WS 2017/18

Efficient Algorithms and Data Structures

Harald Räcke

Fakultät für Informatik
TU München

http://www14.in.tum.de/lehre/2017WS/ea/

Winter Term 2017/18

Part I

Organizational Matters

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- Modul: IN2003
- Name: "Efficient Algorithms and Data Structures"
  "Effiziente Algorithmen und Datenstrukturen"
- ECTS: 8 Credit points
- Lectures:
  - 4 SWS
    Mon 10:00–12:00 (Room Interim2)
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The Lecturer

- Harald Räcke
- Email: raecke@in.tum.de
- Room: 03.09.044
- Office hours: (by appointment)

Tutorials

A01 Monday, 12:00–14:00, 00.08.038 (Schmid)

A02 Monday, 12:00–14:00, 00.09.038 (Stotz)

A03 Monday, 14:00–16:00, 02.09.023 (Liebl)

B04 Tuesday, 10:00–12:00, 00.08.053 (Schmid)

B05 Tuesday, 12:00–14:00, 03.11.018 (Kraft)

B06 Tuesday, 14:00–16:00, 00.08.038 (Somogyi)

D07 Thursday, 10:00–12:00, 03.11.018 (Liebl)

E08 Friday, 12:00–14:00, 00.13.009 (Stotz)

E09 Friday, 14:00–16:00, 00.13.009 (Kraft)

Assignment sheets

In order to pass the module you need to pass an exam.

Assessment

Assignment Sheets:

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- Don't forget name and student id number for each group member.

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- If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \leq 4 \\ x & \text{otw.} \end{cases}$$

- It will improve by 0.3 or 0.4, respectively. Examples:
  - $3.3 \to 3.0$
  - $2.0 \to 1.7$
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Assessment

Requirements for Bonus

- 50% of the points are achieved on submissions 2–8,
- 50% of the points are achieved on submissions 9–14,
- each group member has written at least 4 solutions.

1 Contents

- Foundations
  - Machine models
  - Efficiency measures
  - Asymptotic notation
  - Recursion
- Higher Data Structures
  - Search trees
  - Hashing
  - Priority queues
  - Union/Find data structures
- Cuts/Flows
- Matchings

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2 Literatur

Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:
The design and analysis of computer algorithms,
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  Springer, 1998

Part II

Foundations

3 Goals

- Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
- Learn how to analyze and judge the efficiency of algorithms.
- Learn how to design efficient algorithms.

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What do you measure?

- Memory requirement
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How do you measure?

- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.

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  - Gives [illegible] like "this algorithm always runs in time $\mathcal{O}(n^2)$".
  - Typically focuses on the [illegible].
  - Can give lower bounds like "any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case".

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Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- the size of the input (number of bits)
- the number of arguments

Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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How to measure performance

- Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
- Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

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Model of Computation

How to measure performance

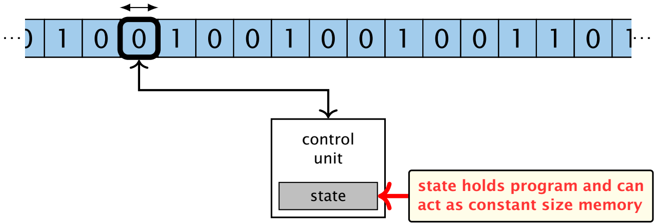
1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

- Very simple model of computation.
- Only the “current” memory location can be altered.
- Very good model for discussing computabiliy, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

⟹ **Not a good model for developing efficient algorithms.**



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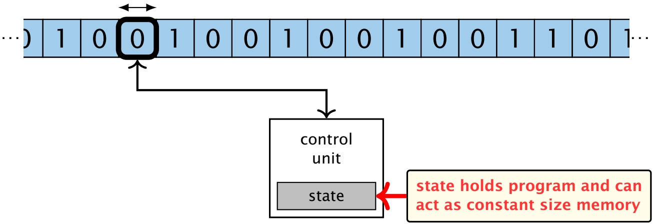

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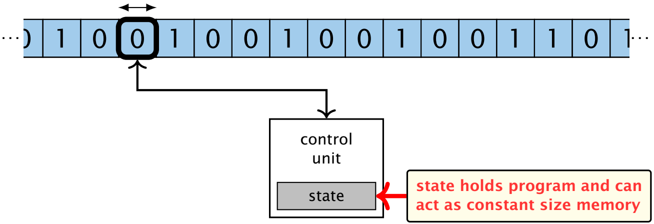

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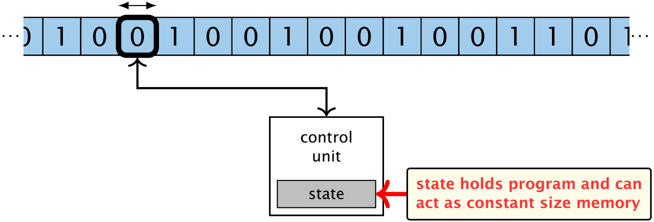

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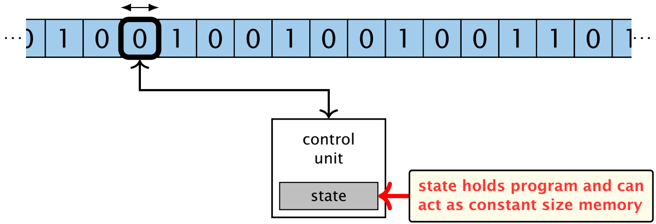
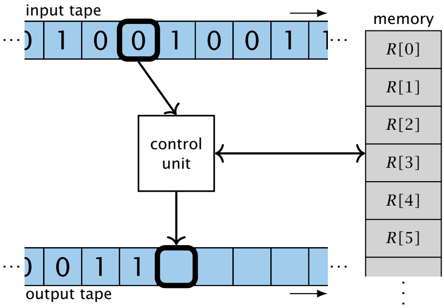


Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.
- Registers hold integers.
- Indirect addressing.



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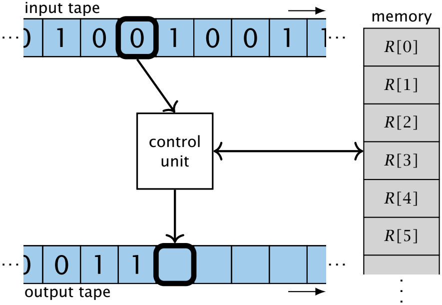

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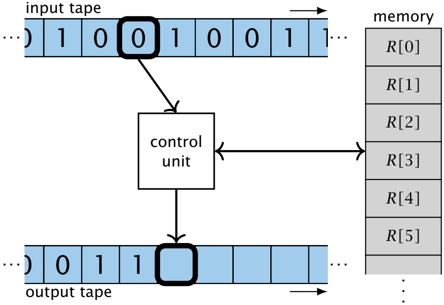

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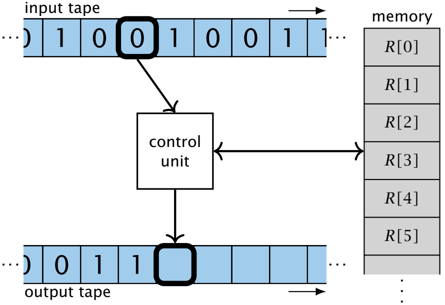

Random Access Machine (RAM)

Operations

- input operations (input tape → $R[i]$)
  - READ i
- output operations ($R[i]$ → output tape)
  - WRITE i
- register-register transfers
  - $R[j] := R[i]$
  - $R[j] := 4$
- **indirect** addressing
  - $R[j] := R[R[i]]$
    loads the content of the $R[i]$-th register into the j-th register
  - $R[R[i]] := R[j]$
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Operations

- branching (including loops) based on comparisons
  - jump x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  - jumpz x $R[i]$
    jump to x if $R[i] = 0$
    if not the instruction counter is increased by 1;
  - jumpi i
    jump to $R[i]$ (indirect jump);
- arithmetic instructions: $+$, $-$, $\times$, $/$
  - $R[i] := R[j] + R[k]$;
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Model of Computation

- uniform cost model
  Every operation takes time 1.
- logarithmic cost model
  The cost depends on the content of memory cells:
  - The time for a step is equal to the largest operand involved;
  - The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

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4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

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r ← 2;
for i = 1 → n do
        r ← r^2
return  r
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running time

space requirement

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- best-case complexity:

$$C_{\mathrm{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

  Usually easy to analyze, but not very meaningful.

- worst-case complexity:

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  Usually moderately easy to analyze; sometimes too pessimistic.

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We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

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Asymptotic Notation

Formal Definition

Let f denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 \colon [g(n) \leq c \cdot f(n)]\}$
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- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
  (set of functions that asymptotically grow not faster than f)
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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.

3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \to \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) = f(n) + z(n)$.

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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an anonymous function in the set $\Theta(n)$ that makes the expression true.

Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Careful!

"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

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Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\left\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \right.$$
$$\left. \text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \right\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

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Lemma 3

Let f, g *be functions with the property*
$\exists n_0 > 0 \, \forall n \geq n_0 : f(n) > 0$ *(the same for* g*). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant* c
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
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- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(\max\{f(n), g(n)\})$

The expressions also hold for Ω*. Note that this means that* $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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Comments

- Do not use asymptotic notation within induction proofs.
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.
- However, suppose that I have two algorithms:
  - Algorithm A. Running time $f(n) = 1000 \log n = O(\log n)$.
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  Clearly $f = o(g)$. However, as long as $\log n \le 1000$ Algorithm B will be more efficient.

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6 Recurrences

Algorithm 2 mergesort(list L)

```
n ← size(L)
if n ≤ 1 return L
L_1 ← L[1 ⋯ ⌊n/2⌋]
L_2 ← L[⌊n/2⌋ + 1 ⋯ n]
mergesort(L_1)
mergesort(L_2)
L ← merge(L_1, L_2)
return L
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6 Recurrences

Algorithm 2 mergesort(list L)

1: $n \leftarrow \text{size}(L)$
2: **if** $n \leq 1$ **return** L
3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
4: $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
5: mergesort(L_1)
6: mergesort(L_2)
7: $L \leftarrow \text{merge}(L_1, L_2)$
8: **return** L

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

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Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \le \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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Suppose we guess $T(n) \leq dn \log n$ for a constant d.

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if we choose $d \ge c$.

Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

How do we get a result for all values of n?

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We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

6.1 Guessing+Induction

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

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$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$

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\end{aligned}
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$\frac{n}{2} + 1 \leq \frac{9}{16} n$

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\boxed{\left\lceil \tfrac{n}{2} \right\rceil \leq \tfrac{n}{2} + 1} \quad &\leq 2\big(d(n/2 + 1) \log(n/2 + 1)\big) + cn \\
\boxed{\tfrac{n}{2} + 1 \leq \tfrac{9}{16}n} \quad &\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn
\end{aligned}
$$

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We also make a guess of $T(n) \leq dn\log n$ and get

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\begin{aligned}
T(n) &\leq 2T\left(\left\lceil\frac{n}{2}\right\rceil\right) + cn \\
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&\leq 2\left(d(n/2+1)\log(n/2+1)\right) + cn \\
&\leq dn\log\left(\frac{9}{16}n\right) + 2d\log n + cn
\end{aligned}
$$

$\left\lceil\frac{n}{2}\right\rceil \leq \frac{n}{2}+1$

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$\log\frac{9}{16}n = \log n + (\log 9 - 4)$

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\boxed{\left\lceil\tfrac{n}{2}\right\rceil \leq \tfrac{n}{2}+1} \quad &\leq 2\big(d(n/2+1)\log(n/2+1)\big) + cn \\
\boxed{\tfrac{n}{2}+1 \leq \tfrac{9}{16}n} \quad &\leq dn\log\left(\frac{9}{16}n\right) + 2d\log n + cn \\
\boxed{\log\tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad &= dn\log n + (\log 9 - 4)dn + 2d\log n + cn
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$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

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$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$

$$\leq 2(d(n/2+1)\log(n/2+1)) + cn$$

$\frac{n}{2} + 1 \leq \frac{9}{16}n$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$\log \frac{9}{16}n = \log n + (\log 9 - 4)$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$\log n \leq \frac{n}{4}$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

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\boxed{\log \tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad &= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\
\boxed{\log n \leq \tfrac{n}{4}} \quad &\leq dn \log n + (\log 9 - 3.5)dn + cn \\
&\leq dn \log n - 0.33dn + cn
\end{aligned}
$$

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We also make a guess of $T(n) \leq dn\log n$ and get

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\boxed{\log n \leq \tfrac{n}{4}} \quad &\leq dn\log n + (\log 9 - 3.5)dn + cn \\
&\leq dn\log n - 0.33dn + cn \\
&\leq dn\log n
\end{aligned}
$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 4

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.
If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.
If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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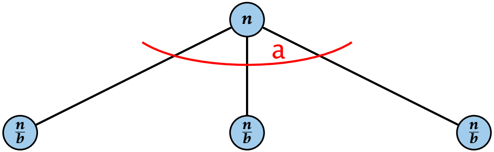

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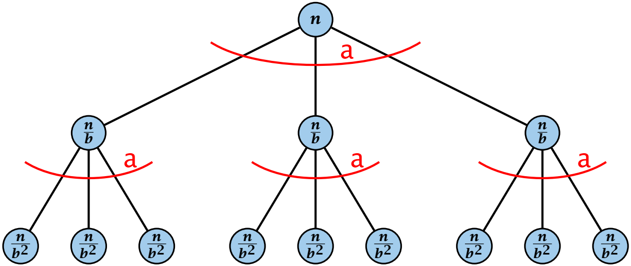

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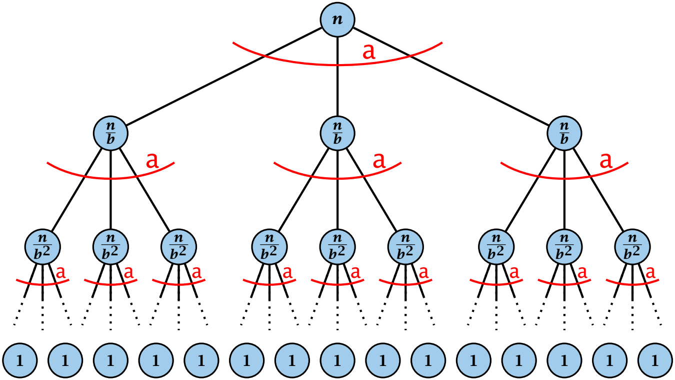

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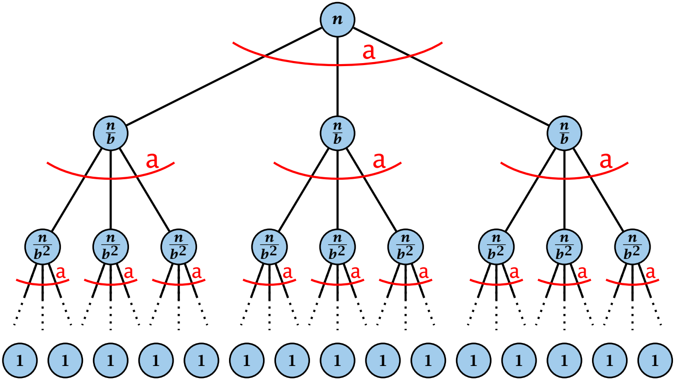

$f(n)$

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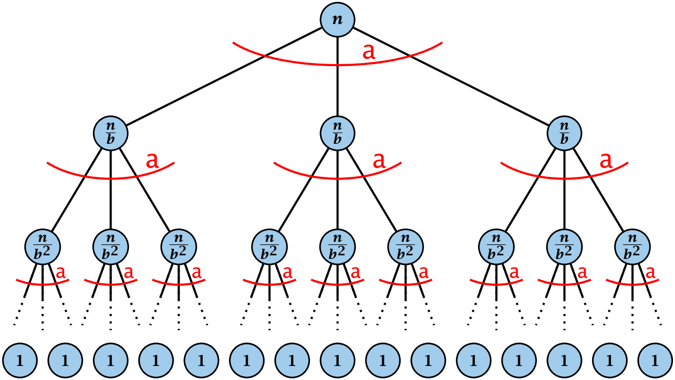

$f(n)$

$af(\frac{n}{b})$

The Recursion Tree

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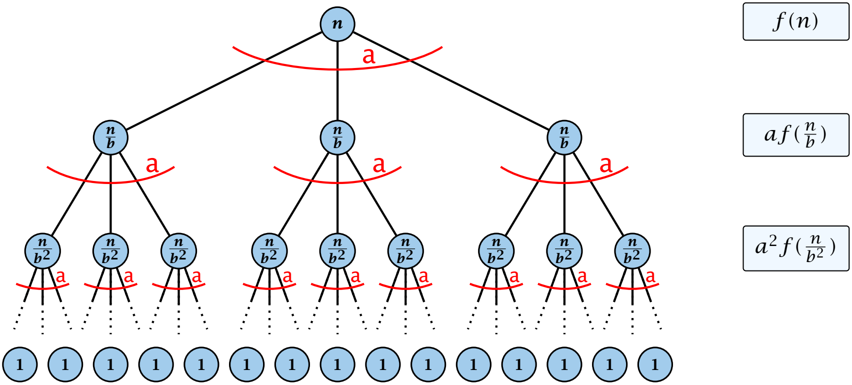

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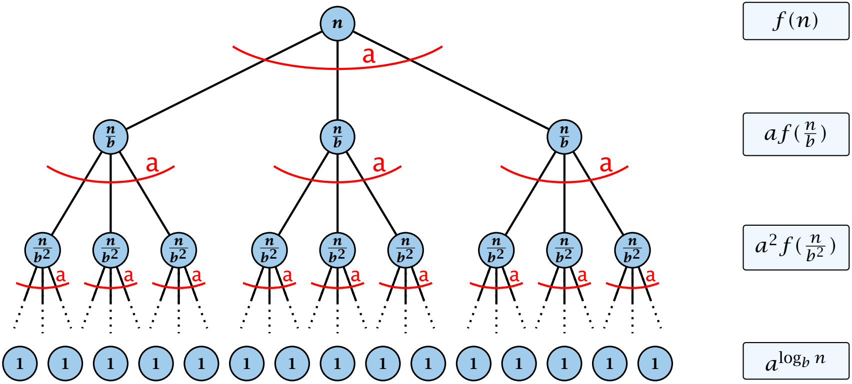

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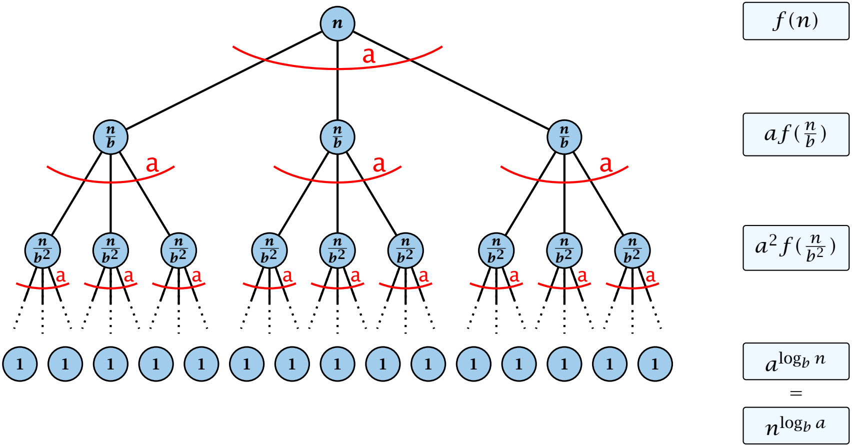

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a}$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\end{aligned}
$$

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$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i
\end{aligned}
$$

$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

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$b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}$

$$= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^{\epsilon})^i$$

$\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$

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$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$$

$$= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)$$

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$$= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(b^{\epsilon}\right)^i \\
&= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \\
&= cn^{\log_b a-\epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\
&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}
$$

$b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

$\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i \\
&= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1) \\
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&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}
$$

$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

$\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$

Hence,

$$
T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}
$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

$$= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$

$$= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}$$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a})$.

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\end{aligned}
$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
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\end{aligned}
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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$$
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n)$.

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1
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Hence,

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\end{aligned}
$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
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&= cn^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k}\ell^{k+1}
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&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
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$n = b^\ell \Rightarrow \ell = \log_b n$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n)$.

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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Hence,

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$$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

$$\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{ccccccccc|c} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline \end{array}$$

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								0	

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$$
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 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \quad A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & \quad B \\
 & & & & & & {}_1 & {}_1 & {}_1 & & \\
\hline
 & & & & & & & 0 & 0 & 0 &
\end{array}
$$

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					1	1	1		
						0	0	0	

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				0	1	1	1		
					1	0	0	0	

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	0	1	1	0	0	1	0	0	0	

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1	0	0	1	1	0	1	1	1		
1	0	1	1	0	0	1	0	0	0	

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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$$
\begin{array}{r}
1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ \boxed{1} \\
\hline
1\ 0\ 0\ 0\ 1
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
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\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0
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\hline
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 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
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 & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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 & & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

1	0	0	0	1	×	1	0	1	1
					1	0	0	0	1
				1	0	0	0	1	0
			0	0	0	0	0	0	0
		1	0	0	0	1	0	0	0

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

1	0	0	0	1	×	1	0	1	1
					1	0	0	0	1
				1	0	0	0	1	0
			0	0	0	0	0	0	0
		1	0	0	0	1	0	0	0
		1	0	1	1	1	0	1	1

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
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$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

1	0	0	0	1	×	1	0	1	1
					1	0	0	0	1
				1	0	0	0	1	0
			0	0	0	0	0	0	0
		1	0	0	0	1	0	0	0
		1	0	1	1	1	0	1	1

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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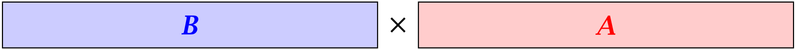

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_{n-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \; b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \; a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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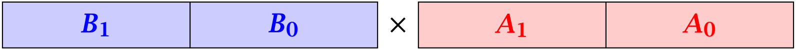

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	$\times$	A_1	A_0

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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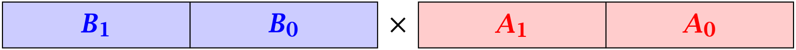

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

```
if |A| = |B| = 1 then
    return a_0 · b_0
split A into A_0 and A_1
split B into B_0 and B_1
Z_2 ← mult(A_1, B_1)
Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
Z_0 ← mult(A_0, B_0)
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $|A| = |B| = 1$ **then** — $\mathcal{O}(1)$
  return $a_0 \cdot b_0$
split A into A_0 and A_1
split B into B_0 and B_1
$Z_2 \leftarrow \text{mult}(A_1, B_1)$
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return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $|A| = |B| = 1$ **then** $\mathcal{O}(1)$

  return $a_0 \cdot b_0$ $\mathcal{O}(1)$

split A into A_0 and A_1

split B into B_0 and B_1

$Z_2 \leftarrow \text{mult}(A_1, B_1)$

$Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

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Example: Multiplying Two Integers

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if $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
    return $a_0 \cdot b_0$ $\mathcal{O}(1)$
split A into A_0 and A_1 $\mathcal{O}(n)$
split B into B_0 and B_1
$Z_2 \leftarrow \text{mult}(A_1, B_1)$
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Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

1:	**if** $\|A\| = \|B\| = 1$ **then**	$\mathcal{O}(1)$
2:	**return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3:	split A into A_0 and A_1	$\mathcal{O}(n)$
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Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
return $a_0 \cdot b_0$	$\mathcal{O}(1)$
split A into A_0 and A_1	$\mathcal{O}(n)$
split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	
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Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

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Example: Multiplying Two Integers

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return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

$\Longrightarrow$ Not better then the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

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$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0 \end{aligned}$$

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Hence,

Algorithm 4 mult(A, B)

1: **if** $|A| = |B| = 1$ **then**
2: **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
4: split B into B_0 and B_1
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$
\begin{aligned}
Z_1 &= A_1B_0 + A_0B_1 \\
&= (A_0 + A_1)\cdot(B_0 + B_1) - \overbrace{A_1B_1}^{=Z_2} - \overbrace{A_0B_0}^{=Z_0}
\end{aligned}
$$

Hence,

Algorithm 4 mult(A, B)

- **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
    - **return** $a_0 \cdot b_0$
- split A into A_0 and A_1
- split B into B_0 and B_1
- $Z_2 \leftarrow \text{mult}(A_1, B_1)$
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Hence,

