least  $2^{s-1}$  children. The total number of nodes is  $n$ .

Each node has at most one node of rank  $s$  as child during the running time of the algorithm.

This being the case, the rank sequence of the roots of the trees is strictly increasing during the running time of the algorithm. Hence, there is at most one node of rank  $s$  for each rank  $s$ .

Therefore, there are at most  $n/2^s$  nodes of rank  $s$ . □

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## Lemma 30

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### Proof.

- ▶ Let's say a node  $v$  sees node  $x$  if  $v$  is in  $x$ 's sub-tree at the time that  $x$  becomes a child.
- ▶ A node  $v$  sees at most one node of rank  $s$  during the running time of the algorithm.
- ▶ This holds because the rank-sequence of the roots of the different trees that contain  $v$  during the running time of the algorithm is a strictly increasing sequence.
- ▶ Hence, every node sees at most one rank  $s$  node, but every rank  $s$  node is seen by at least  $2^s$  different nodes. □

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We define

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## Theorem 31

*Union find with path compression fulfills the following amortized running times:*

- ▶  $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- ▶  $\text{find}(x) : \mathcal{O}(\log^*(n))$
- ▶  $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

# Amortized Analysis

In the following we assume  $n \geq 2$ .

rank-group:

A node with rank  $r$  belongs to the rank-group  $r$ .

The rank-group  $r$  contains only nodes with rank  $\geq r$ .

rank

rank-group:  $\geq$  rank

The maximum number of rank-groups

is  $\lceil \lg n \rceil$  (rank of root is  $\lceil \lg n \rceil$ )

The total number of rank-groups is at most

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In the following we assume  $n \geq 2$ .

## rank-group:

- ▶ A node with rank  $\text{rank}(v)$  is in **rank group**  $\log^*(\text{rank}(v))$ .
- ▶ The rank-group  $g = 0$  contains only nodes with rank 0 or rank 1.
- ▶ A rank group  $g \geq 1$  contains ranks  $\text{tow}(g-1) + 1, \dots, \text{tow}(g)$ .
- ▶ The maximum non-empty rank group is  $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$  (which holds for  $n \geq 2$ ).
- ▶ Hence, the total number of rank-groups is at most  $\log^* n$ .

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# Amortized Analysis

## Accounting Scheme:

- create an account for every find-operation
- create an account for every node

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from  $v$  to  $\text{parent}[v]$  as follows:

- if  $v$  is the root we charge the cost to the root's account
- if  $v$  is the root's child we charge the cost to the root's account
- if the grand-number of  $v$  is 0 (the same as that of  $v$ 's parent) we charge the cost to the grand-number's account (before starting path compression) we charge the cost to the node's account of  $v$ 's parent
- otherwise we charge the cost to the grand-number's account

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- ▶ if  $v$  is the root we charge the cost to the account of the root
- ▶ if  $v$  is not the root we charge the cost to the account of  $\text{parent}[v]$
- ▶ if the grand-father of  $v$  is the root we charge the cost to the account of the root
- ▶ if  $v$  is not the root and before starting path compression we charge the cost to the account of  $\text{parent}[v]$
- ▶ if  $v$  is not the root and after path compression we charge the cost to the account of  $\text{parent}[v]$

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- ▶ if the node  $v$  has a credit, we charge the cost to the credit
- ▶ if the credit is not enough, we charge the cost to the account of the parent
- ▶ if we are working with path compression, we charge the cost to the account of the root

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- ▶ If the group-number of  $\text{rank}(v)$  is the same as that of  $\text{rank}(\text{parent}[v])$  (before starting path compression) we charge the cost to the node-account of  $v$ .
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## Observations:

- The number of changed elements is bounded from above by the number of elements in the array, which grows when increasing the size of the array.
- The number of changed elements is bounded from below by half of the current array size.
- The time needed to insert the parent will be in a worst case scenario, when it will need to be inserted again.
- The time needed to insert a node in rank group  $i$  is at most

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- ▶ A find-account is charged at most  $\log^*(n)$  times (once for the root and at most  $\log^*(n) - 1$  times when increasing the rank-group).
- ▶ After a node  $v$  is charged its parent-edge is re-assigned. The rank of the parent strictly increases.
- ▶ After some charges to  $v$  the parent will be in a larger rank-group.  $\Rightarrow v$  will **never** be charged again.
- ▶ The total charge made to a node in rank-group  $g$  is at most  $\text{tow}(g) - \text{tow}(g - 1) - 1 \leq \text{tow}(g)$ .

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Without loss of generality we can assume that all **makeset**-operations occur at the start.

This means if we inflate the cost of **makeset** to  $\log^* n$  and add this to the node account of  $v$  then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is  $\mathcal{O}(\alpha(m, n))$ , where  $\alpha(m, n)$  is the inverse Ackermann function which grows a lot lot slower than  $\log^* n$ . (Here, we consider the average running time of  $m$  operations on at most  $n$  elements).

There is also a lower bound of  $\Omega(\alpha(m, n))$ .

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$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- ▶  $A(0, y) = y + 1$
- ▶  $A(1, y) = y + 2$
- ▶  $A(2, y) = 2y + 3$
- ▶  $A(3, y) = 2^{y+3} - 3$
- ▶  $A(4, y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$

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