Algorithm 4 mult(A, B)

```
if |A| = |B| = 1 then                          O(1)
    return a_0 · b_0                           O(1)
split A into A_0 and A_1
split B into B_0 and B_1
Z_2 ← mult(A_1, B_1)
Z_0 ← mult(A_0, B_0)
Z_1 ← mult(A_0 + A_1, B_0 + B_1) − Z_2 − Z_0
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

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We can use the following identity to compute Z_1:

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Z_1 &= A_1B_0 + A_0B_1 \\
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if $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
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split A into A_0 and A_1	$\mathcal{O}(n)$
split B into B_0 and B_1	
$Z_2 \leftarrow \text{mult}(A_1, B_1)$	
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We can use the following identity to compute Z_1:

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Hence,

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if $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
return $a_0 \cdot b_0$	$\mathcal{O}(1)$
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split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	
$Z_1 \leftarrow \mathrm{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1 B_0 + A_0 B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0} \end{aligned}$$

Hence,

Algorithm 4 mult(A, B)

1: **if** $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
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Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

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6.3 The Characteristic Polynomial

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$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

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This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of $T[n]$'s.
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Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

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The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \;\middle|\; \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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Lemma 5
Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

Proof.
There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\prod_{i=2}^{k}(\lambda_i-\lambda_1)\cdot\begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

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$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n\lambda_i^{n}}_{T[n]} + c_1\underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k\underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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Suppose λ_i has multiplicity j. We know that

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(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

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The Homogeneous Case

Lemma 6

Let $P[\lambda]$ *denote the characteristic polynomial to the recurrence*

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ *be the (complex) roots of* $P[\lambda]$ *with multiplicities* ℓ_i. *Then the general solution to the recurrence is given by*

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\Longrightarrow\alpha-\beta=\frac{2}{\sqrt{5}}$$

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Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

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$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

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Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$

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$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

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and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

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1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

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6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

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What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n\right) = 1 \ .$$

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

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$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.

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6. The coefficients of the resulting power series are the a_n.

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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   This leads to the following conditions:

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This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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$$= \frac{7}{4} \cdot \sum_{n \ge 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \ge 0} z^n - \frac{1}{2} \cdot \sum_{n \ge 0} (n+1) z^n$$

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$$= \sum_{n\geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big) z^n$$

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&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\
&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
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\end{aligned}
$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 9

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.$$

6.5 Transformation of the Recurrence

Example 9

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$$g_n := \log f_n \;.$$

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$$f_n = 2^{F_n}$$

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Then:

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6 Recurrences

We get

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\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k
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6 Recurrences

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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2}
\end{aligned}$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- The value can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

- **S. search(k)**: Returns pointer to object x from S with $\text{key}[x] = k$ or null.
- **S. insert(x)**: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.
- **S. delete(x)**: Given pointer to object x from S, delete x from the set.
- **S. minimum()**: Return pointer to object with smallest key-value in S.
- **S. maximum()**: Return pointer to object with largest key-value in S.
- **S. successor(x)**: Return pointer to the next larger element in S or null if x is maximum.
- **S. predecessor(x)**: Return pointer to the next smaller element in S or null if x is minimum.

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Dynamic Set Operations

- **S. union(S')**: Sets $S := S \cup S'$. The set S' is destroyed.
- **S. merge(S')**: Sets $S := S \cup S'$. Requires $S \cap S' = \emptyset$.
- **S. split(k, S')**:
  $S := \{x \in S \mid \text{key}[x] \le k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.
- **S. concatenate(S')**: $S := S \cup S'$.
  Requires $\text{key}[S.\text{maximum}()] \le \text{key}[S'.\text{minimum}()]$.
- **S. decrease-key(x, k)**: Replace $\text{key}[x]$ by $k \le \text{key}[x]$.

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- $\boldsymbol{S}.\,\mathbf{decrease\text{-}key}(\boldsymbol{x}, \boldsymbol{k})$: Replace $\mathrm{key}[x]$ by $k \leq \mathrm{key}[x]$.

Examples of ADTs

Stack:

- $\boldsymbol{S}.\,\mathbf{push}(\boldsymbol{x})$: Insert an element.
- $\boldsymbol{S}.\,\mathbf{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- $\boldsymbol{S}.\,\mathbf{empty}()$: Tell if S contains any object.

Queue:

- $\boldsymbol{S}.\,\mathbf{enqueue}(\boldsymbol{x})$: Insert an element.
- $\boldsymbol{S}.\,\mathbf{dequeue}()$: Return the element that is longest in the structure; delete it from S.
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Priority-Queue:

- $\boldsymbol{S}.\,\mathbf{insert}(\boldsymbol{x})$: Insert an element.
- $\boldsymbol{S}.\,\mathbf{delete\text{-}min}()$: Return the element with lowest key-value; delete it from S.

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- $S.\,\mathbf{insert}(x)$: Insert an element.
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7 Dictionary

Dictionary:

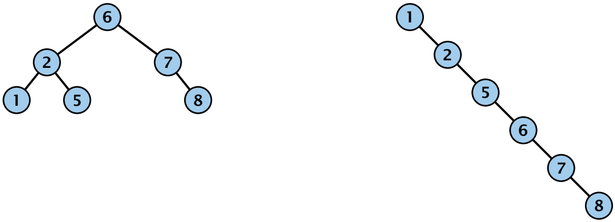
- $\boldsymbol{S.\,\mathrm{insert}(x)}$: Insert an element x.
- $\boldsymbol{S.\,\mathrm{delete}(x)}$: Delete the element pointed to by x.
- $\boldsymbol{S.\,\mathrm{search}(k)}$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:

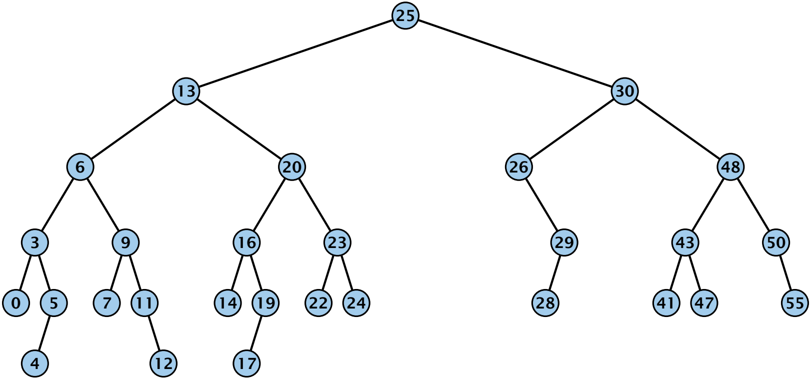


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\mathrm{insert}(x)$
- $T.\mathrm{delete}(x)$
- $T.\mathrm{search}(k)$
- $T.\mathrm{successor}(x)$
- $T.\mathrm{predecessor}(x)$
- $T.\mathrm{minimum}()$
- $T.\mathrm{maximum}()$

Binary Search Trees: Searching

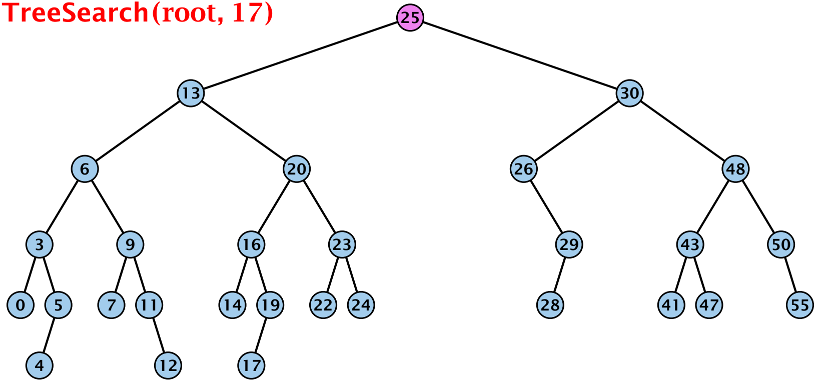


Algorithm 1 TreeSearch(x, k)

1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
2: **if** $k < \text{key}[x]$ **return** TreeSearch$(\text{left}[x], k)$
3: **else return** TreeSearch$(\text{right}[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 17)



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Binary Search Trees: Searching

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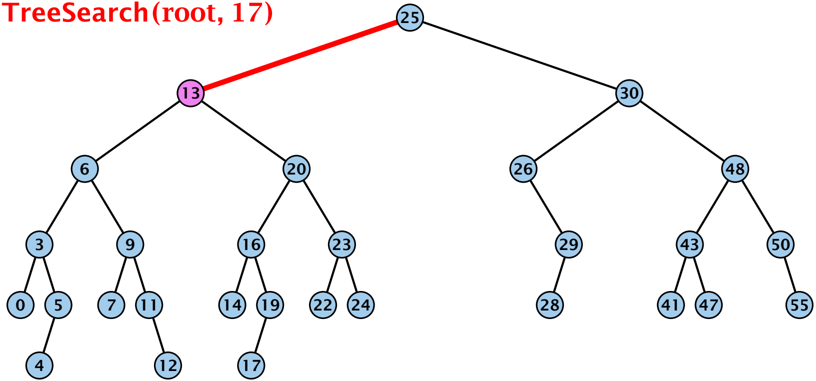

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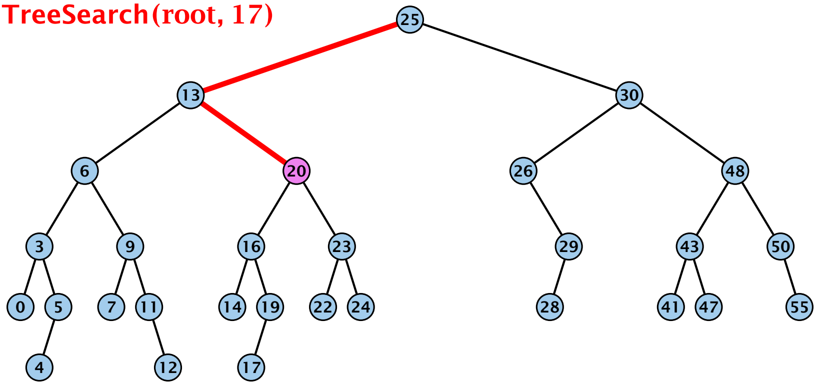
Binary Search Trees: Searching



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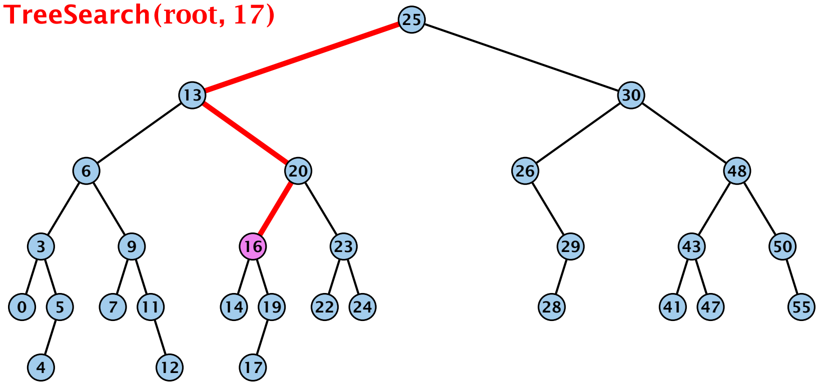

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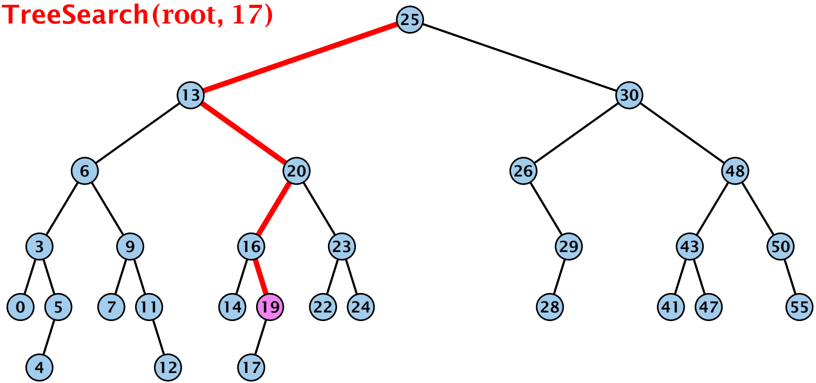
Binary Search Trees: Searching



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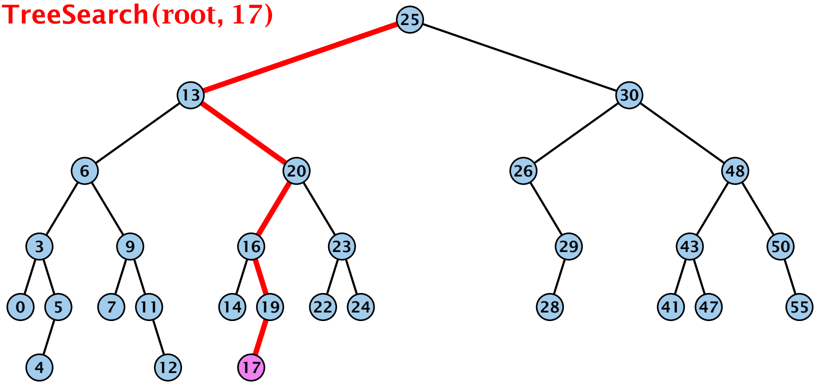

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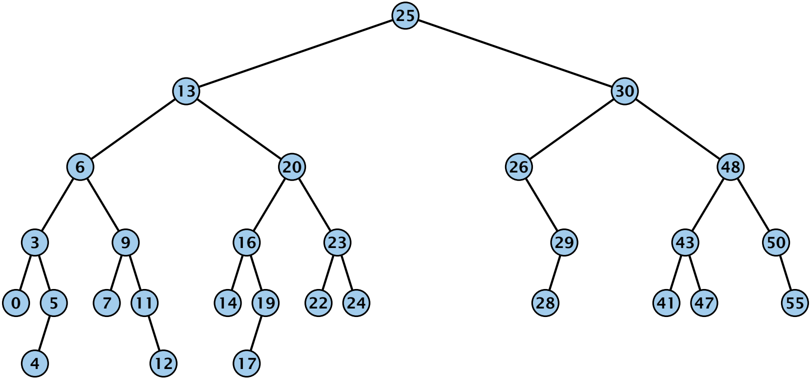

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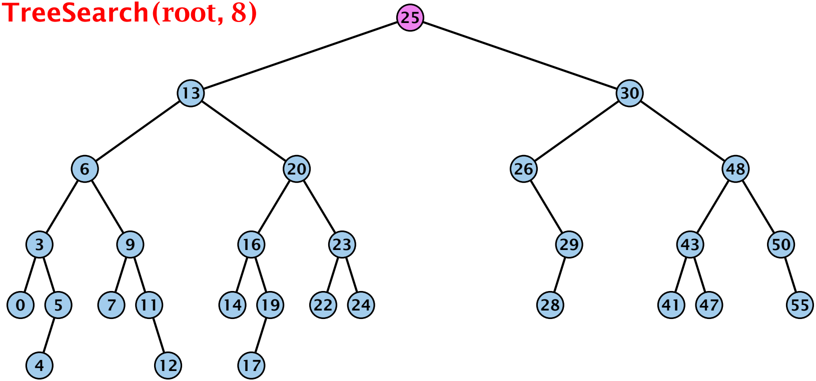
Binary Search Trees: Searching

TreeSearch(root, 8)



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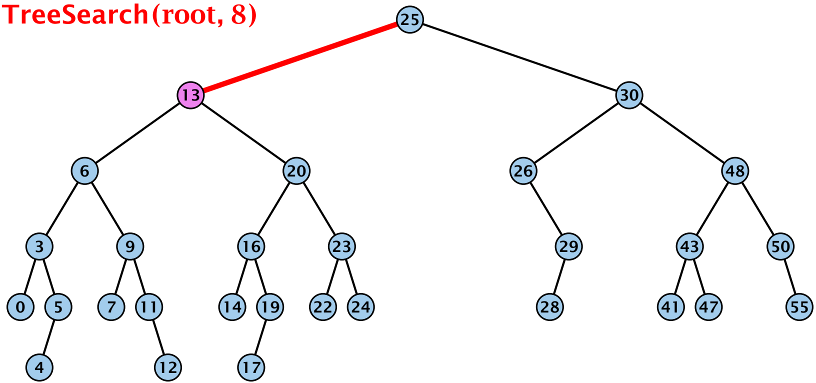
Binary Search Trees: Searching



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Binary Search Trees: Searching

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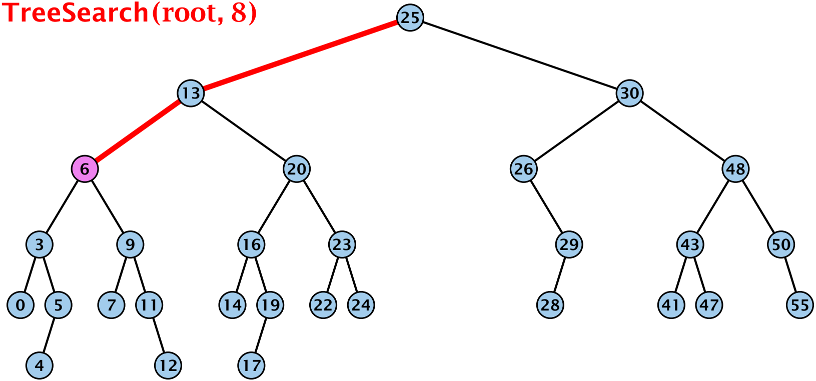

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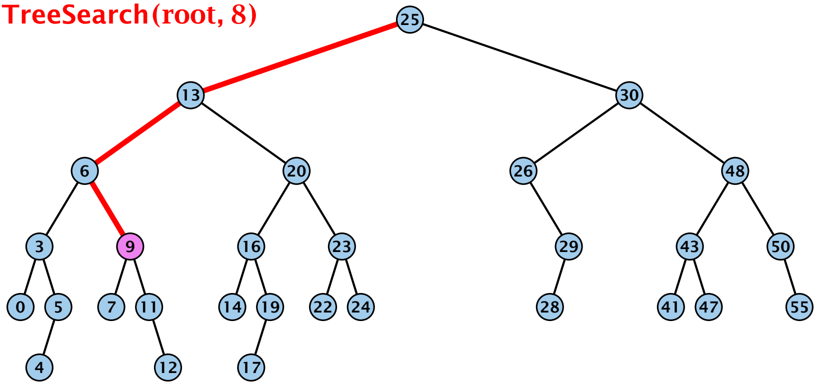

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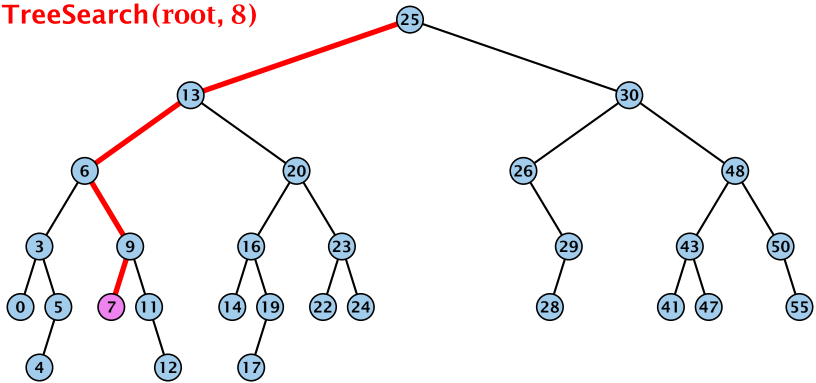

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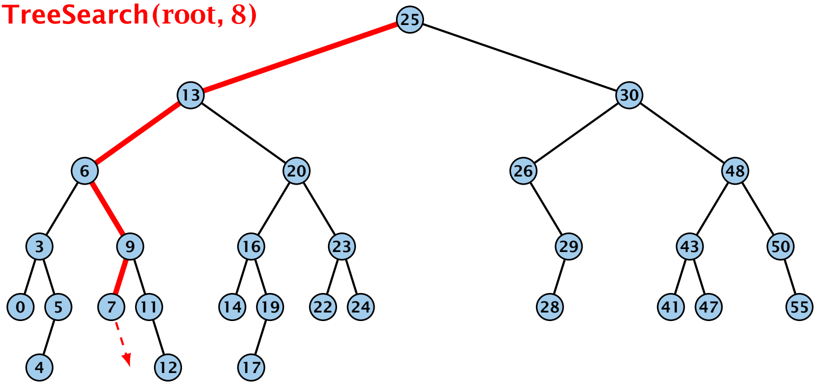

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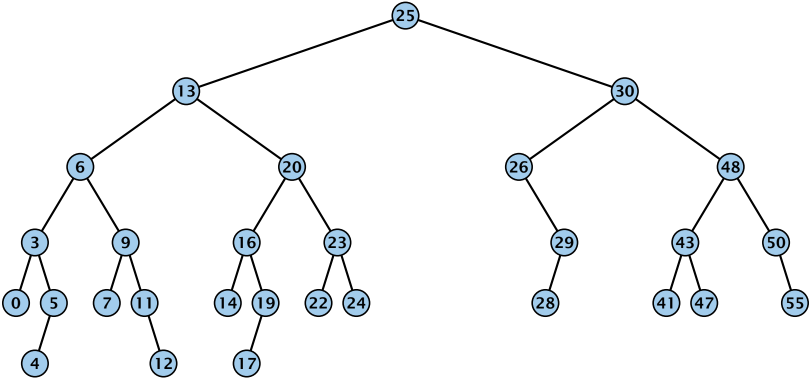
Binary Search Trees: Minimum

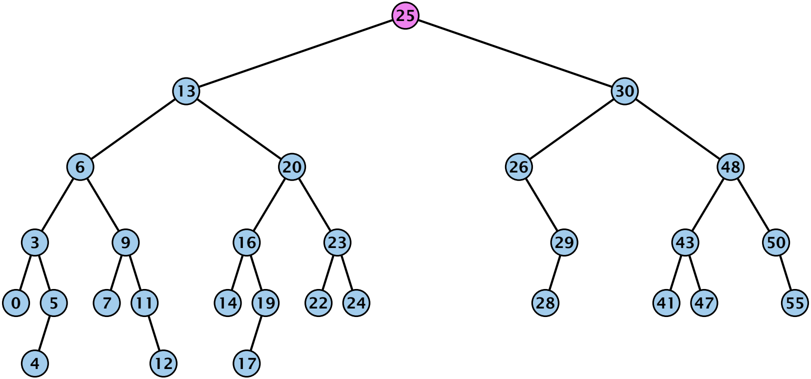


Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x

2: **return** TreeMin(left[x])

Binary Search Trees: Minimum



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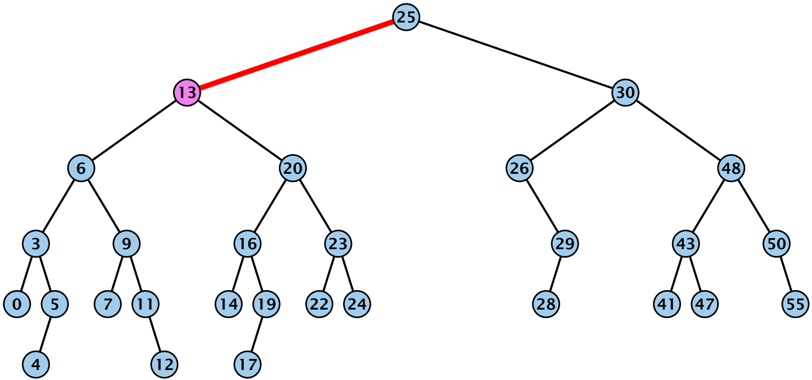

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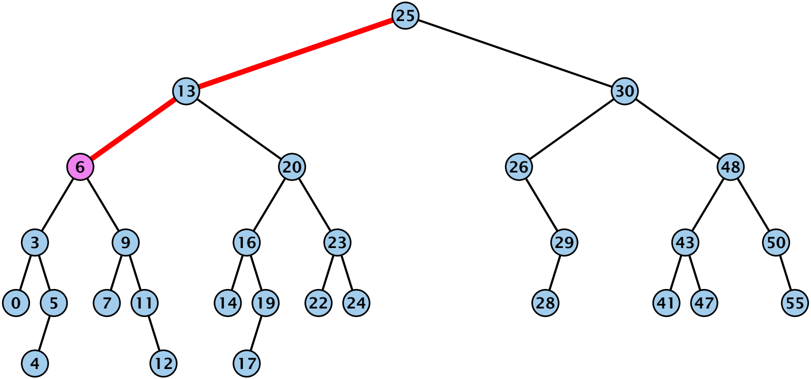
Binary Search Trees: Minimum



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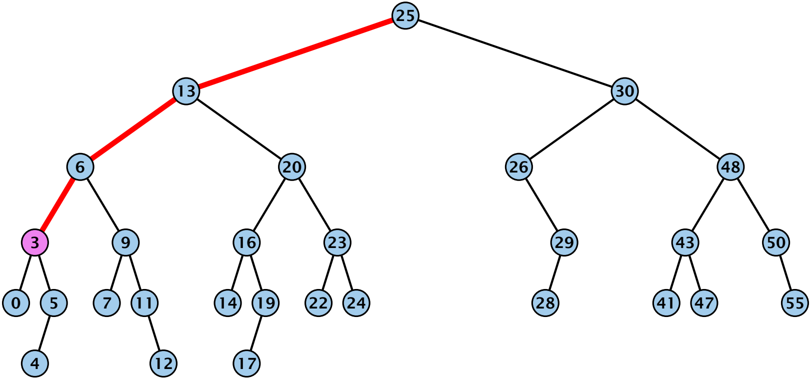

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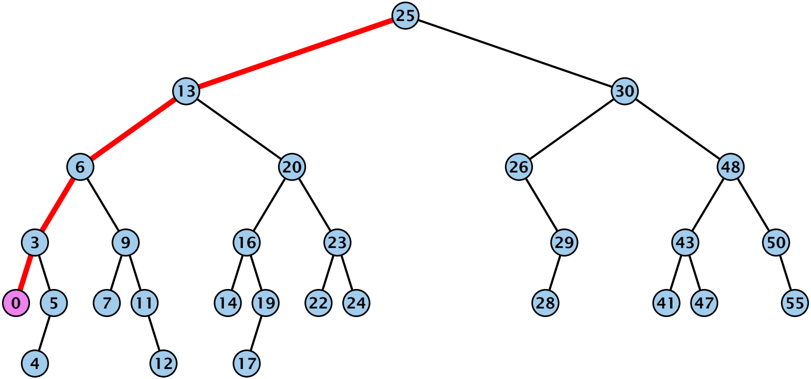
2: **return** TreeMin(left[x])

Binary Search Trees: Minimum

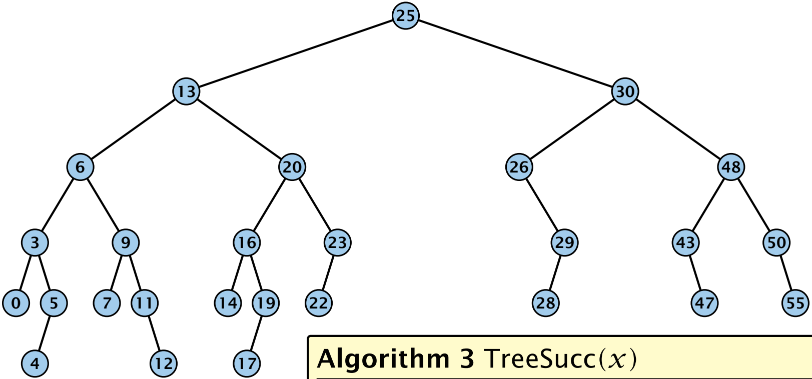


Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x

2: **return** TreeMin(left[x])

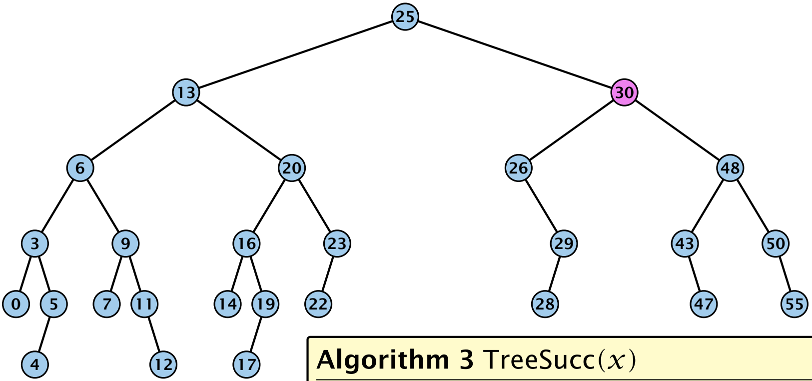
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

```
if right[x] ≠ null return TreeMin(right[x])
y ← parent[x]
while y ≠ null and x = right[y] do
    x ← y; y ← parent[x]
return y;
```

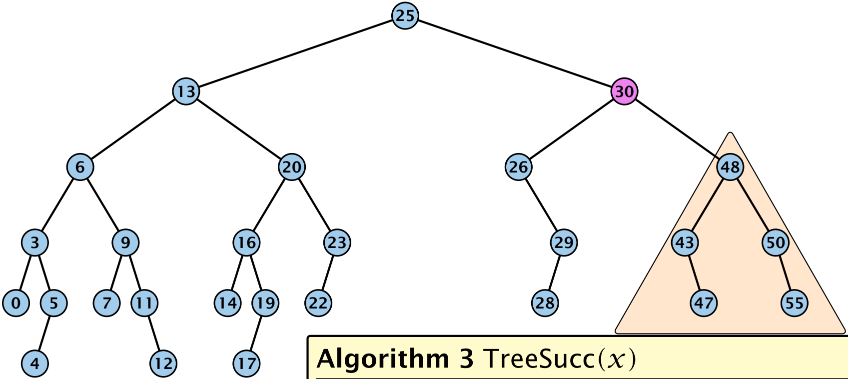
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

if right[x] $\neq$ null **return** TreeMin(right[x])
$y \leftarrow$ parent[x]
while $y \neq$ null **and** $x =$ right[y] **do**
  $x \leftarrow y; y \leftarrow$ parent[x]
return y;

Binary Search Trees: Successor



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if right$[x] \neq$ null **return** TreeMin(right$[x]$)

$y \leftarrow$ parent$[x]$

while $y \neq$ null **and** $x =$ right$[y]$ **do**

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return y;

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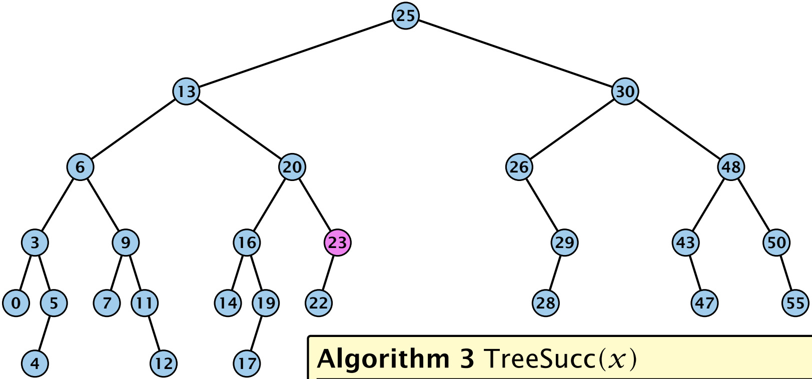

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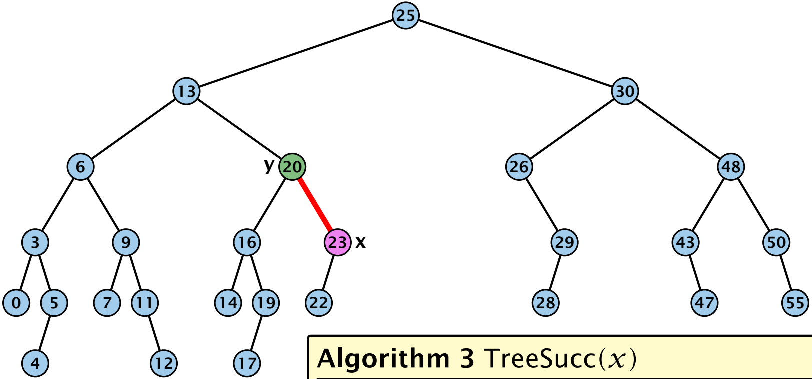

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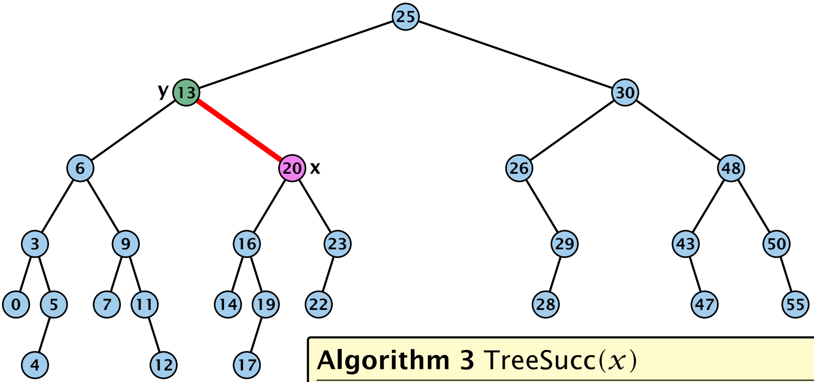

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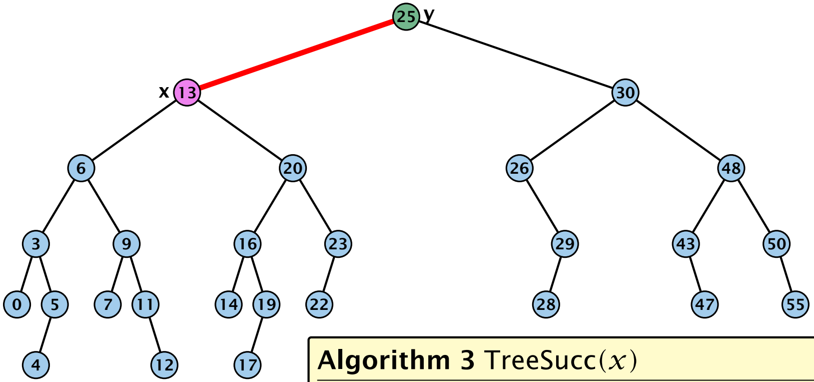

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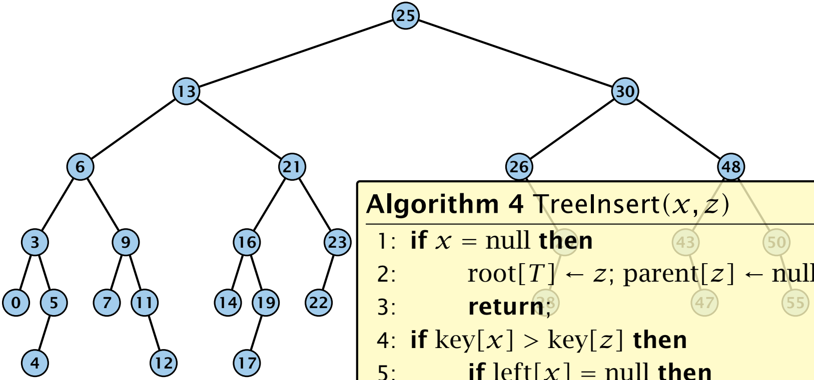
Binary Search Trees: Insert



Algorithm 4 TreeInsert(x, z)

if $x = \text{null}$ **then**
    root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
    return;
if key$[x] >$ key$[z]$ **then**
    if left$[x] = $ null **then**
        left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
    else TreeInsert(left$[x], z$);
else
    if right$[x] = $ null **then**
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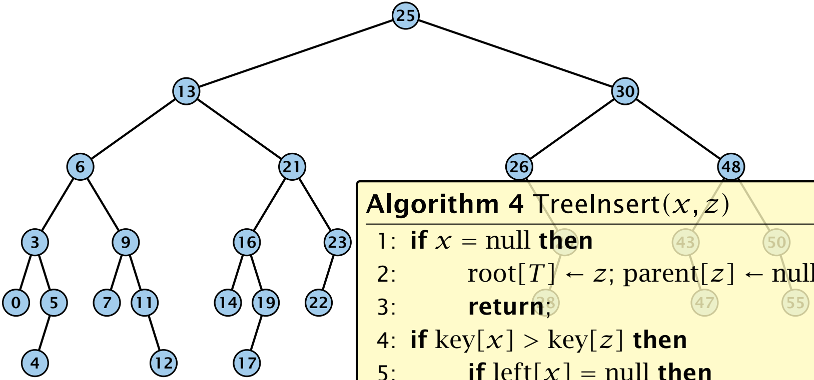
Binary Search Trees: Insert

Insert element **not** in the tree.



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    if $\text{left}[x] = \text{null}$ **then**
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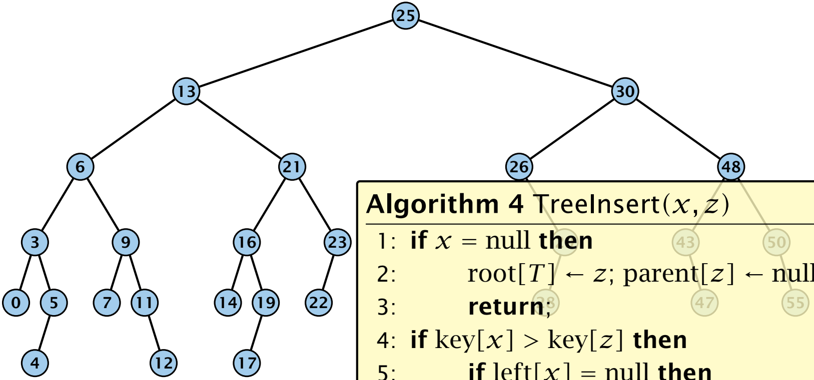

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

```
if x = null then
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if key[x] > key[z] then
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else
    if right[x] = null then
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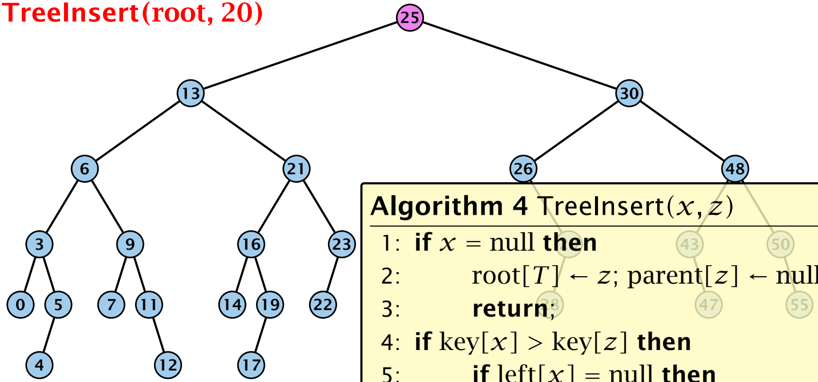
```

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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    return;
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    if $\text{left}[x] = \text{null}$ **then**
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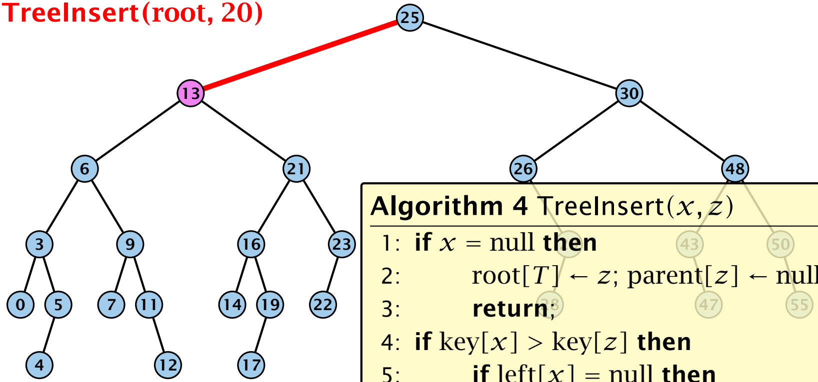

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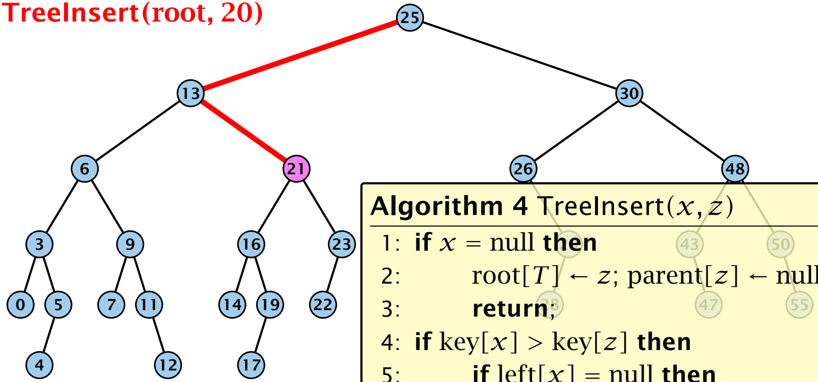

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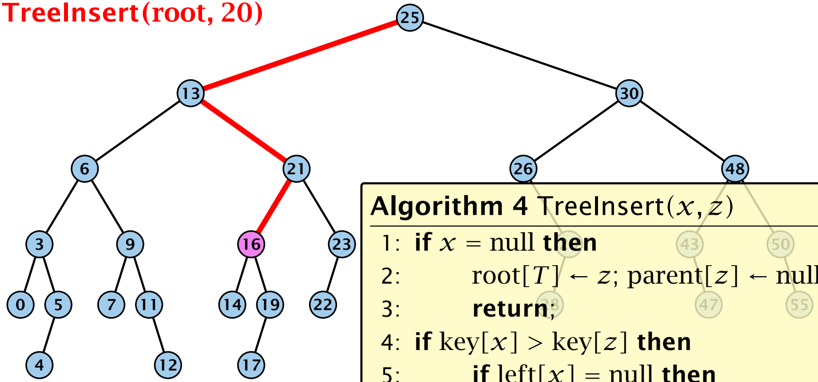

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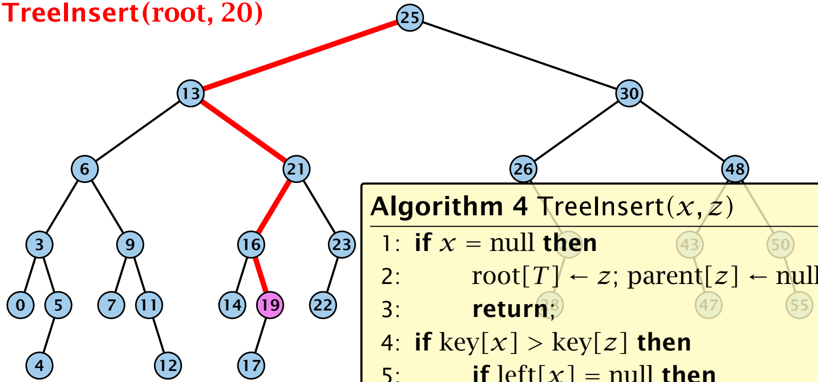

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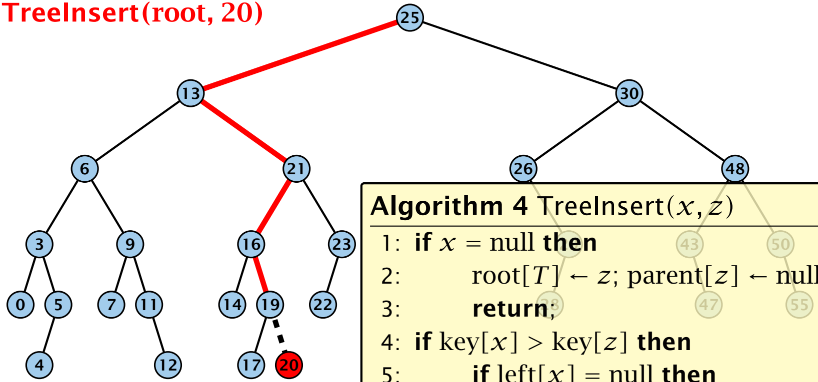

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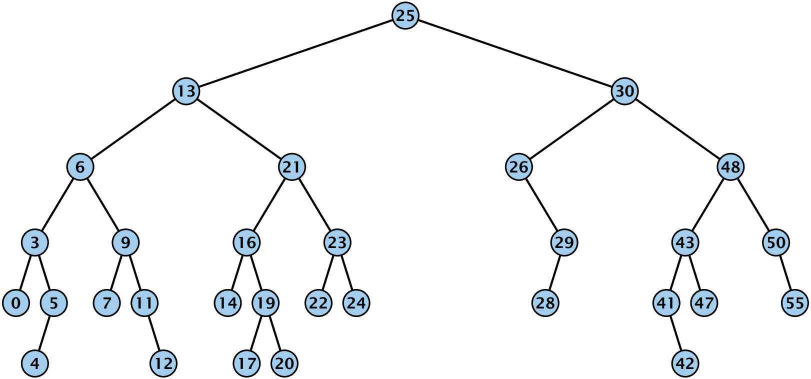
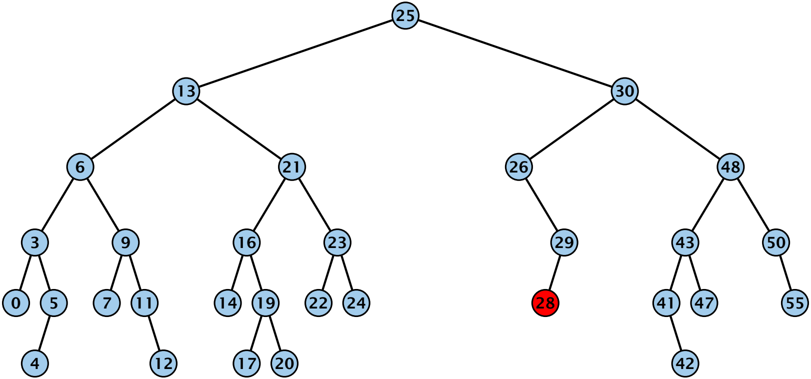
```

Binary Search Trees: Delete



Binary Search Trees: Delete

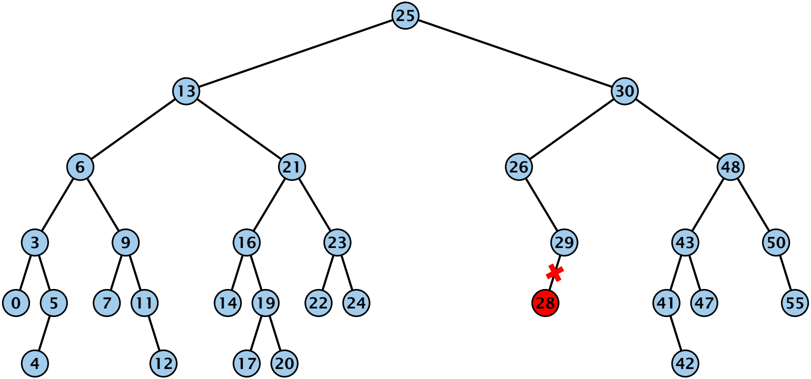


Case 1:

Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete



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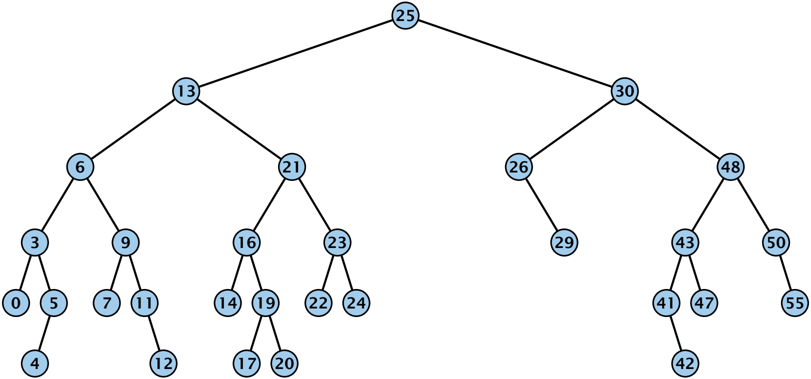

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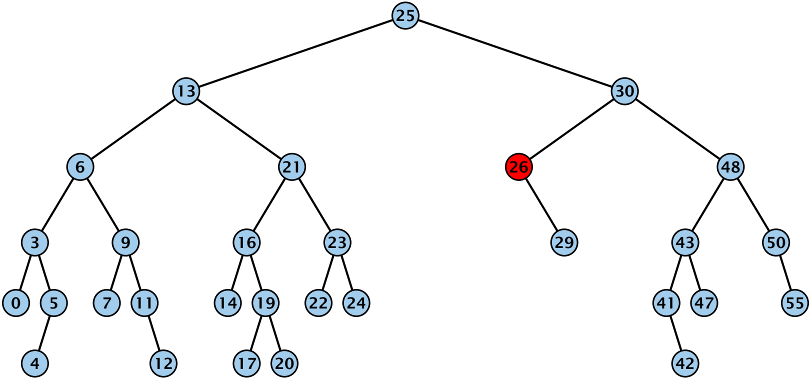
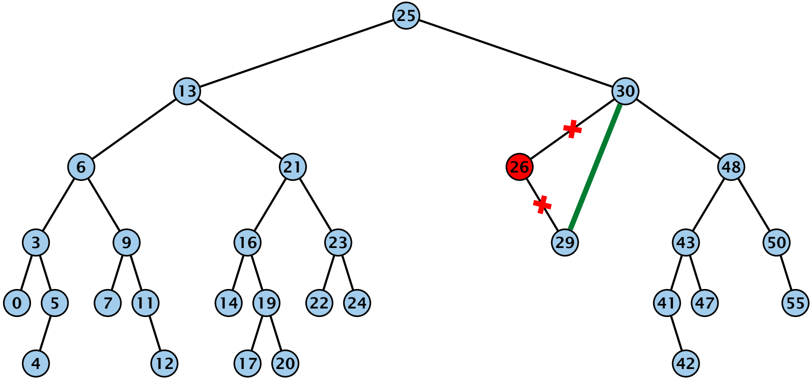


Case 2:

Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete



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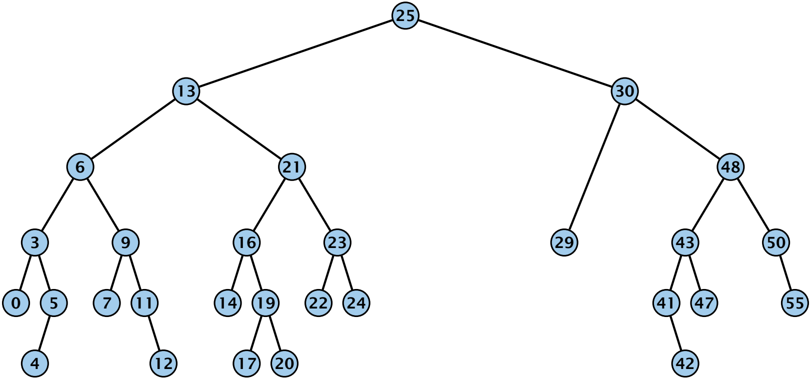

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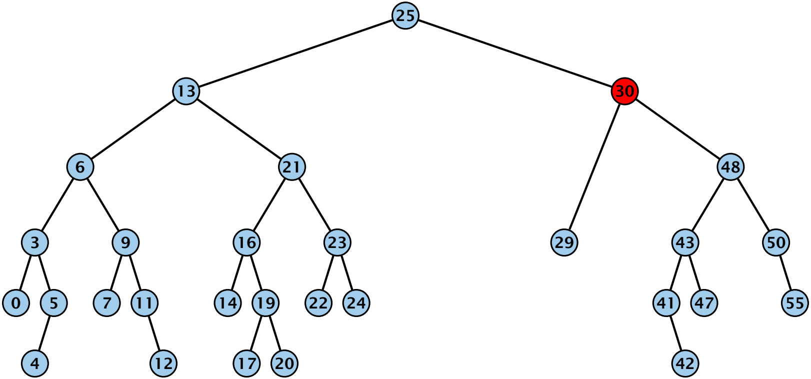
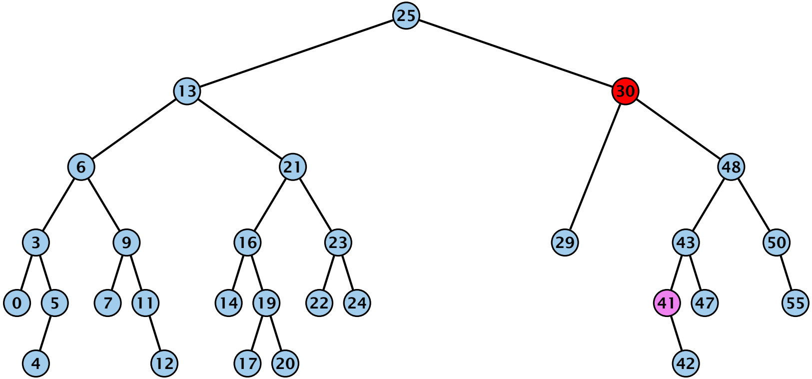


Case 3:

Element has two children

- Find the successor of the element
- Splice successor out of the tree
- Replace content of element by content of successor

Binary Search Trees: Delete



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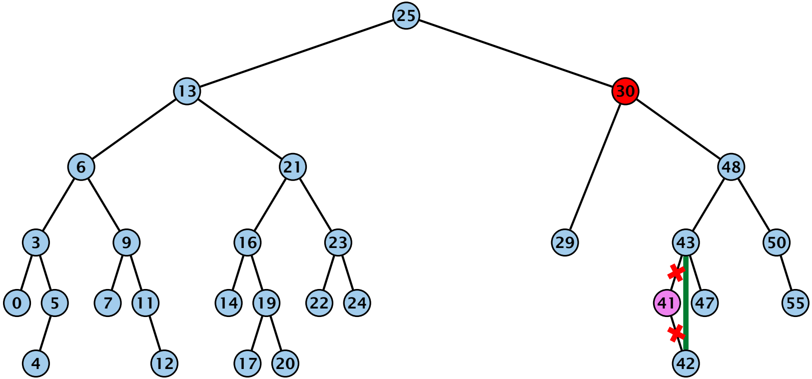

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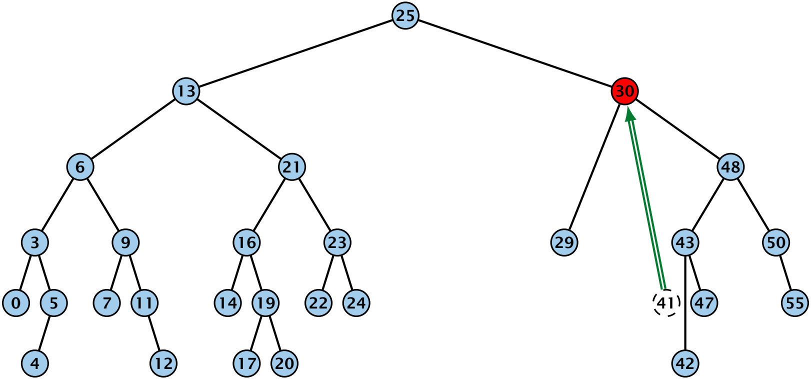

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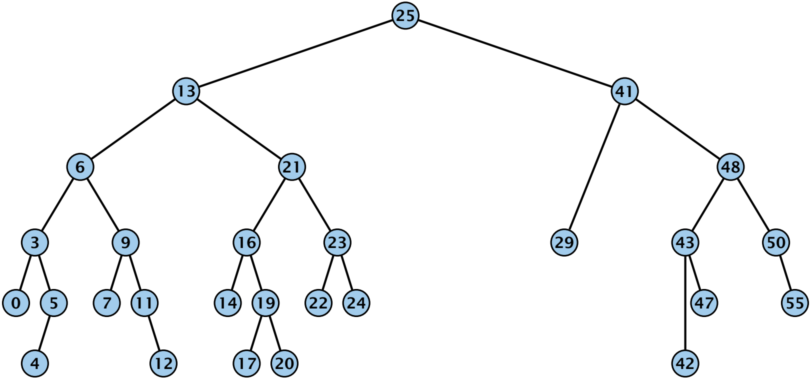

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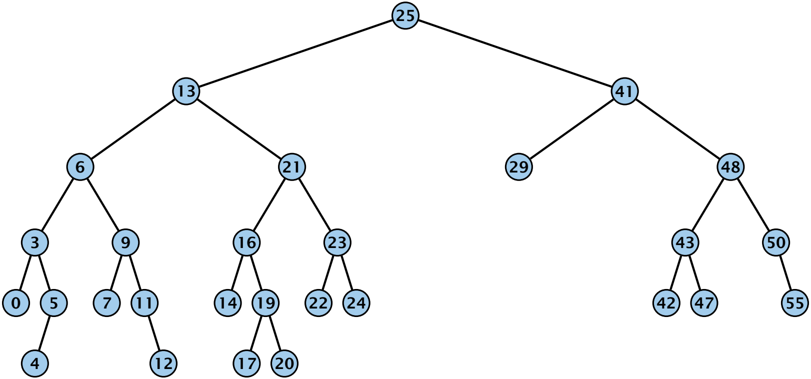

Case 3:

Element has two children

- Find the successor of the element
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- Replace content of element by content of successor

Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
if left[z] = null or right[z] = null
    then y ← z else y ← TreeSucc(z);                  select y to splice out
if left[y] ≠ null
    then x ← left[y] else x ← right[y];      x is child of y (or null)
if x ≠ null then parent[x] ← parent[y];                 parent[x] is correct
if parent[y] = null then
    root[T] ← x
else
    if y = left[parent[y]] then
        left[parent[y]] ← x                                 fix pointer to x
    else
        right[parent[y]] ← x
if y ≠ z then copy y-data to z
```

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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7.2 Red Black Trees

Definition 11

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

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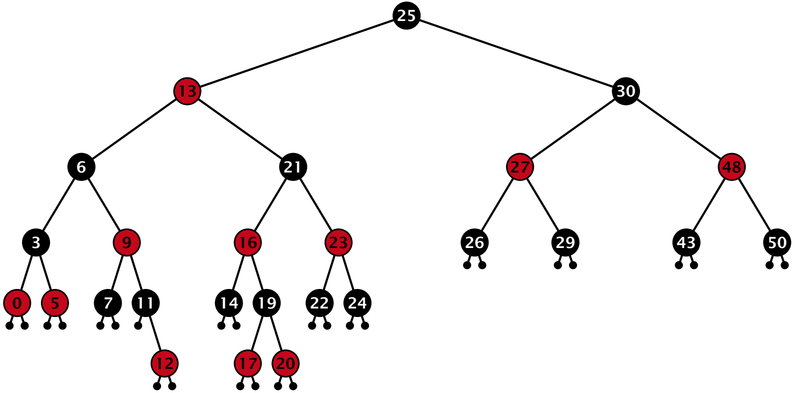
Red Black Trees: Example



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Lemma 12

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The black height $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

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Proof of Lemma 14.

Induction on the height of v.

base case ($\text{height}(v) = 0$)

- If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.
- The black height of v is 0.
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- Supose v is a node with $\text{height}(v) > 0$.
- v has two children with strictly smaller height.
- These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

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Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. □

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7.2 Red Black Trees

Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

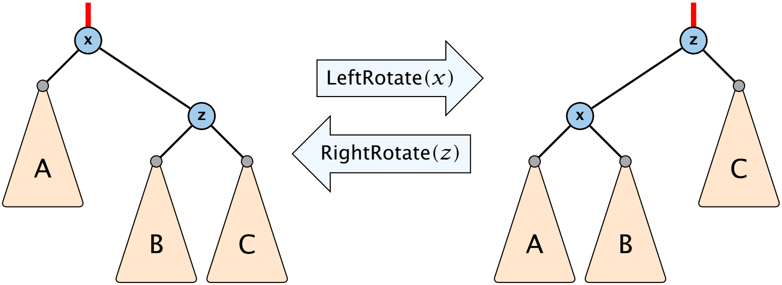
The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

7.2 Red Black Trees

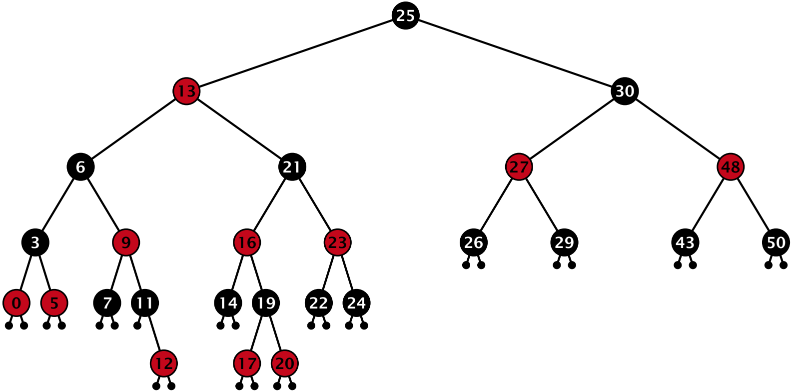
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

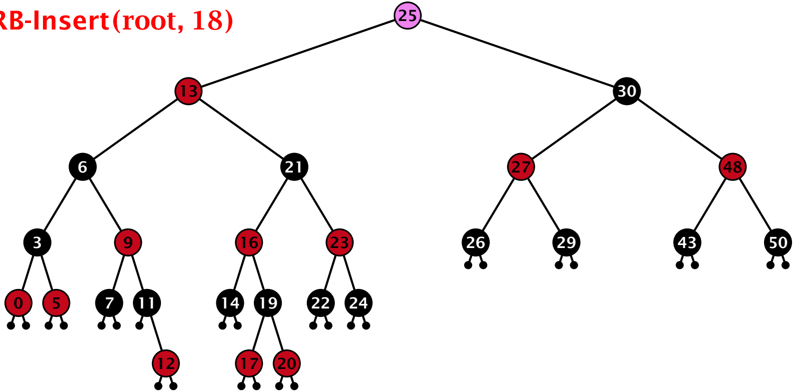


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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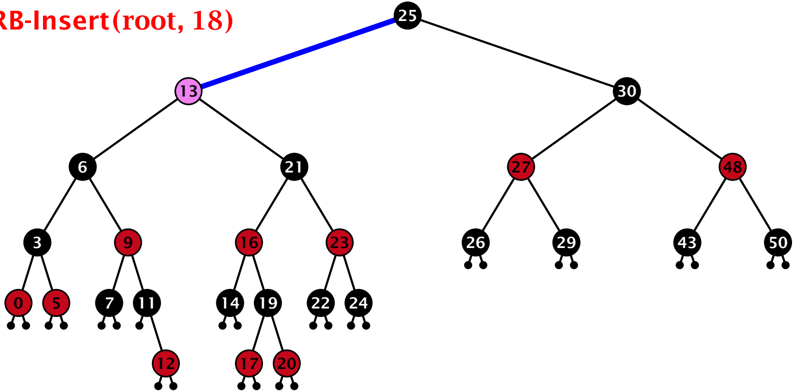

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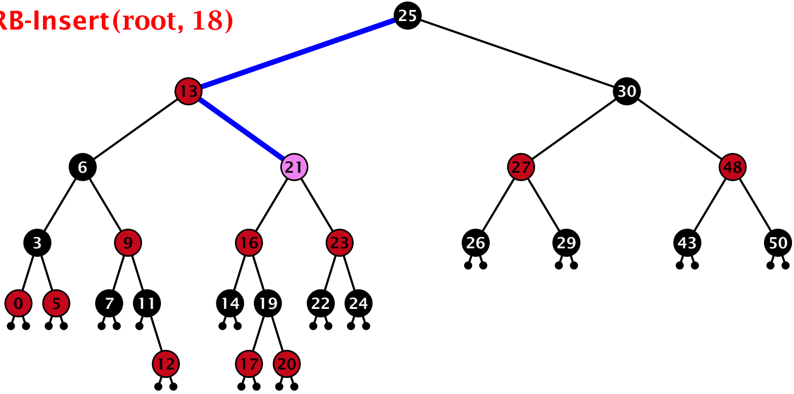

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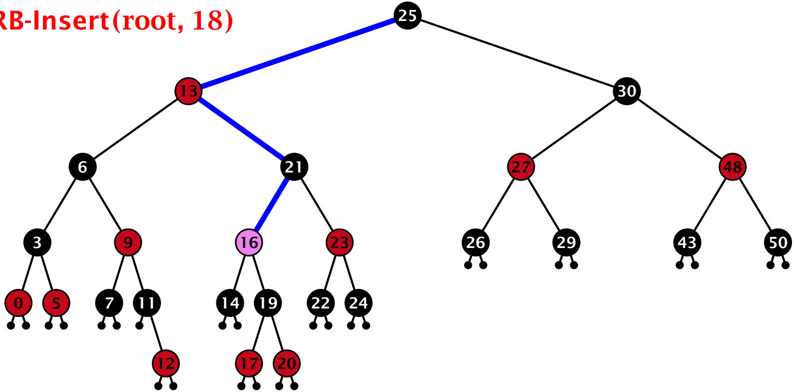

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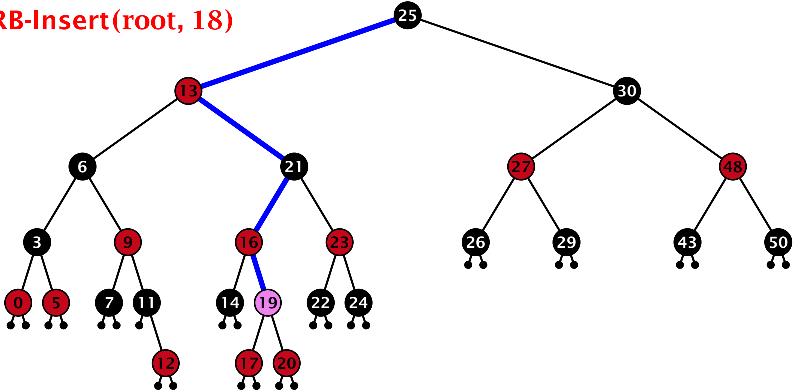

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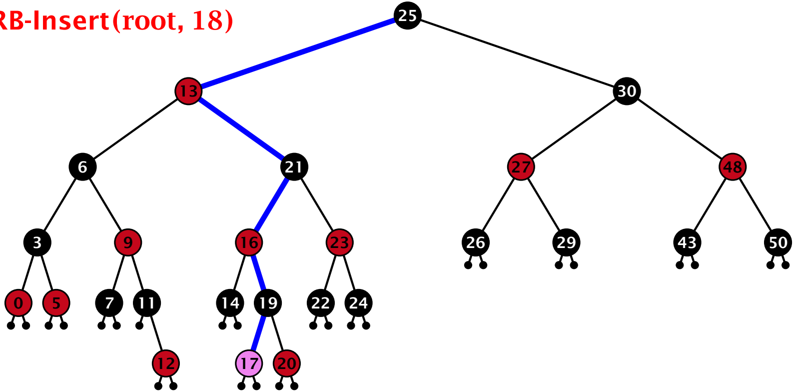

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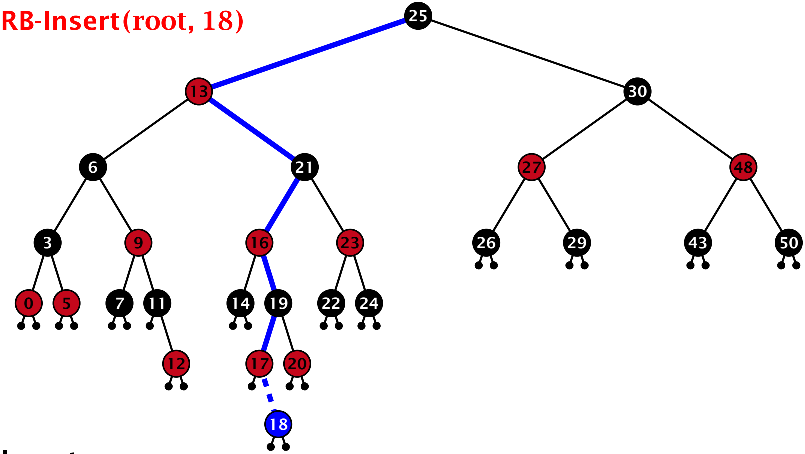

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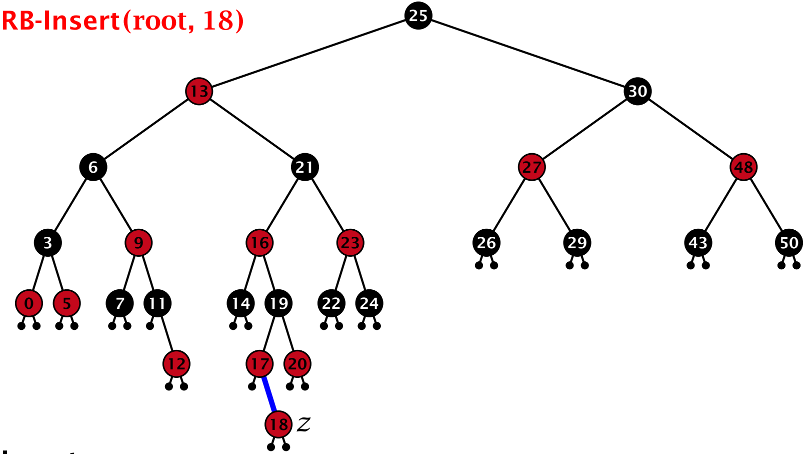

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Red Black Trees: Insert

Invariant of the fix-up algorithm:

- z is a red node
- the black-height property is fulfilled at every node
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  - either both of them are red (most important case)
  - or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
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Red Black Trees: Insert

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            col[p[z]] ← black; col[u] ← black;
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        else                                        Case 2: uncle black
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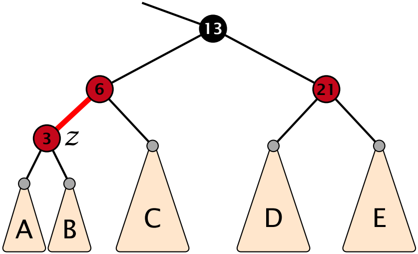
Case 1: Red Uncle



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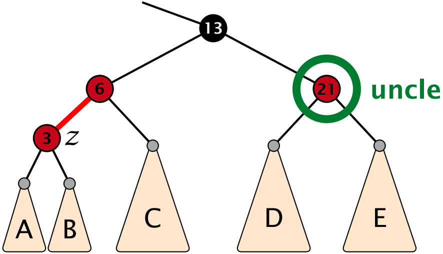

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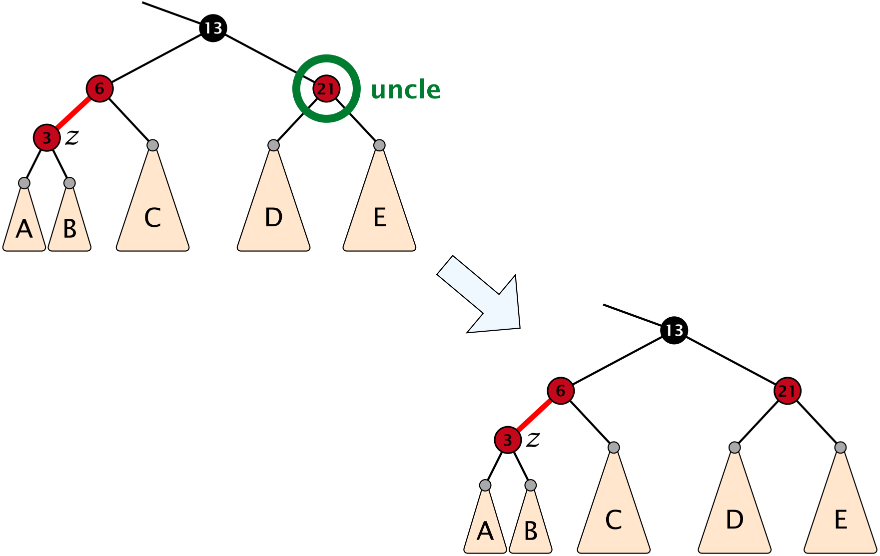

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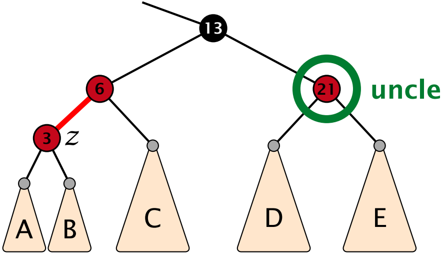
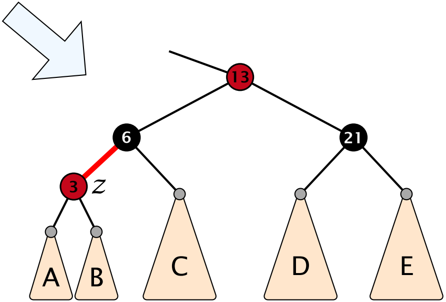


1. recolour



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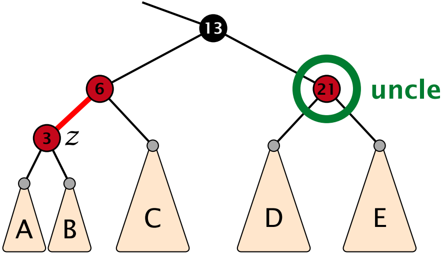

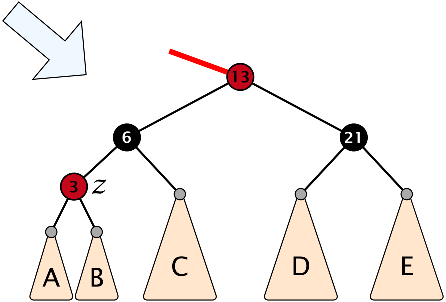

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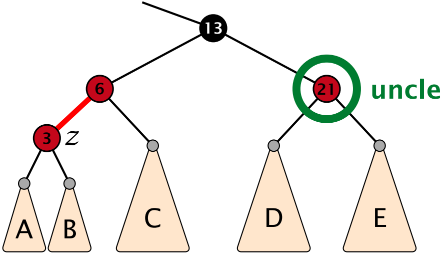
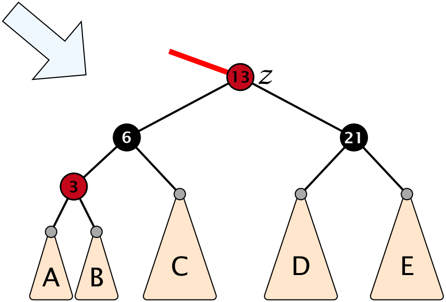


1. recolour
2. move z to grand-parent



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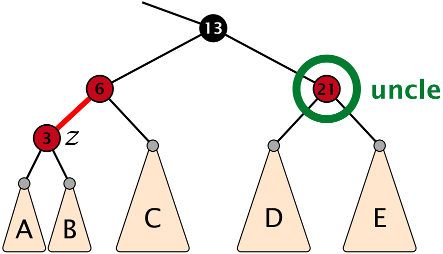

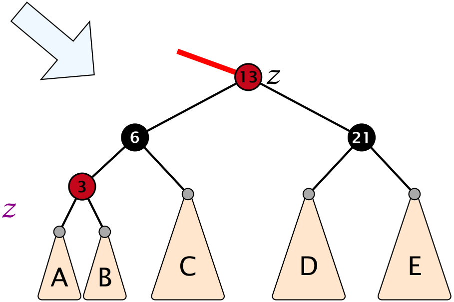

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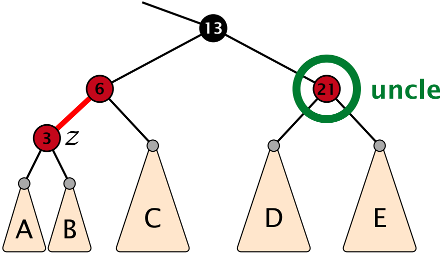
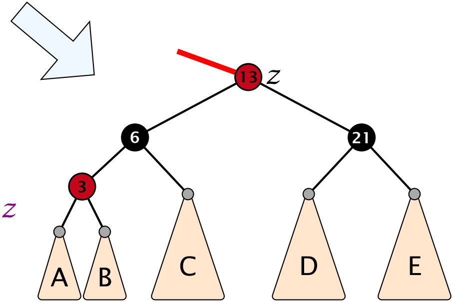
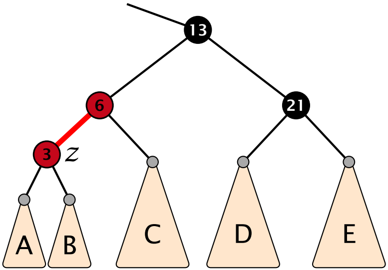




1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



Case 2b: Black uncle and z is left child

1. rotate around grandparent
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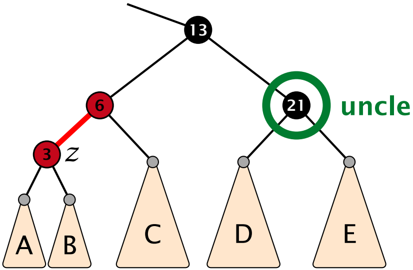

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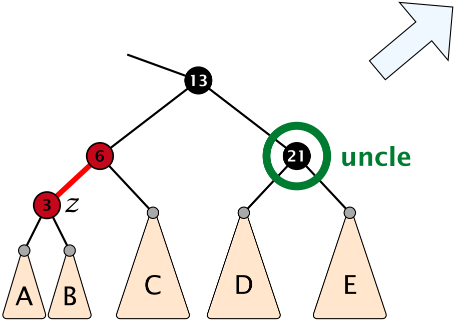

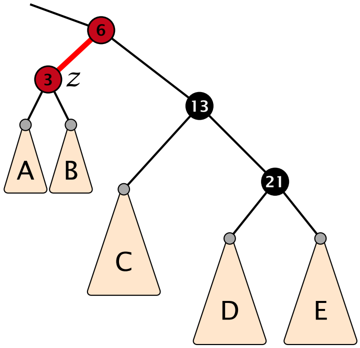

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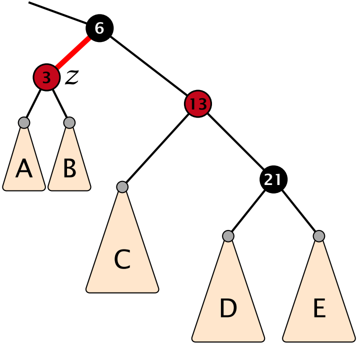

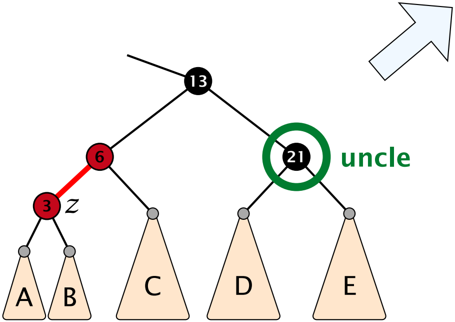

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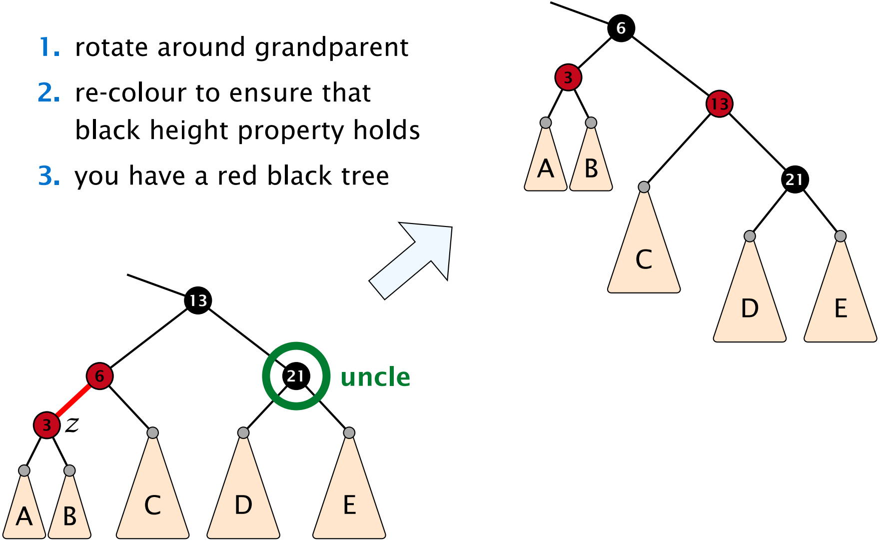

Case 2a: Black uncle and z is right child

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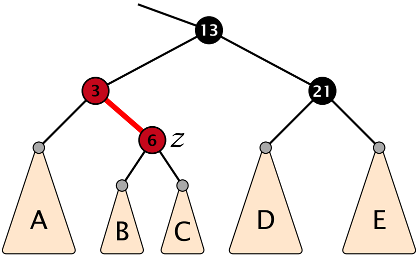
3. you have Case 2b.



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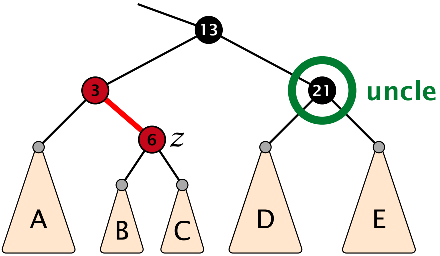

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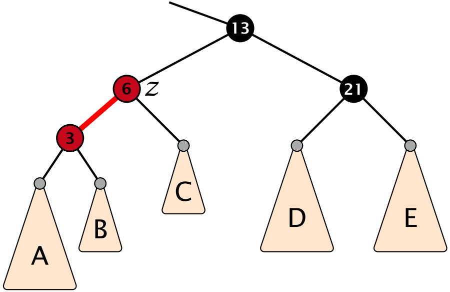

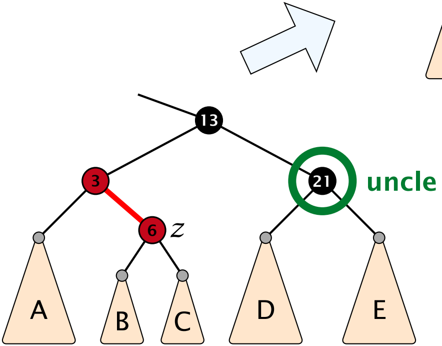

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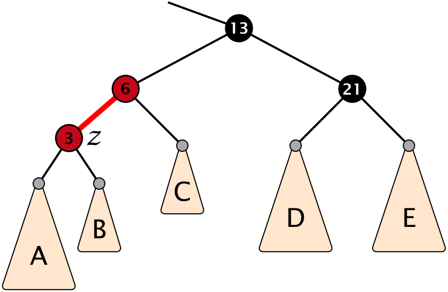

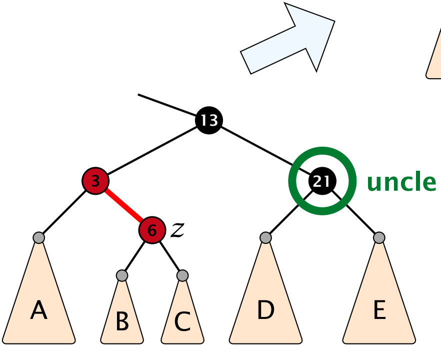

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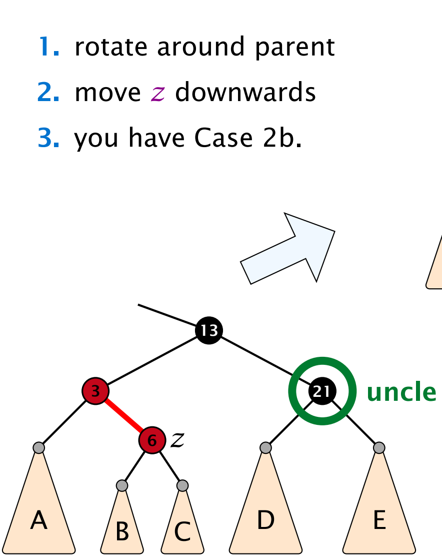

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a → Case 2b → red-black tree
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Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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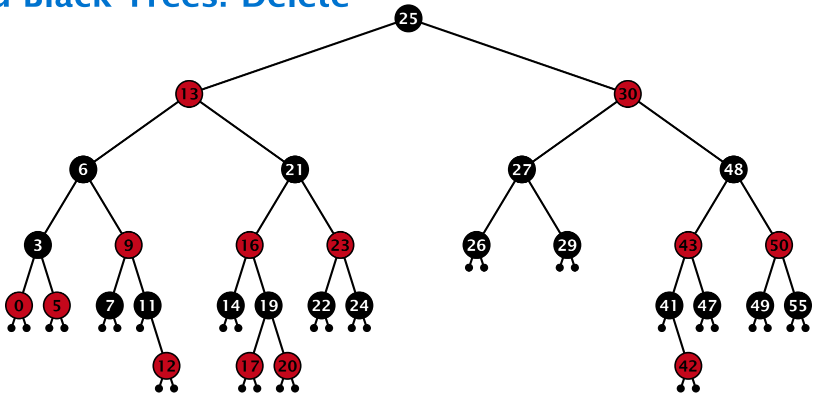

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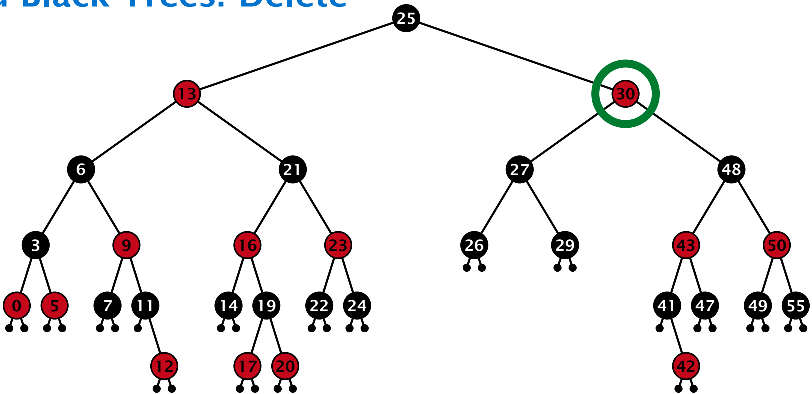

Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

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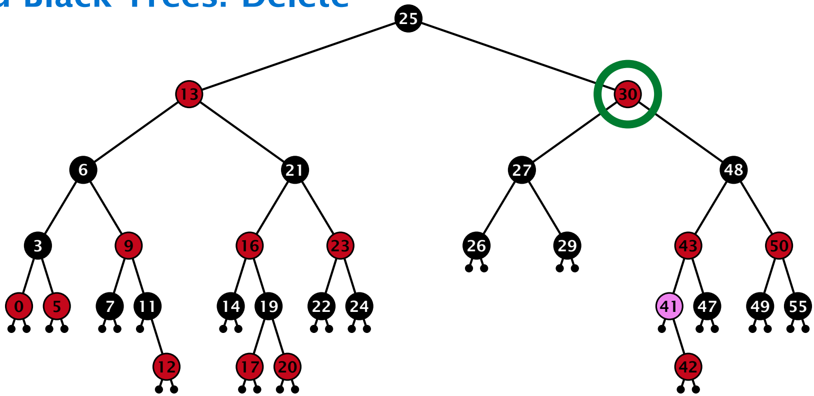

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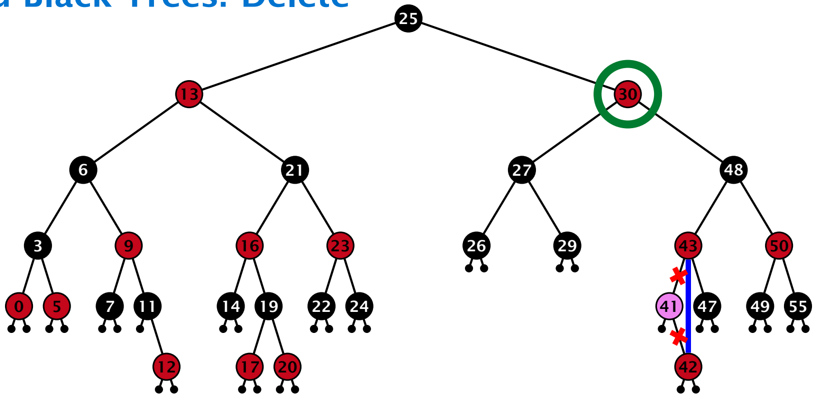

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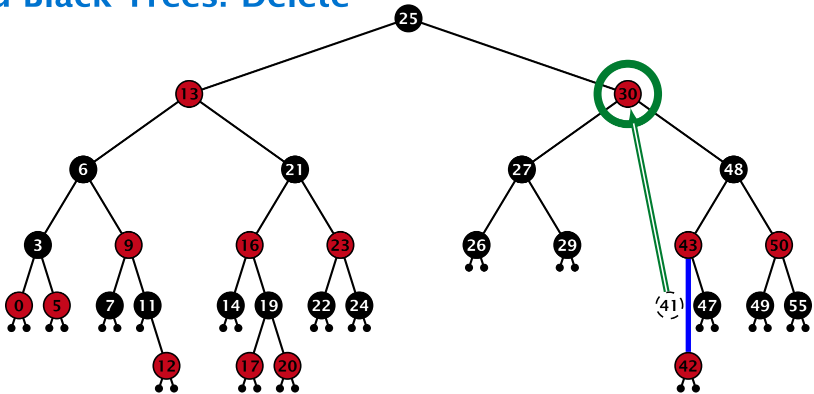

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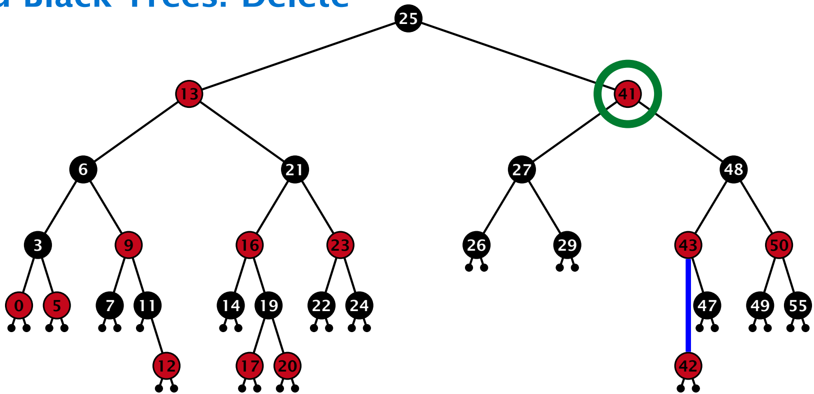

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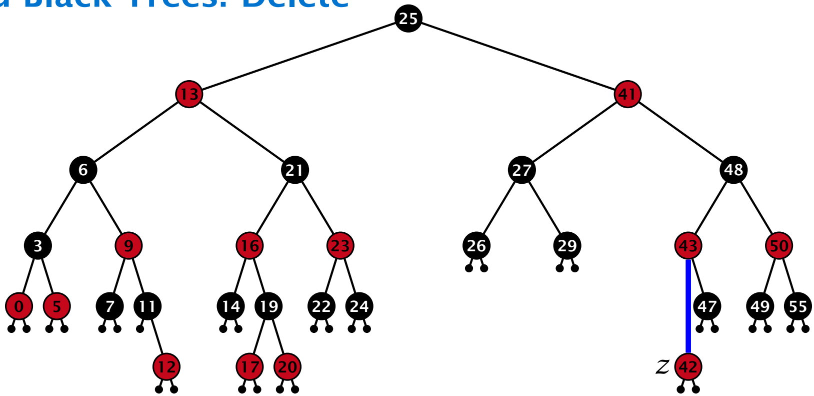

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- deleting black node messes up black-height property
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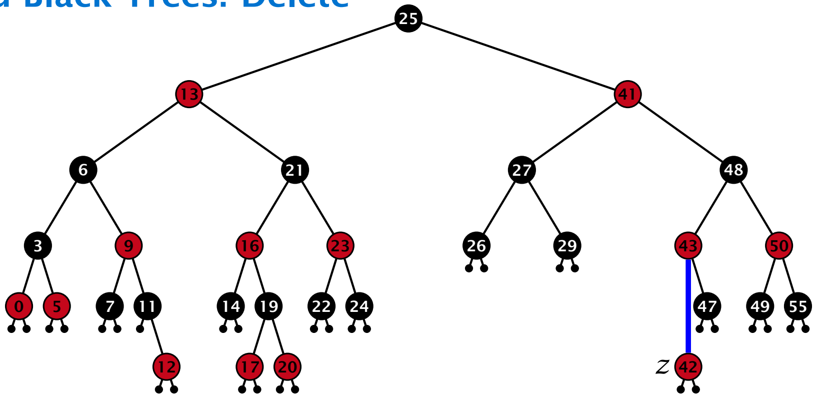

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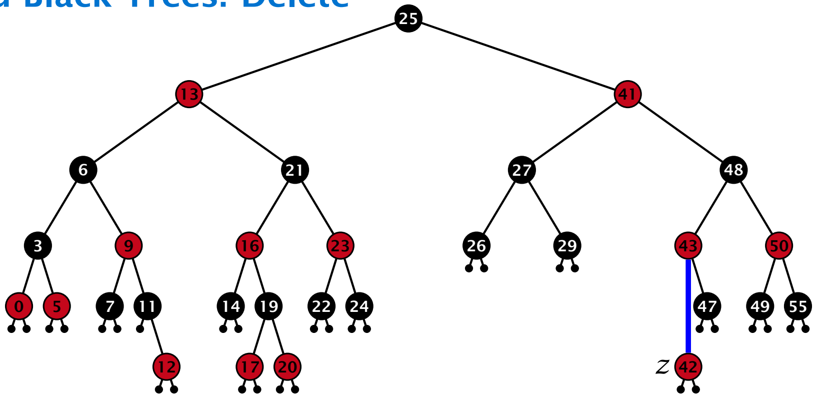

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Invariant of the fix-up algorithm

- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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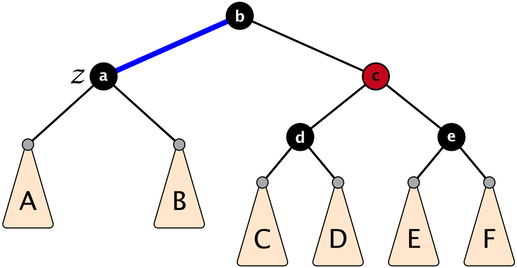

1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4

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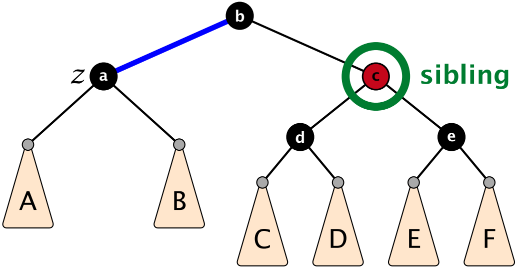

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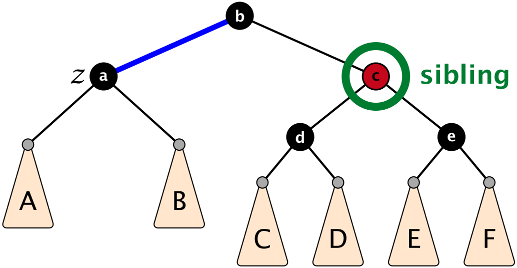

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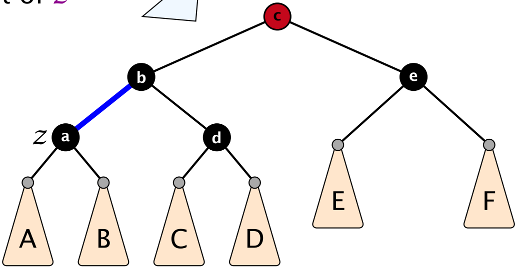

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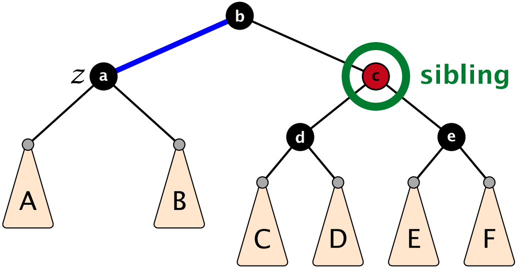

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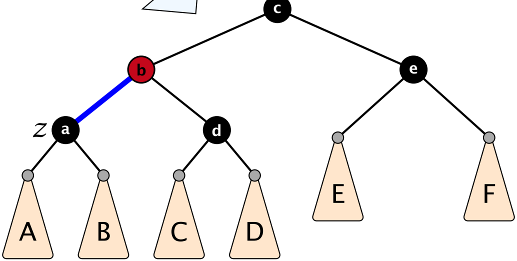

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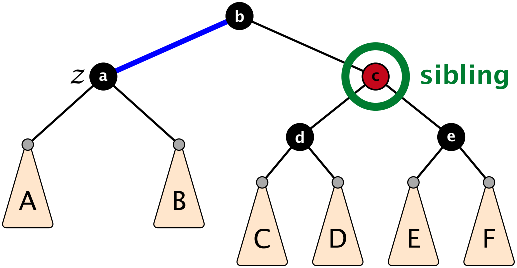

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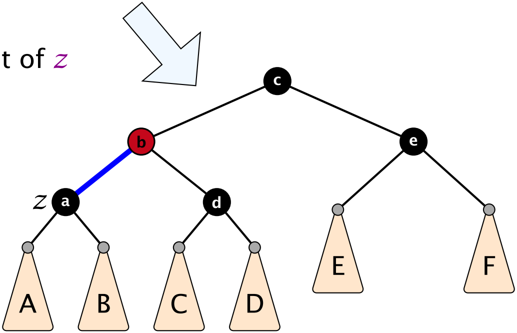

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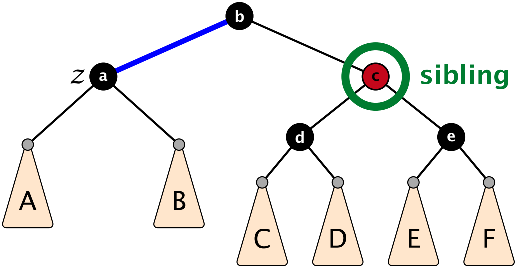

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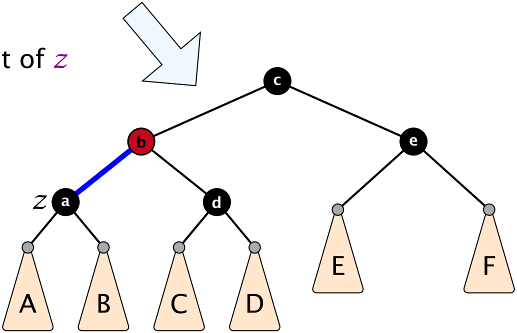
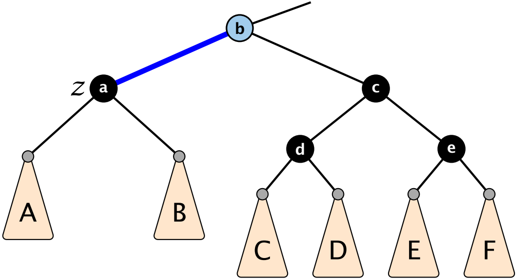


Case 2: Sibling is black with two black children



1. re-color node c
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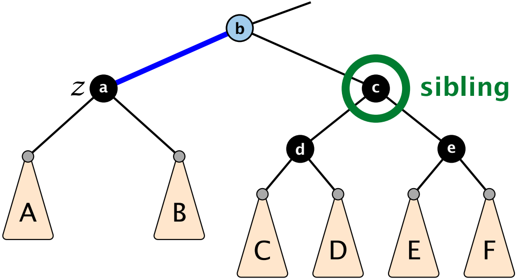

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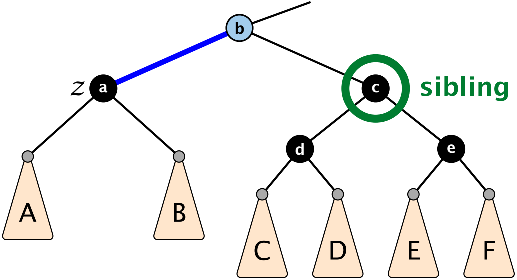

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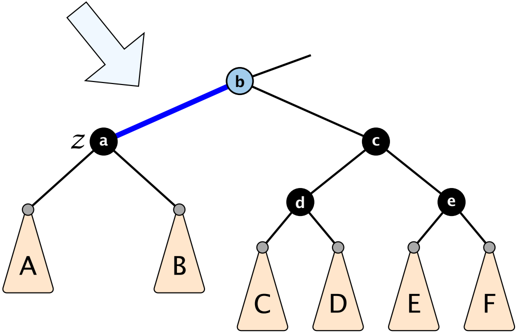

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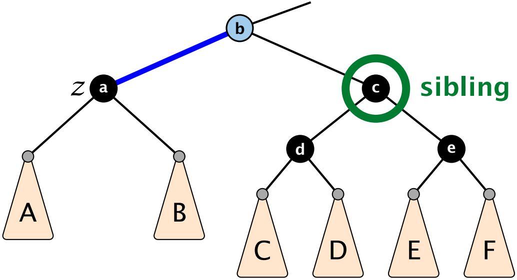

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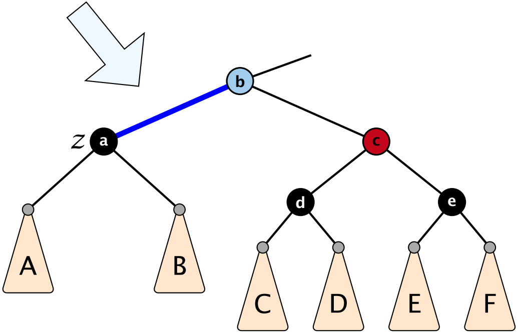

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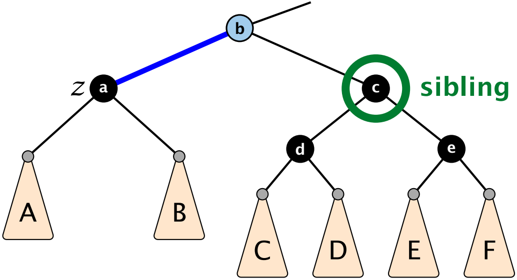

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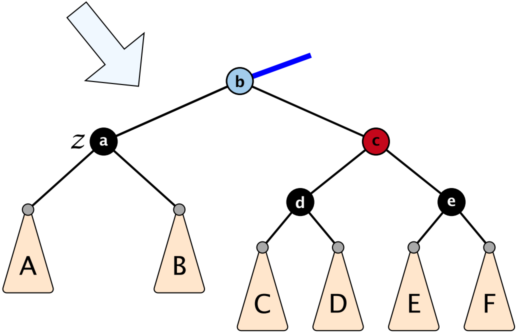

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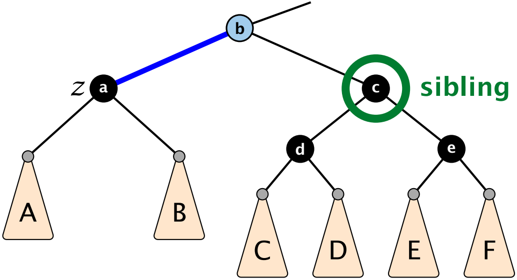

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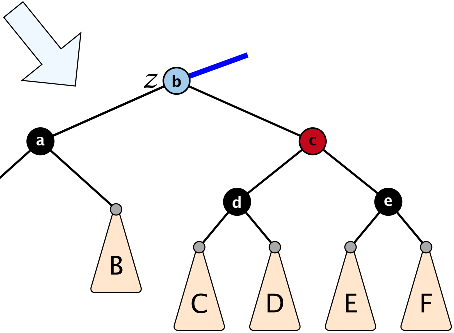

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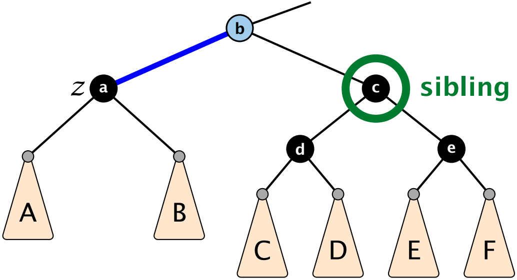

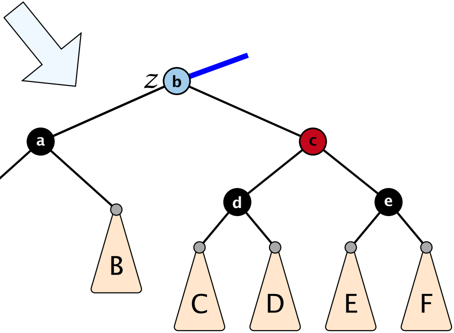

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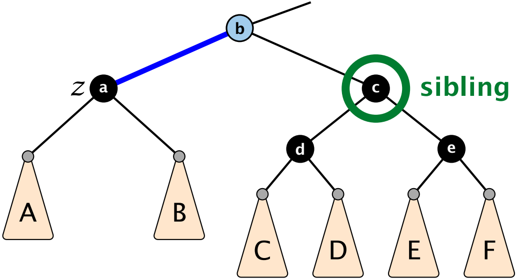

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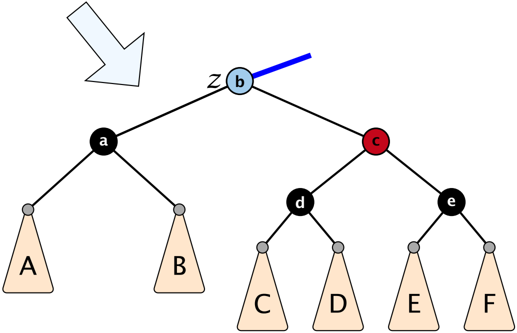
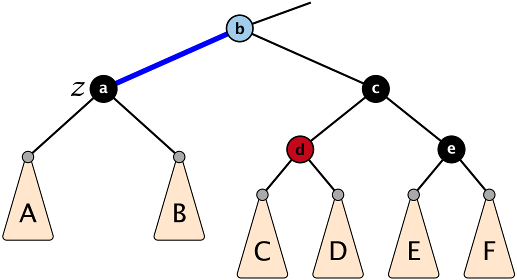


Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



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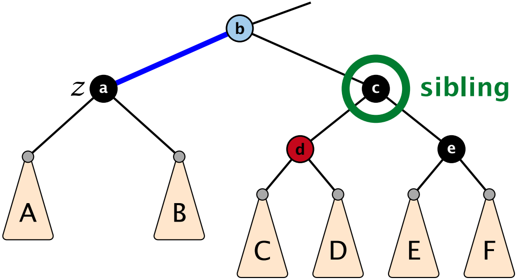

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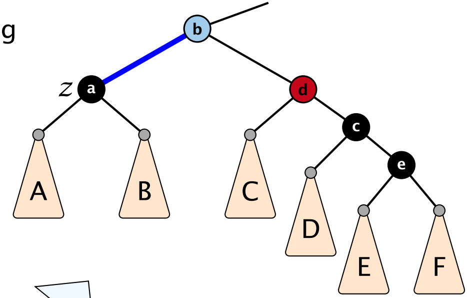

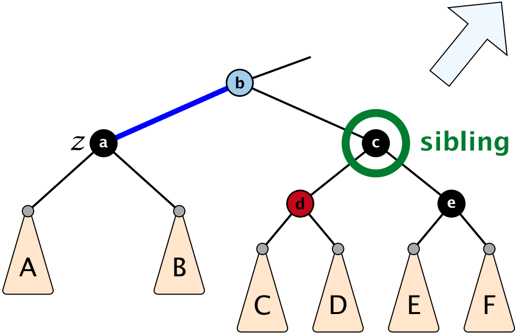

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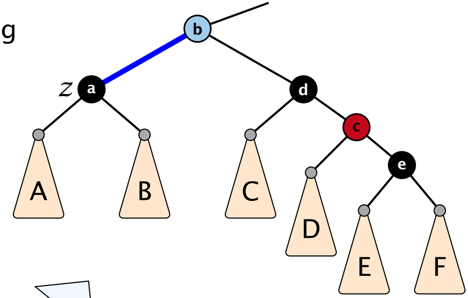

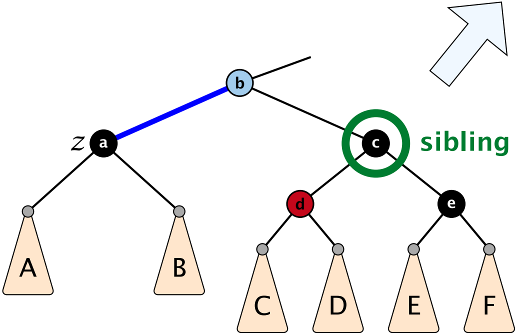

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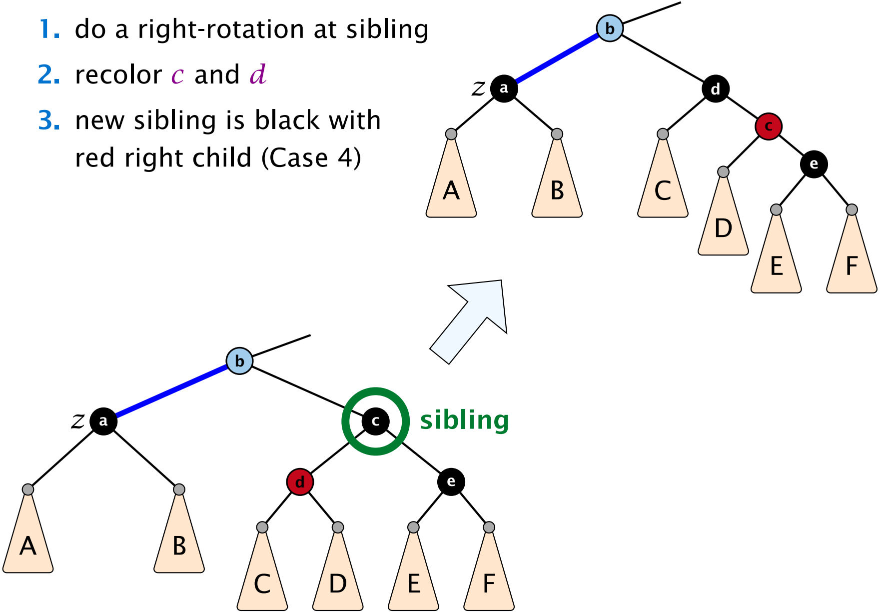

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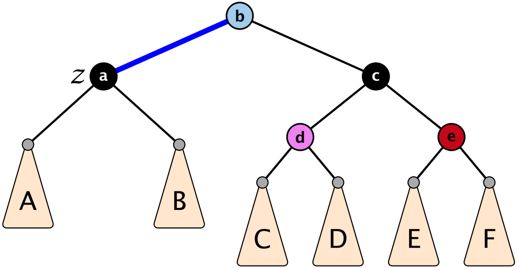

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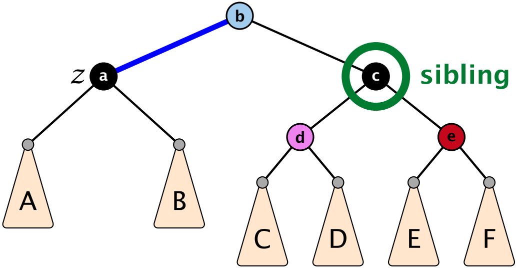

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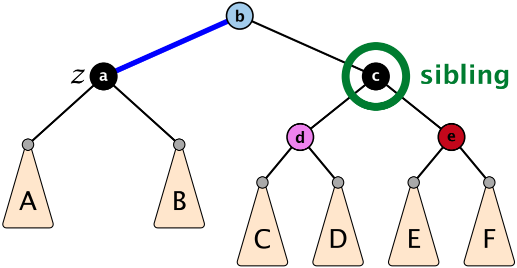

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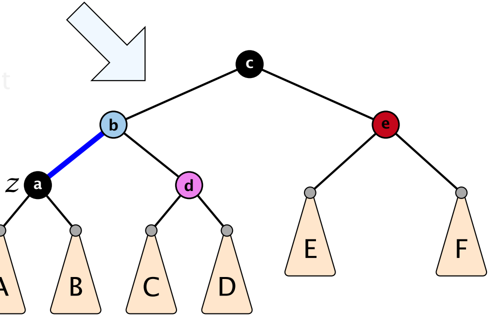

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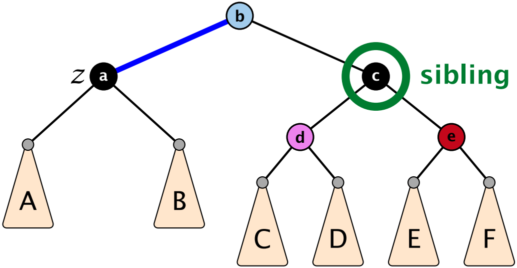

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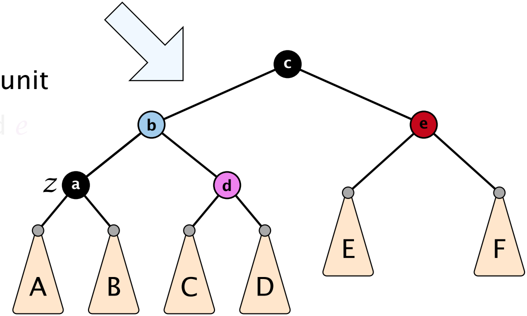

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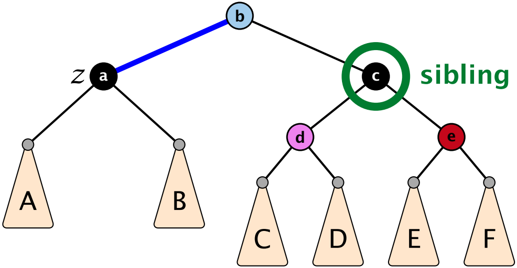

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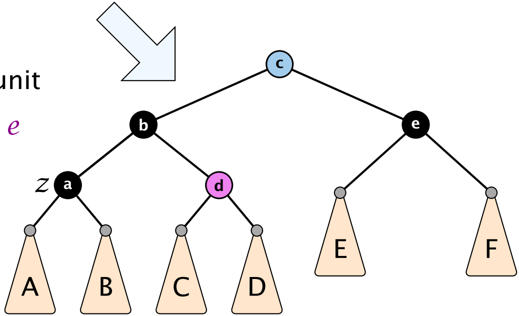

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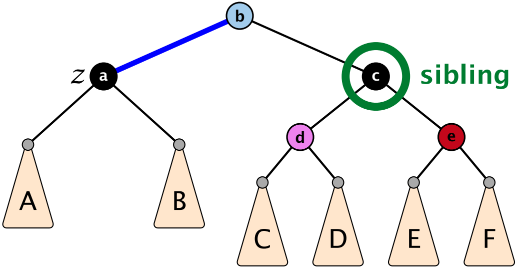

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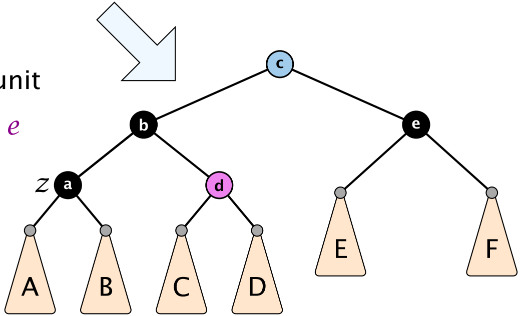

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after access, an element is moved to the root; splay(x)
- repeated accesses are faster
- only amortized guarantee
- read-operations change the tree

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Splay Trees

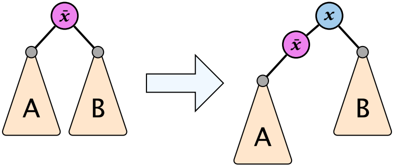
find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay$(\bar{x})$

Splay Trees

insert(x)

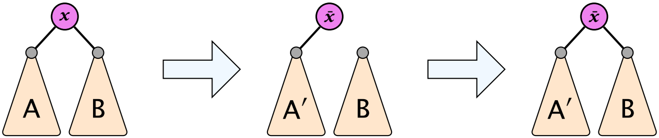
- search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- splay($\bar{x}$) moves $\bar{x}$ to the root
- insert x as new root



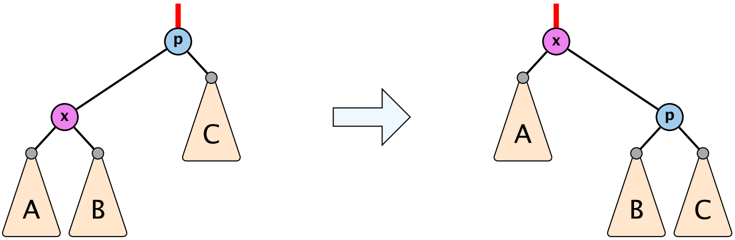
Splay Trees

delete(x)

- search for x; splay(x); remove x
- search largest element $\bar{x}$ in A
- splay($\bar{x}$) (on subtree A)
- connect root of B as right child of $\bar{x}$



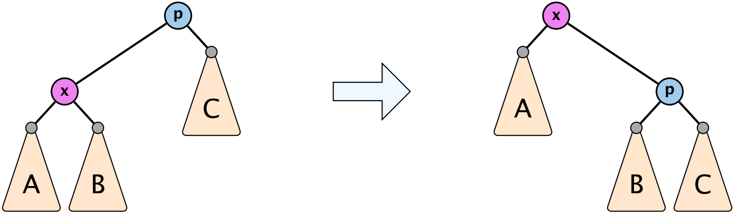
Move to Root



How to bring element to root?

- one (bad) option: moveToRoot(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation

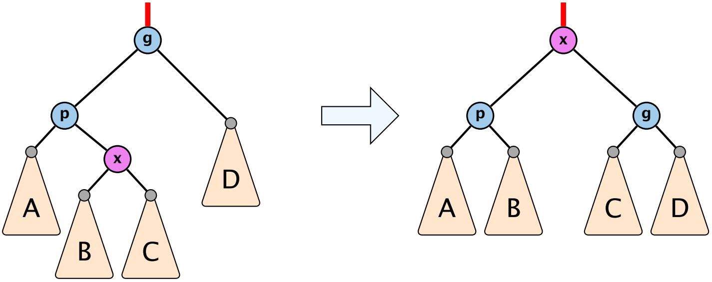
Splay: Zig Case



better option splay(x):

- zig case: if x is child of root do left rotation or right rotation around parent

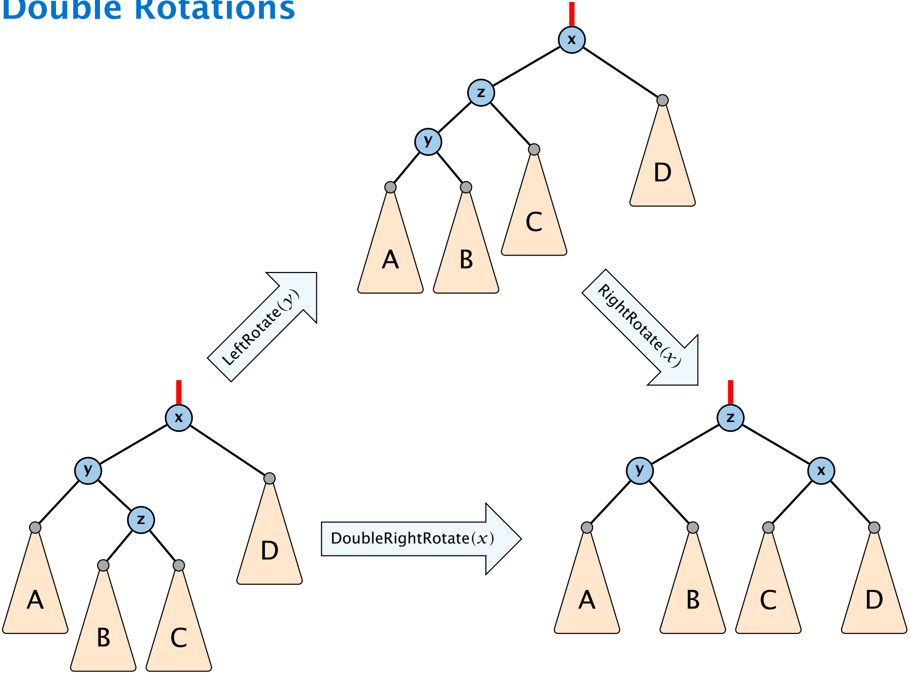
Splay: Zigzag Case



better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



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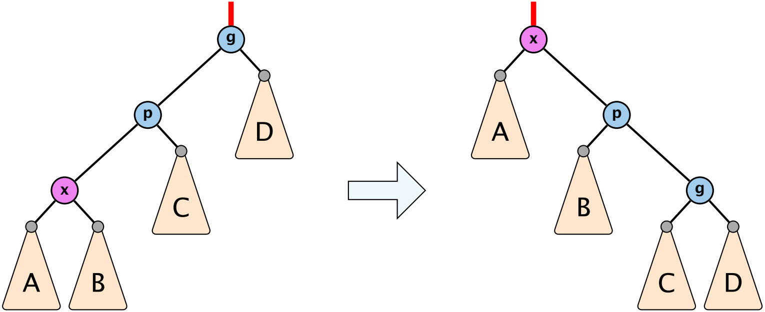
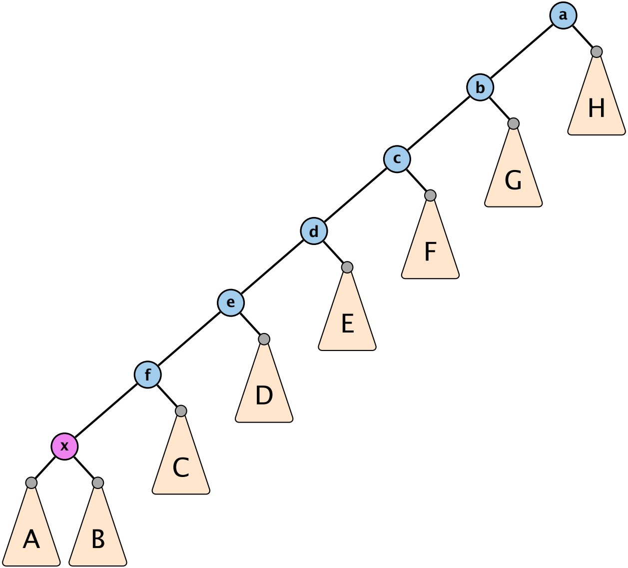


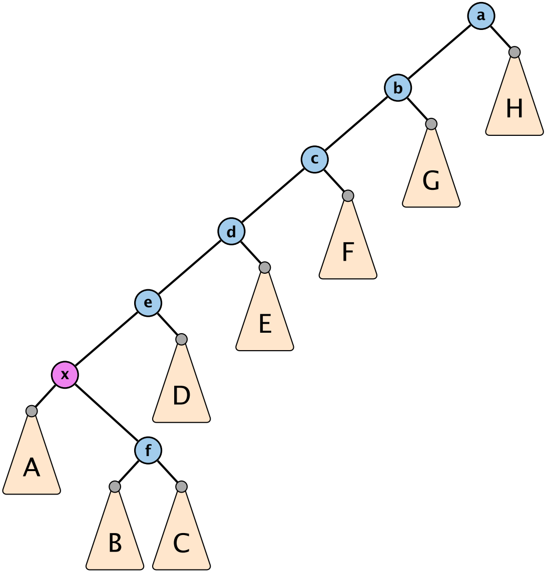
better option splay(x):

- zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



Splay vs. Move to Root



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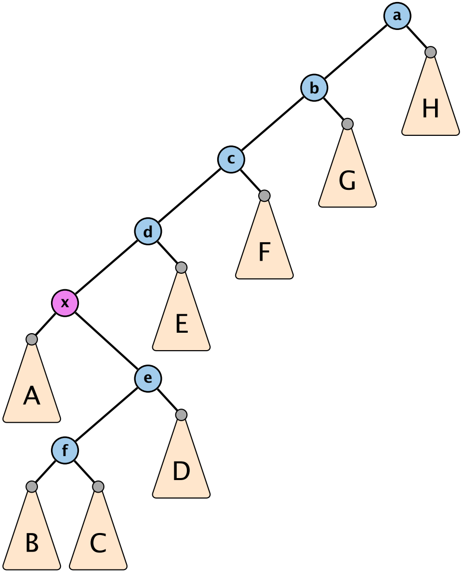

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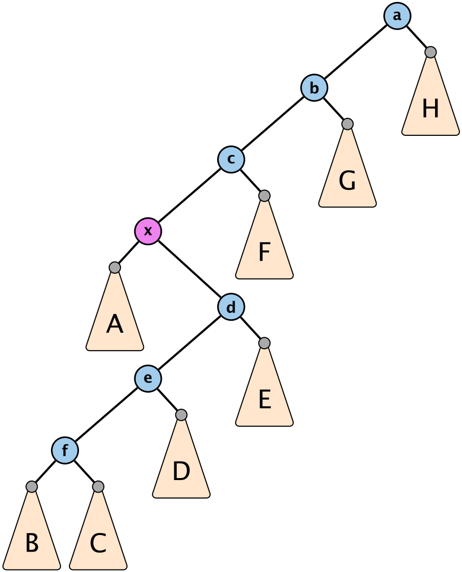

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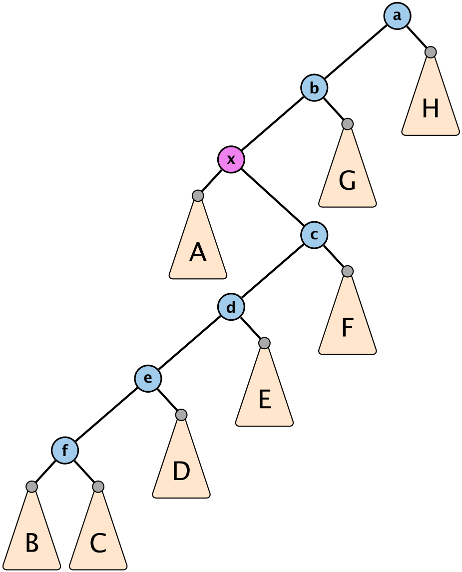

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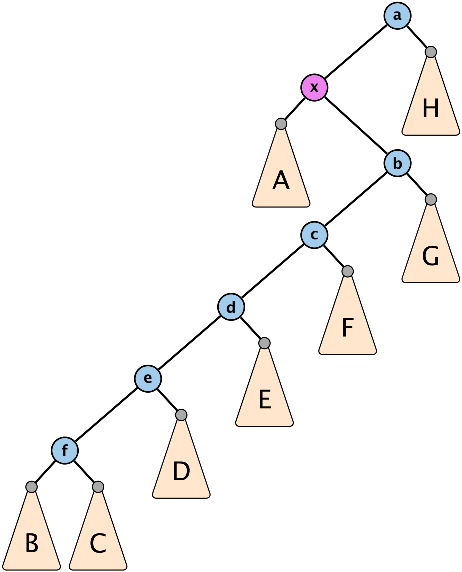

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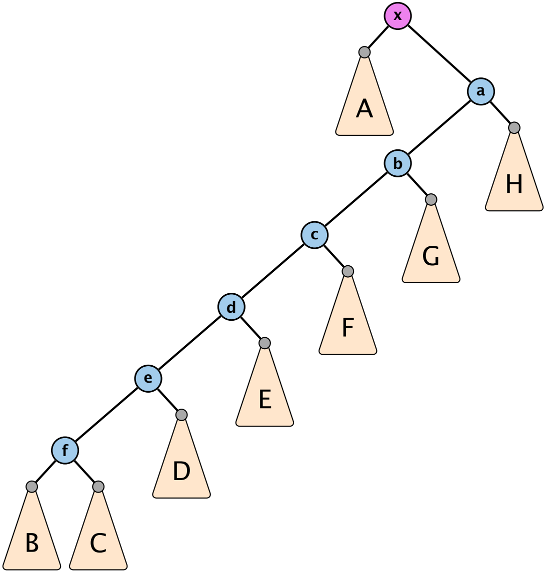

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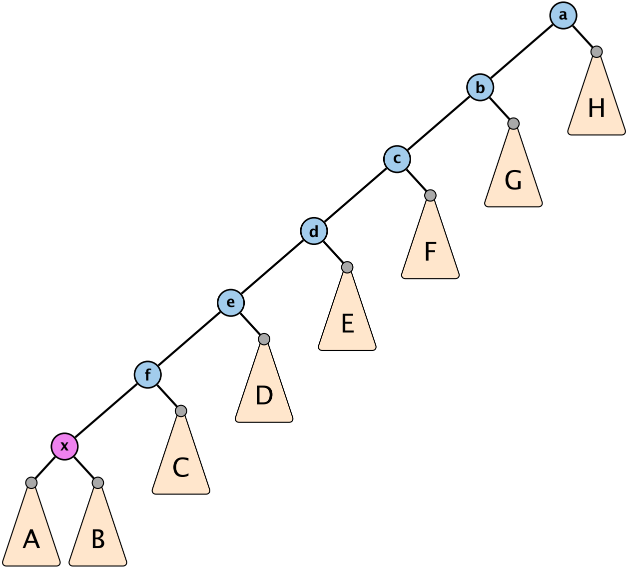

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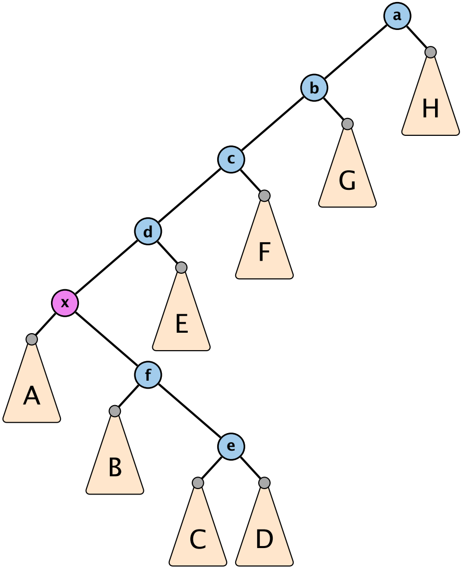

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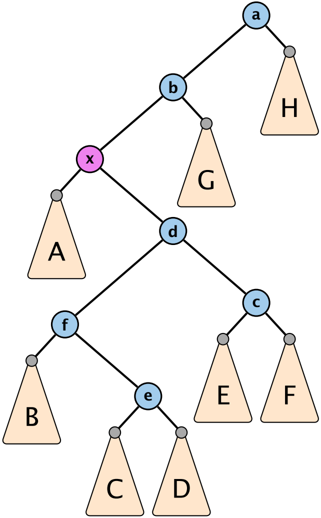

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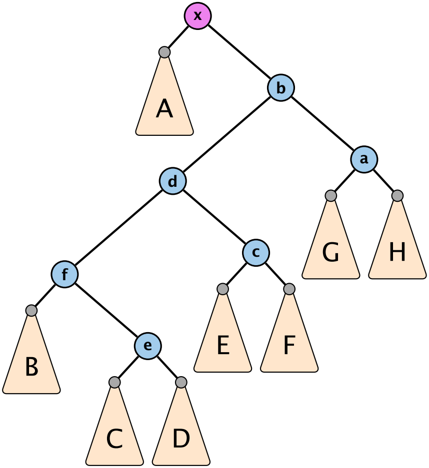

Static Optimality

Suppose we have a sequence of m find-operations. find(x) appears h_x times in this sequence.

The cost of a static search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \,\text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\text{min}}))$, where T_{min} is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- the cost for accessing element x is $1 + \text{depth}(x)$;
- after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S.

Lemma 15

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ for all operations.

Amortized Analysis

Definition 16

A data structure with operations $\text{op}_1(), \ldots, \text{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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$$\sum_{i=1}^{k} c_i$$

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\mathbf{push}()$
- $S.\mathbf{pop}()$
- $S.\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $S.\mathbf{push}()$: cost 1.
- $S.\mathbf{pop}()$: cost 1.
- $S.\mathbf{multipop}(k)$: cost $\min\{\text{size}, k\} = k$.

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Example: Stack

Use potential function $\Phi(S)$ = number of elements on the stack.

Amortized cost:

$S.\text{push}()$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \le 2 .$$

$S.\text{pop}()$: cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \le 0 .$$

$S.\text{multipop}(k)$: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min(\text{size}, k) - \min(\text{size}, k) \le 0 .$$

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- $\mathbf{S.\,pop()}$: cost

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- $\mathbf{S.\,multipop(\mathit{k})}$: cost

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- $\mathbf{S.\,push()}$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \le 2 \ .$$

- $\mathbf{S.\,pop()}$: cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \le 0 \ .$$

- $\mathbf{S.\,multipop(k)}$: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \le 0 \ .$$

Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

Actual cost:

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

$0 \to 1$:

$$\hat{C}_{0\to1} = C_{0\to1} + \Delta\Phi = 1 + 1 \le 2$$

$1 \to 0$:

$$\hat{C}_{1\to0} = C_{1\to0} + \Delta\Phi = 1 - 1 \le 0$$

Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

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Splay Trees

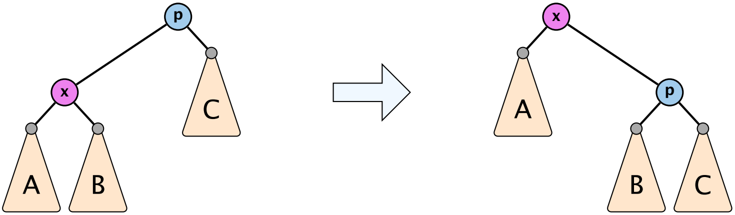
potential function for splay trees:

- size $\mathrm{s}(x) = |T_x|$
- rank $\mathrm{r}(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

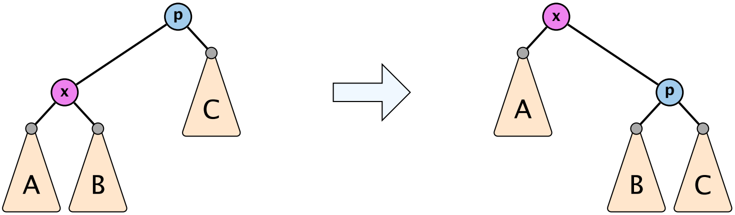
Splay: Zig Case



$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\
&= r'(p) - r(x) \\
&\leq r'(x) - r(x)
\end{aligned}
$$

$$\mathrm{cost}_{\mathrm{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zig Case



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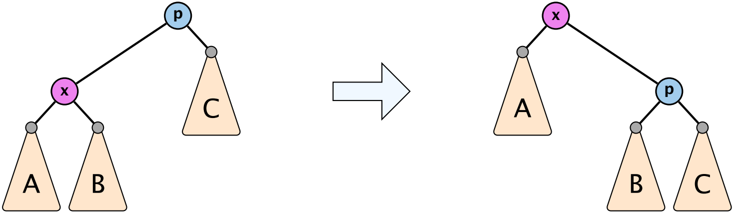

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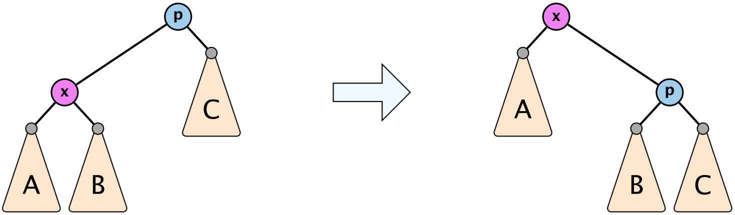

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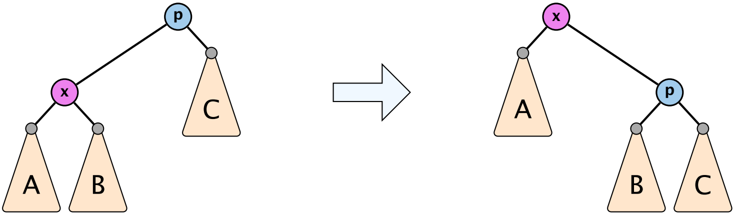

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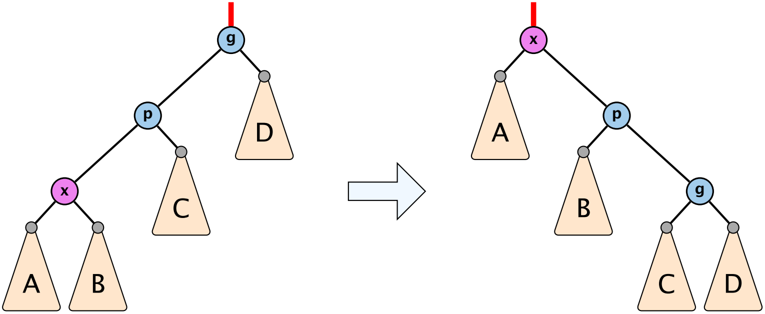

$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\le r'(x) + r'(g) - r(x) - r(x)\\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\\ &\le -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzig}} \le 3(r'(x) - r(x))\end{aligned}$$

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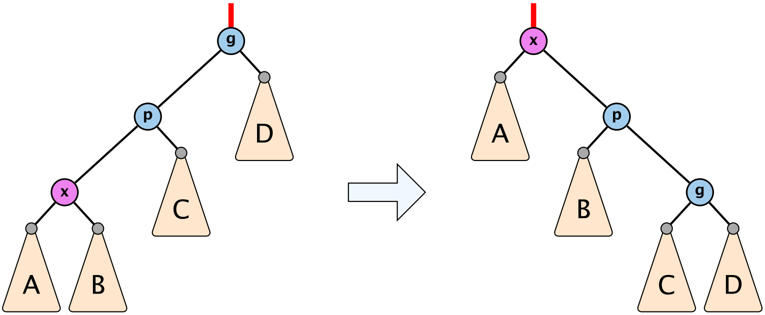

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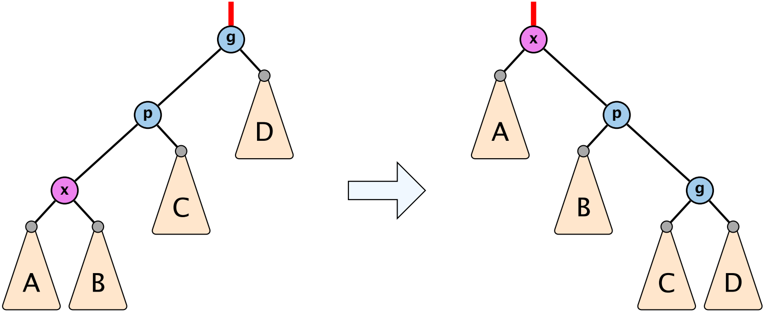

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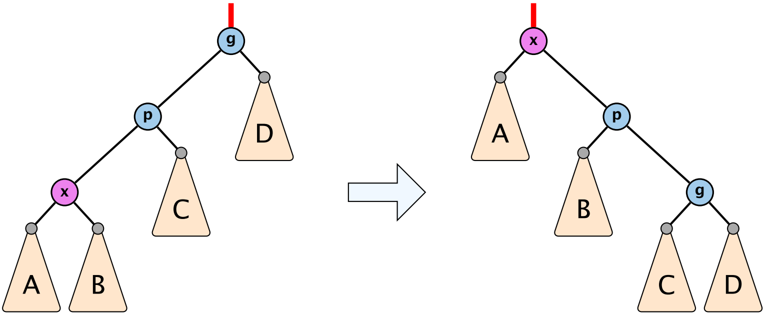

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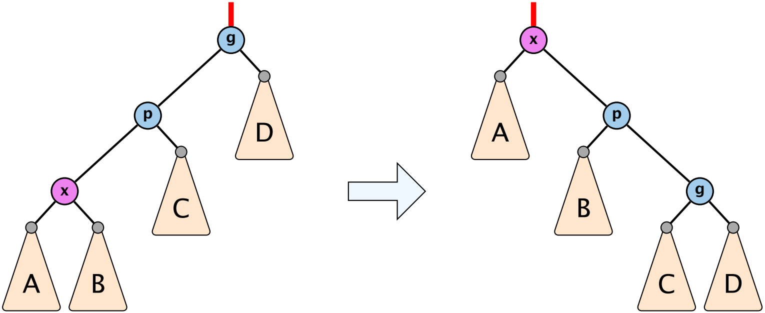

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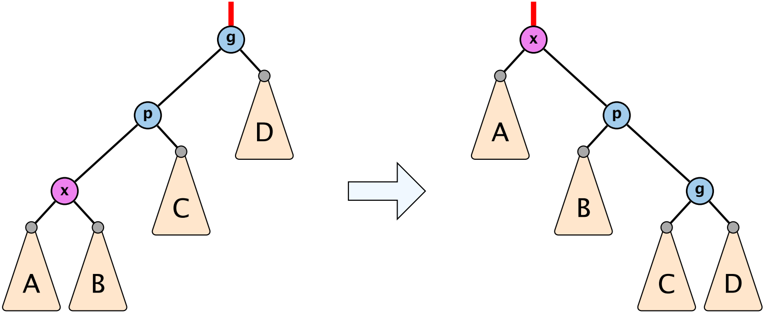

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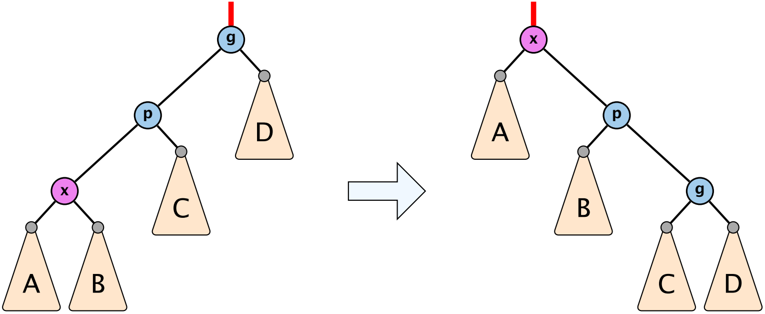

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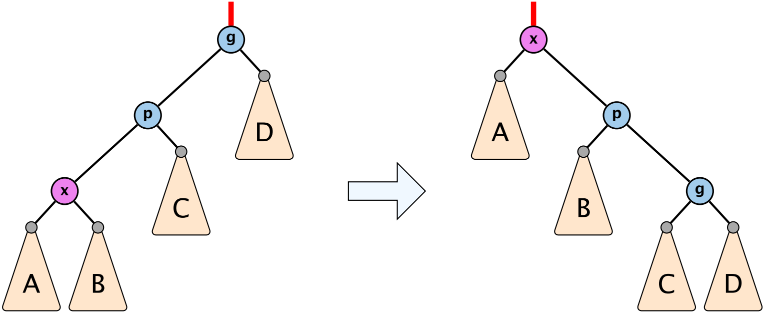

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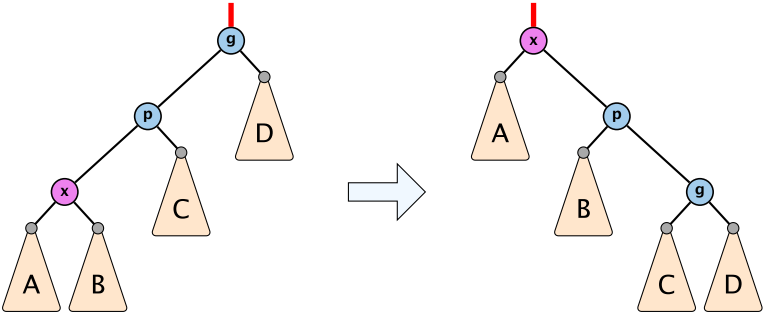

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

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$$\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big) = -1$$

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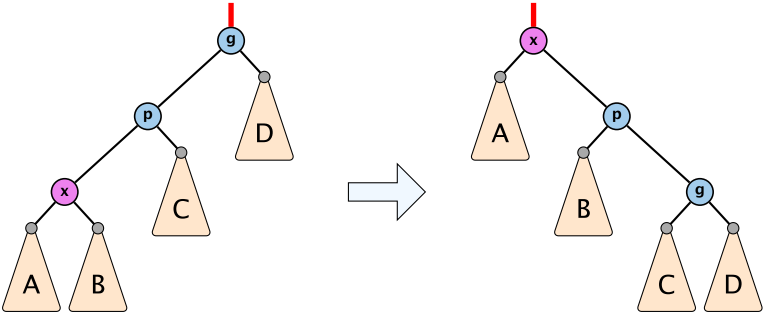

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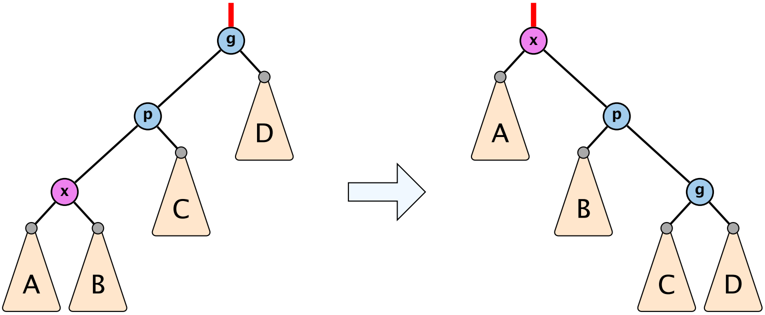

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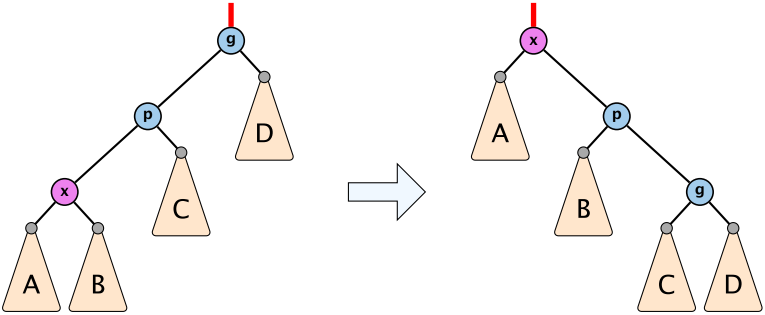

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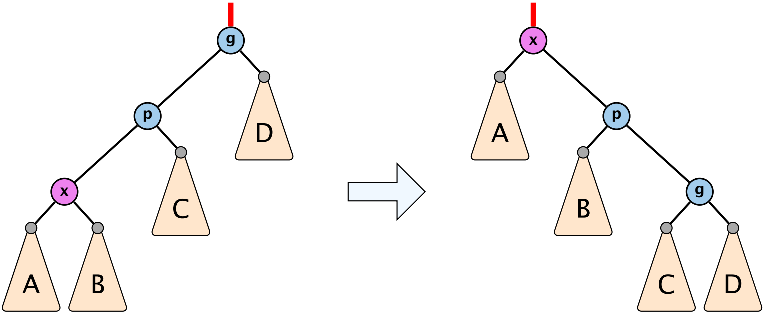

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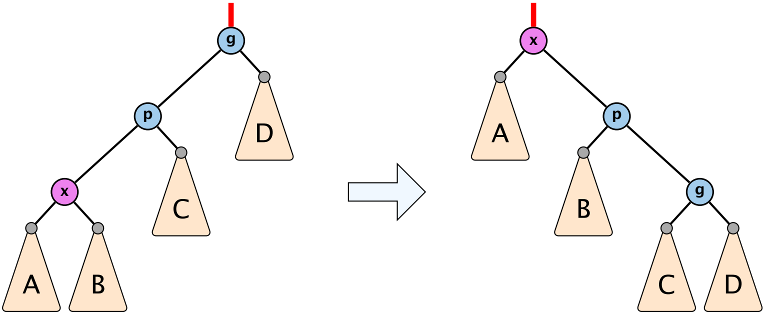
Splay: Zigzig Case



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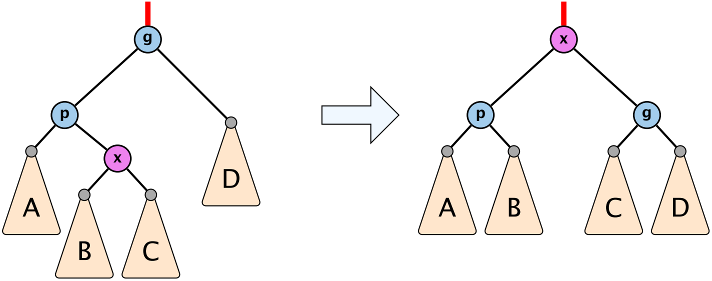
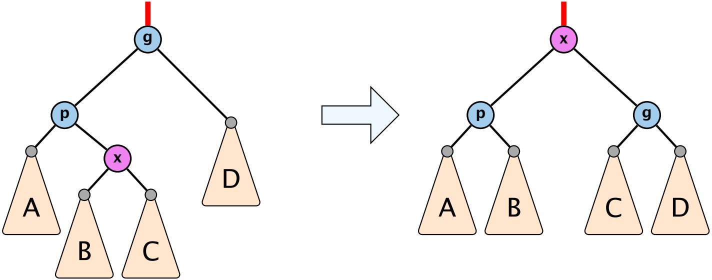
Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\le r'(p) + r'(g) - r(x) - r(x)\\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)\\ &\le -2 + 2(r'(x) - r(x)) \qquad \Rightarrow \text{cost}_{\text{zigzag}} \le 3(r'(x) - r(x))\end{aligned}$$

Splay: Zigzag Case



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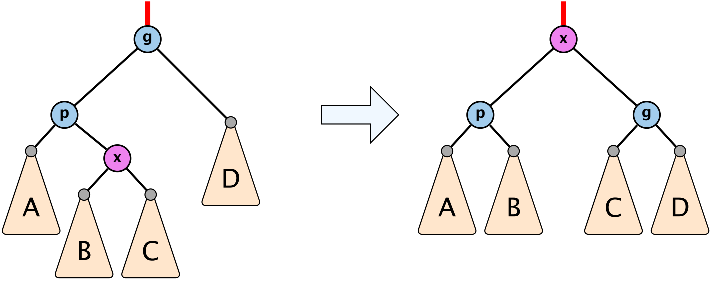
Splay: Zigzag Case



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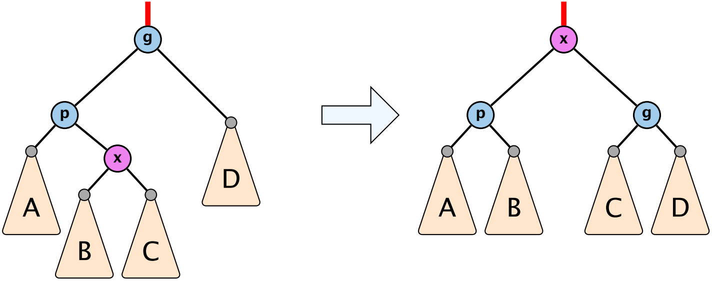
Splay: Zigzag Case



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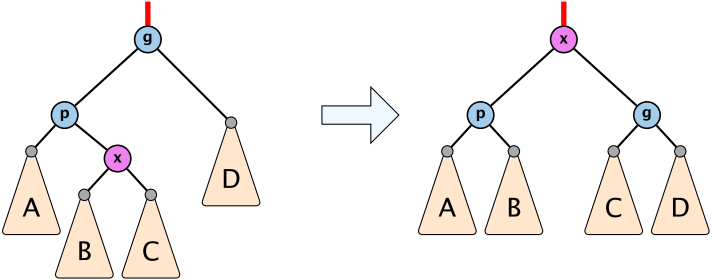
Splay: Zigzag Case



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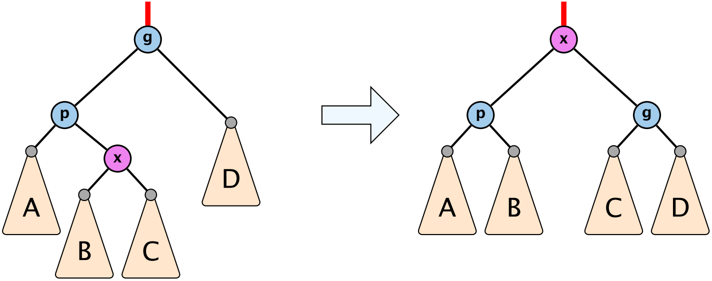

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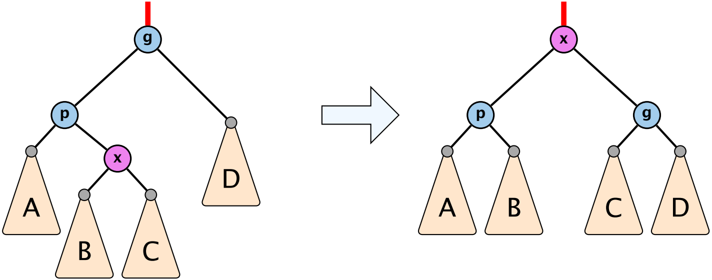

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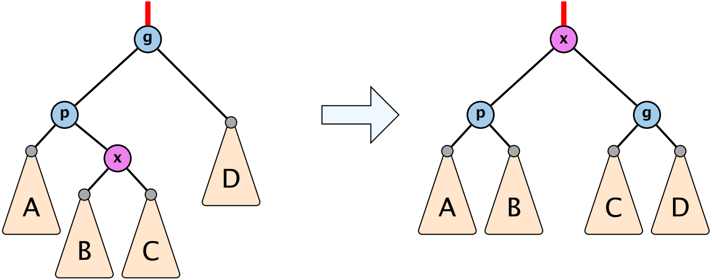

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

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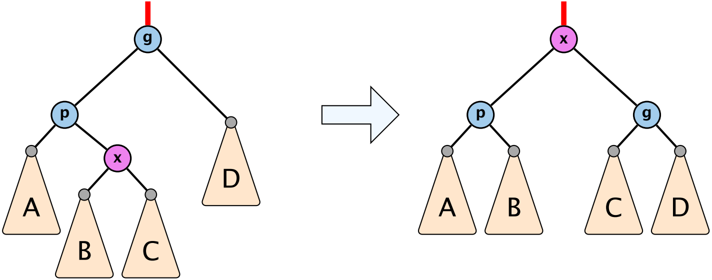

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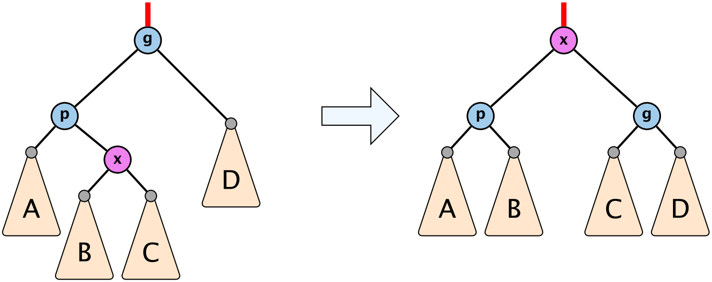

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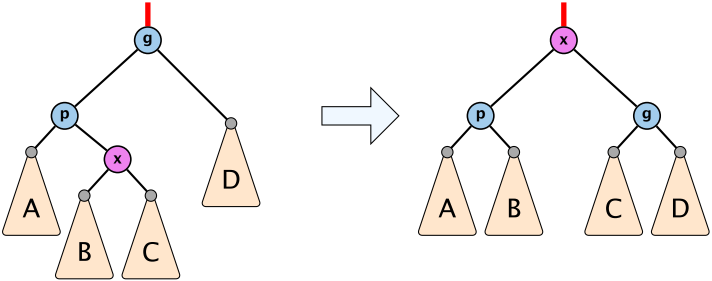

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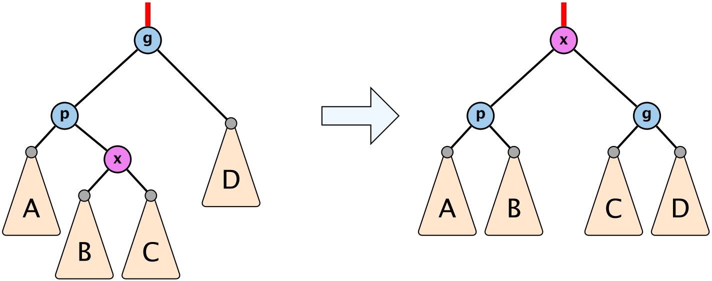

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Amortized cost of the whole splay operation:

$$\begin{aligned}
&\le 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
&= 2 + r(\text{root}) - r_0(x) \\
&\le \mathcal{O}(\log n)
\end{aligned}$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

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7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v.
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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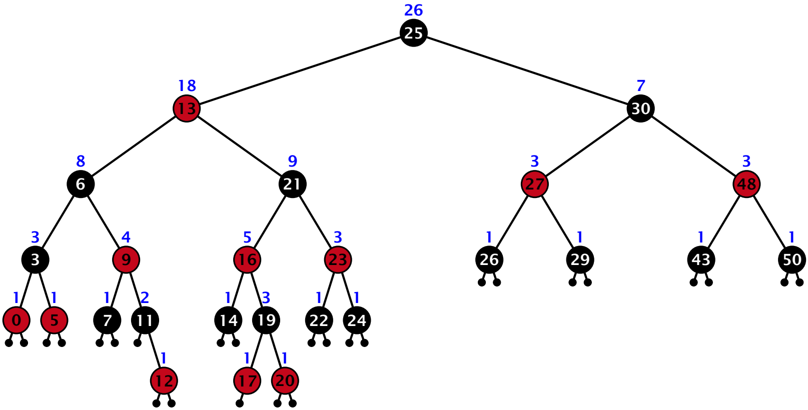
4. How does find-by-rank work?
   Find-by-rank(k) := Select(root, k) with

Algorithm 11 Select(x, i)

```
if x = null then return error
if left[x] ≠ null then r ← left[x].size + 1 else r ← 1
if i = r then return x
if i < r then
    return Select(left[x], i)
else
    return Select(right[x], i − r)
```

Select(x, i)

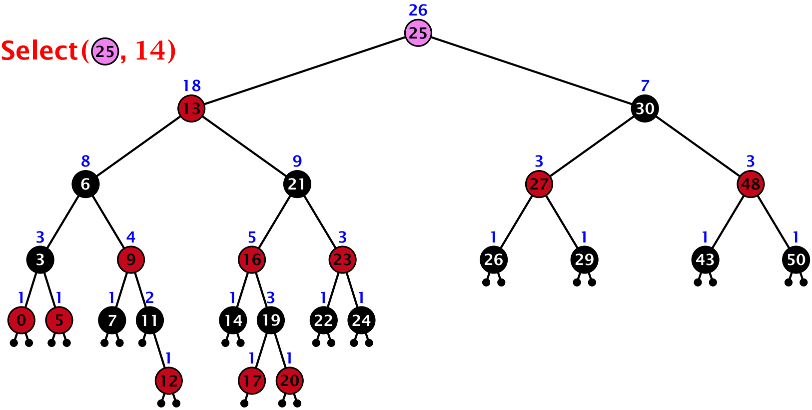


Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)



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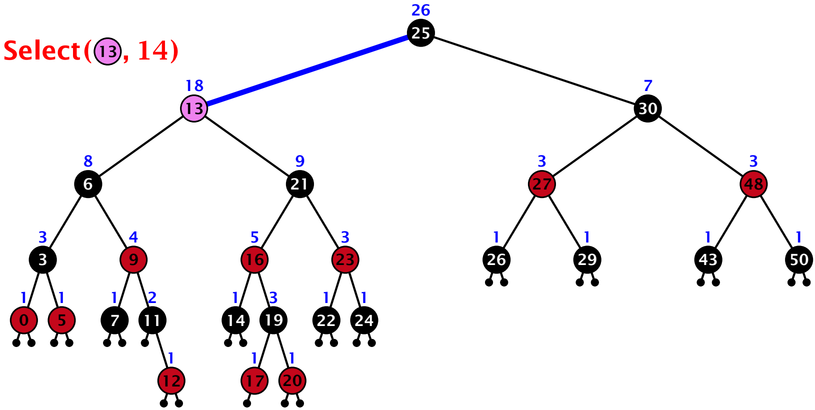

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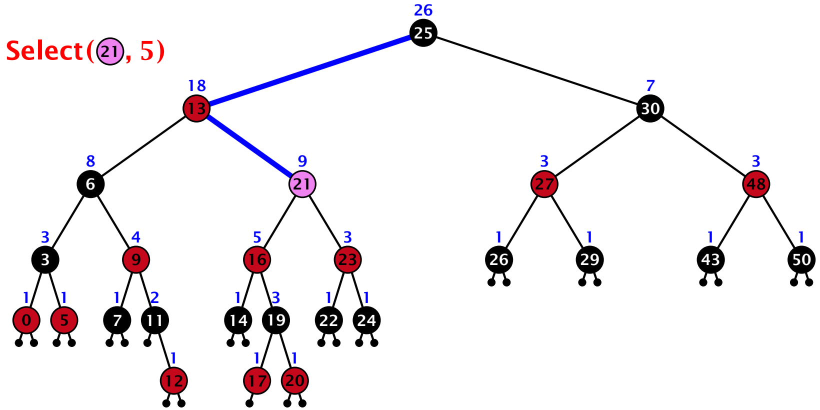

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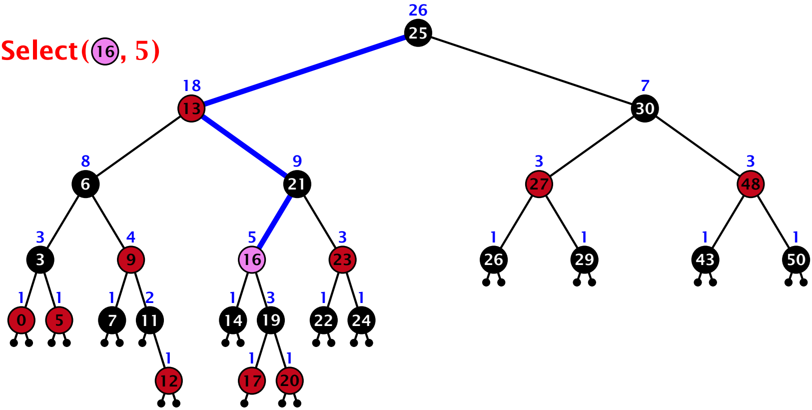

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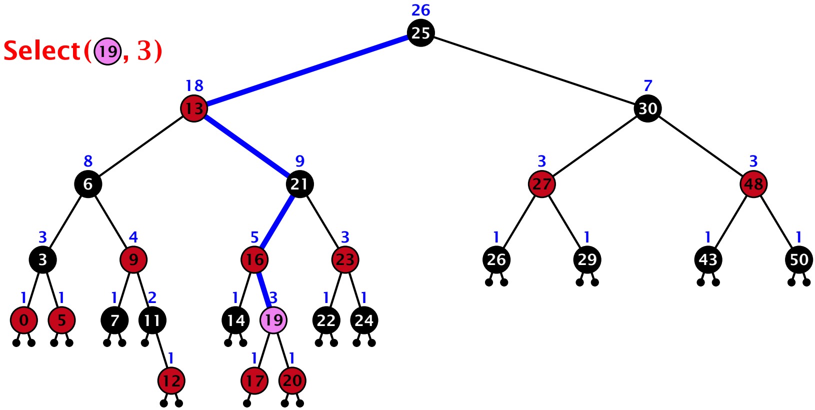

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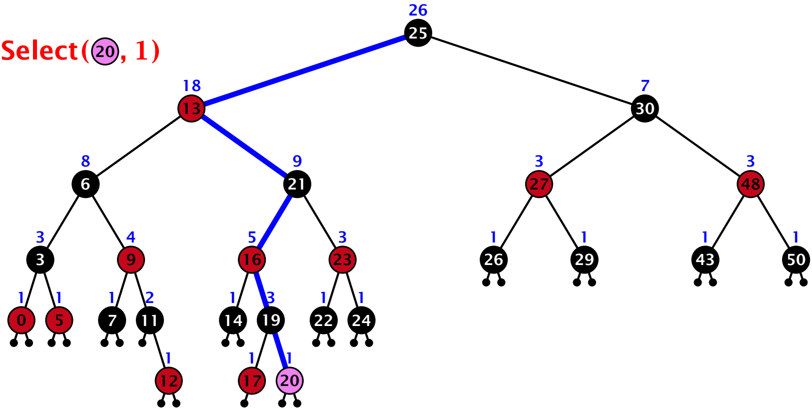

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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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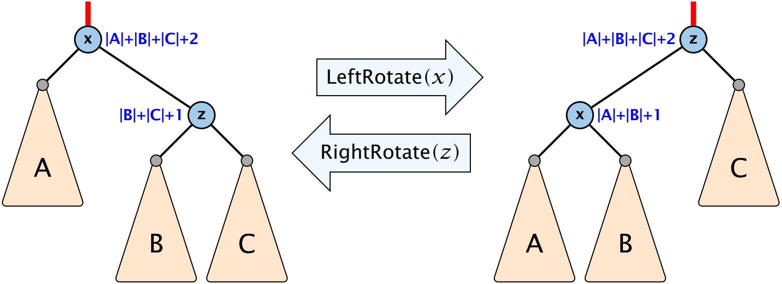
Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 (a, b)-trees

Definition 17

For $b \geq 2a - 1$ an (a, b)-tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex v has at least a and at most b children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
5. there is a special dummy leaf node with key-value ∞

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7.5 (a, b)-trees

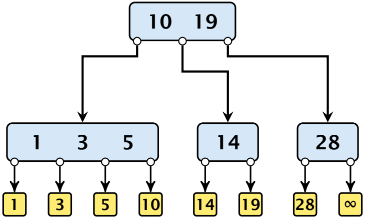
Each internal node v with $d(v)$ children stores $d - 1$ keys $k_1, \dots, k_{d-1}$. The i-th subtree of v fulfills

$$k_{i-1} < \text{ key in } i\text{-th sub-tree } \leq k_i \ ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a, b)-trees

Example 18



7.5 (a, b)-trees

Variants

- The dummy leaf element may not exist; it only makes implementation more convenient.
- Variants in which $b = 2a$ are commonly referred to as B-trees.
- A B-tree usually refers to the variant in which keys and data are stored at internal nodes.
- A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- A B^* tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a B-tree).

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Lemma 19

Let T be an (a,b)-tree for $n > 0$ elements (i.e., $n+1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \le n+1 \le b^h$
2. $\log_b(n+1) \le h \le 1 + \log_a(\frac{n+1}{2})$

Proof.

If $n > 0$ the root has degree at least 2 and all other nodes have degree at least a. This gives that the number of leaf nodes is at least $2a^{h-1}$.

Analogously, the degree of any node is at most b and hence, the number of leaf nodes at most b^h.

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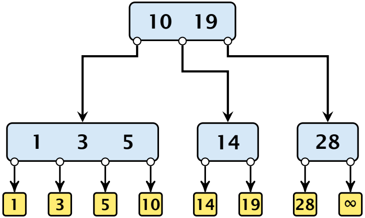
Search



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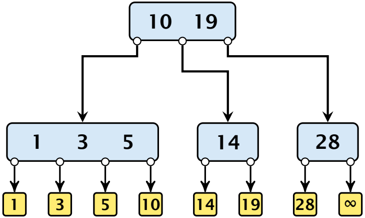
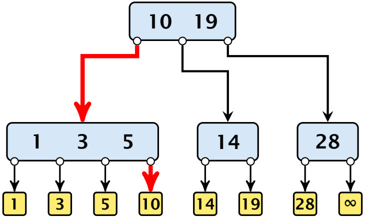
Search(8)



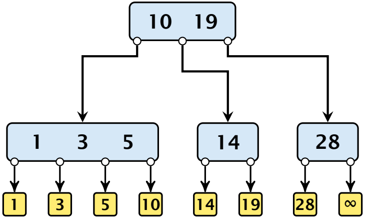
Search

Search(8)



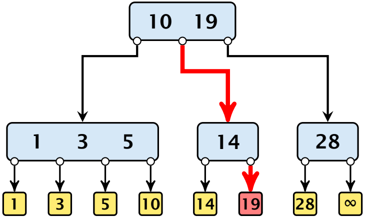
Search

Search(19)

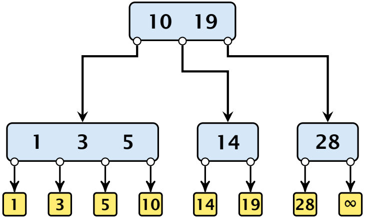


Search

Search(19)



Search



The search is straightforward. It is only important that you need to go all the way to the leaf.

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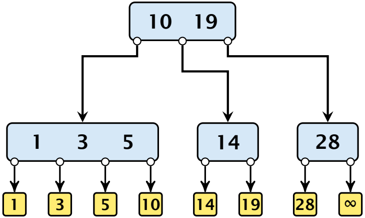

The search is straightforward. It is only important that you need to go all the way to the leaf.

Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

Insert

Insert element x:

- Follow the path as if searching for $\text{key}[x]$.
- If this search ends in leaf ℓ, insert x before this leaf.
- For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.
- If after the insert v contains b nodes, do Rebalance(v).

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Insert

Rebalance(v):

- Let k_i, $i = 1, \dots, b$ denote the keys stored in v.
- Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- Create two nodes v_1, and v_2. v_1 gets all keys $k_1, \dots, k_{j-1}$ and v_2 gets keys $k_{j+1}, \dots, k_b$.
- Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
- They get at most $\lceil \frac{b-1}{2} \rceil$ keys, and have therefore degree at most $\lceil \frac{b-1}{2} \rceil + 1 \leq b$ (since $b \geq 2$).
- The key k_j is promoted to the parent of v. The current pointer to v is altered to point to v_1, and a new pointer (to the right of k_j) in the parent is added to point to v_2.
- Then, re-balance the parent.

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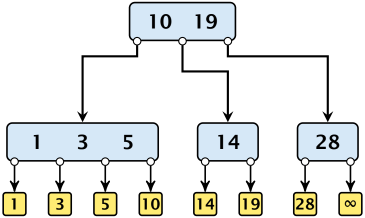
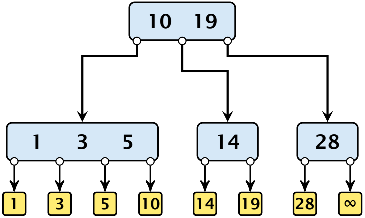


Insert

Insert(8)



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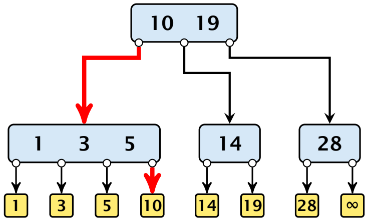

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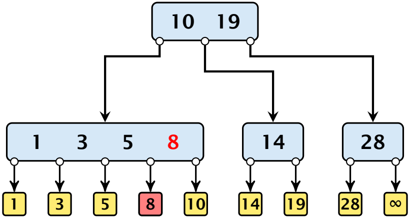

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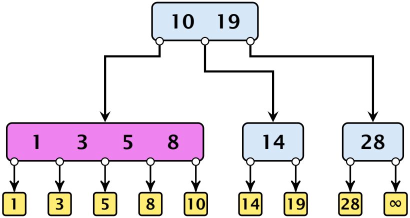

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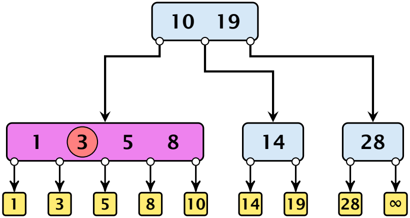

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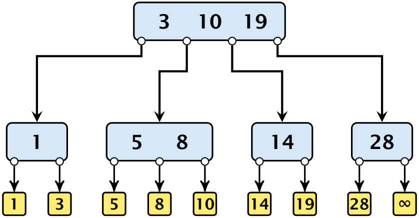

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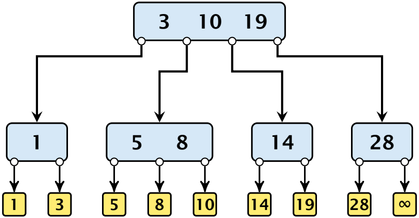

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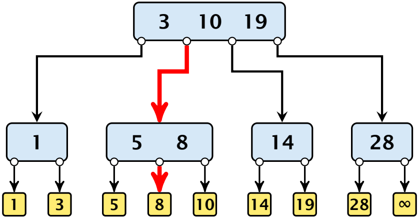

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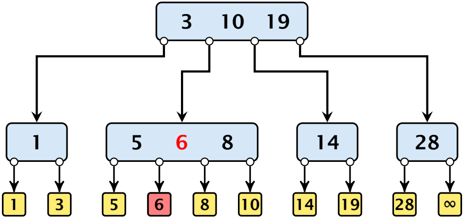

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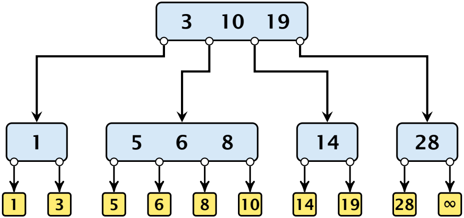

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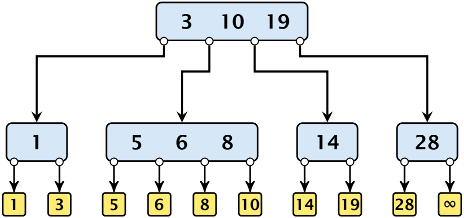

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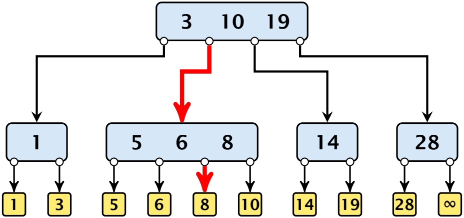

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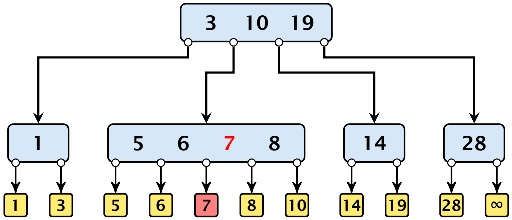

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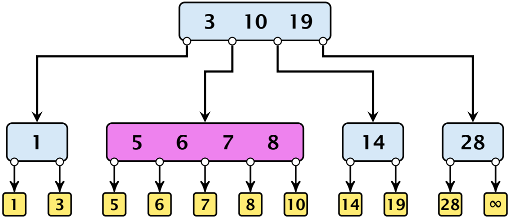

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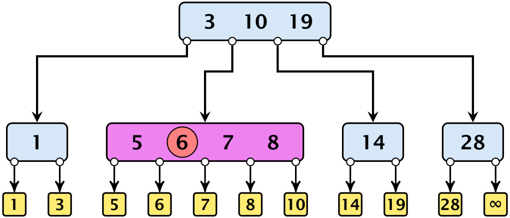

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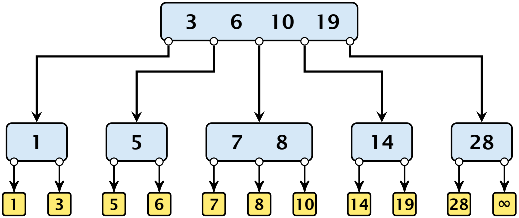

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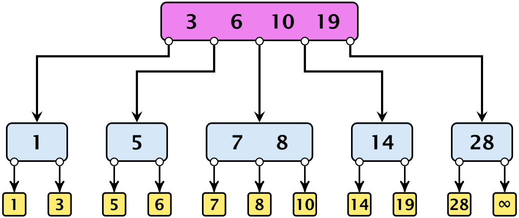

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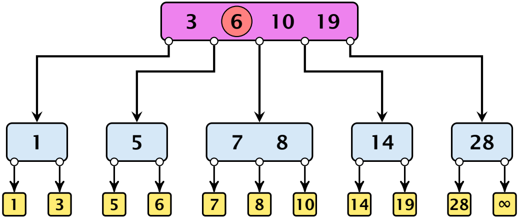

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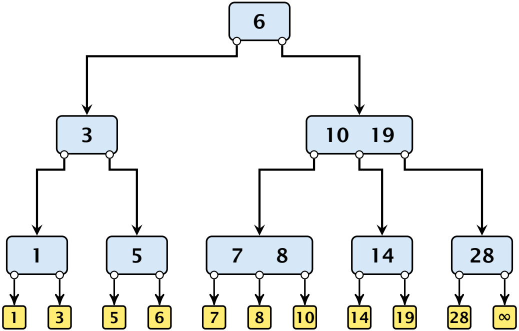

Delete

Delete element x (pointer to leaf vertex):

- Let v denote the parent of x. If $\text{key}[x]$ is contained in v, remove the key from v, and delete the leaf vertex.
- Otherwise delete the key of the predecessor of x from v; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- If now the number of keys in v is below $a-1$ perform Rebalance$'(v)$.

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Rebalance'(v):

- If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- If not: merge v with one of its neighbours.
- The merged node contains at most $(a-2)+(a-1)+1$ keys, and has therefore at most $2a-1 \leq b$ successors.
- Then rebalance the parent.
- During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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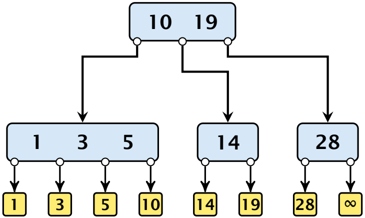
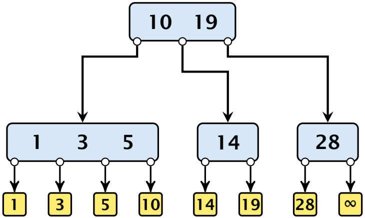


Delete

Delete(10)



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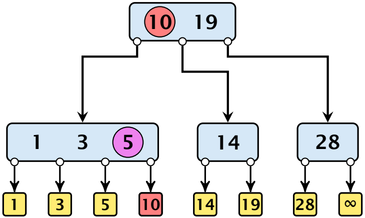
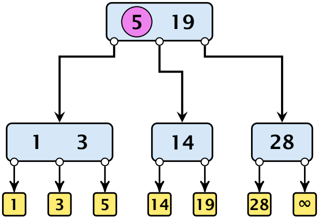


Delete

Delete(10)



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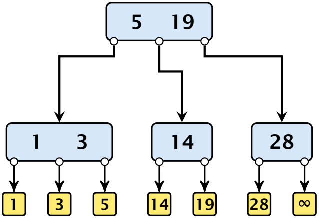
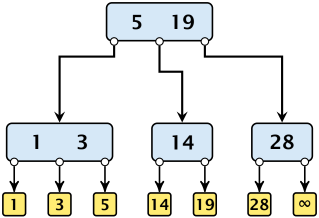


Delete

Delete(14)



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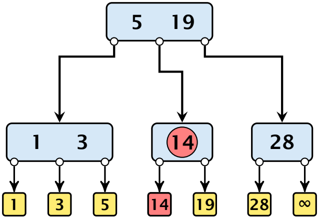

Delete

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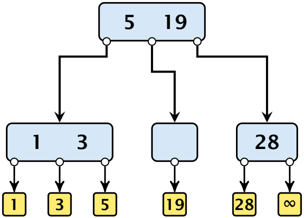
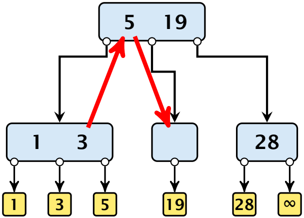


Delete

Delete(14)



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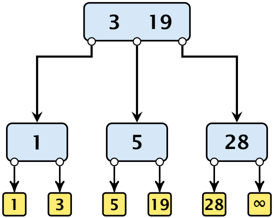

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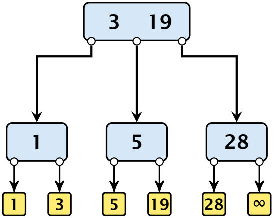
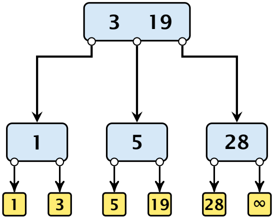


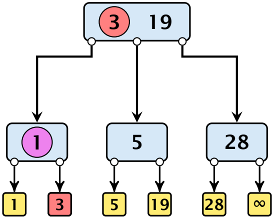
Delete

Delete(3)



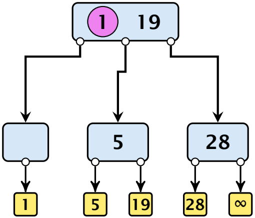
Delete

Delete(3)



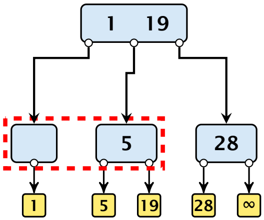
Delete

Delete(3)



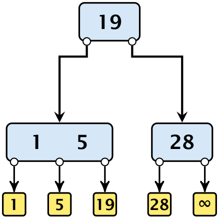
Delete

Delete(3)



Delete

Delete(3)



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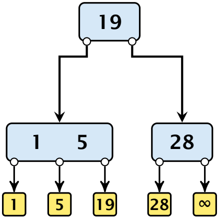

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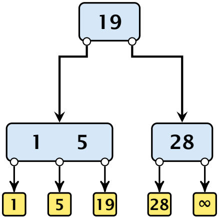

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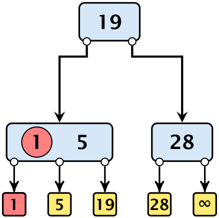

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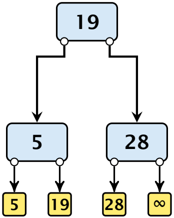

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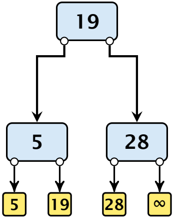

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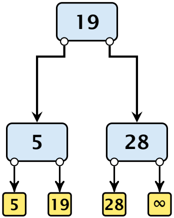

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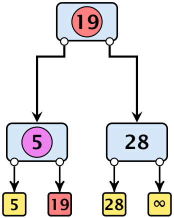

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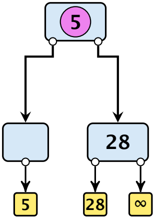

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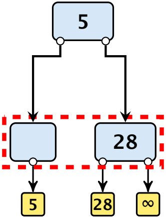

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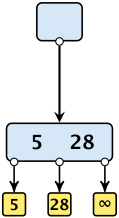

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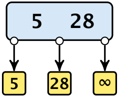
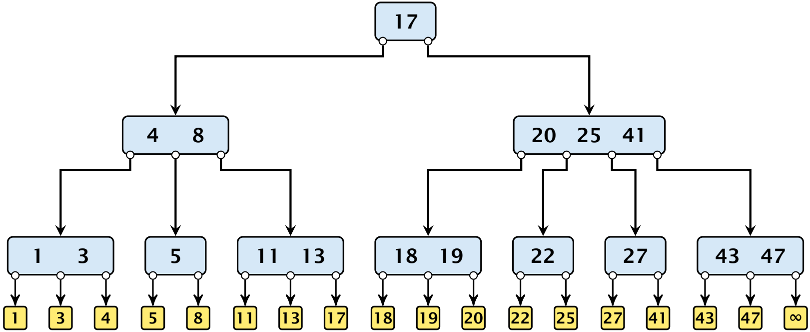


$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:



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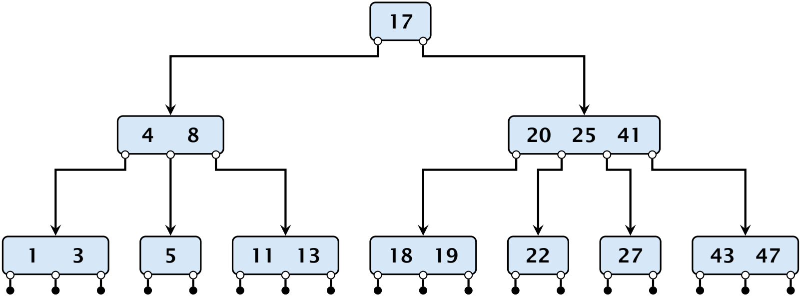

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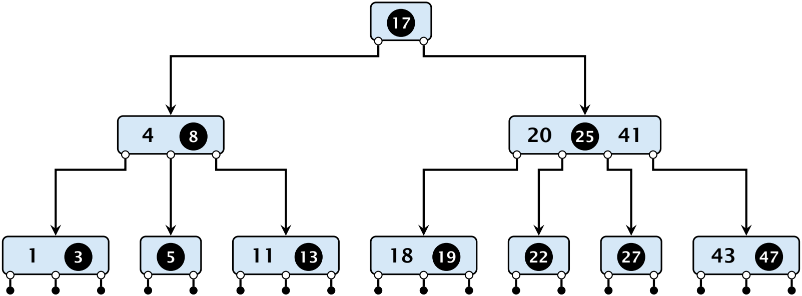

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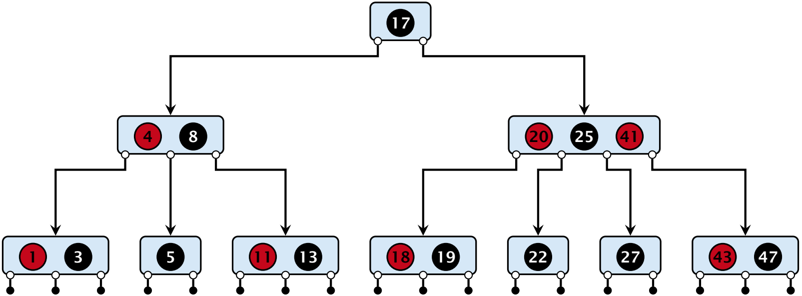

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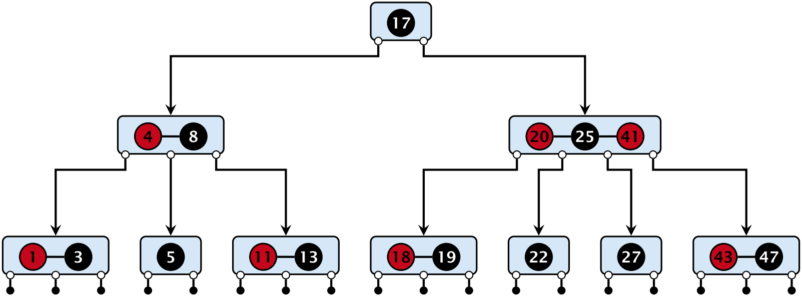

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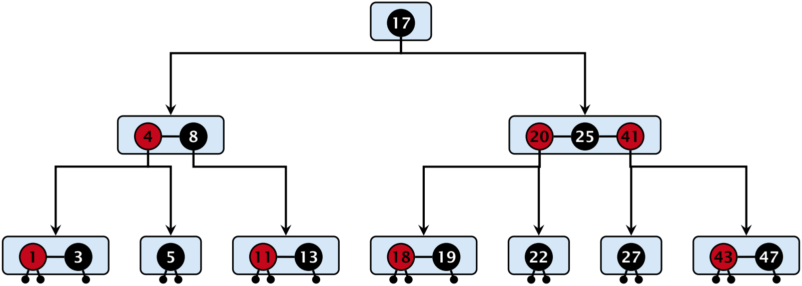

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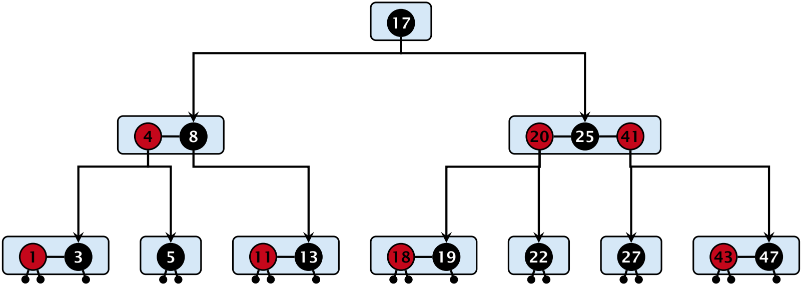

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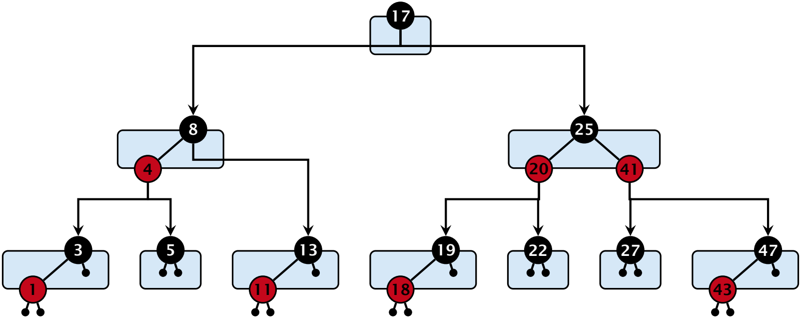

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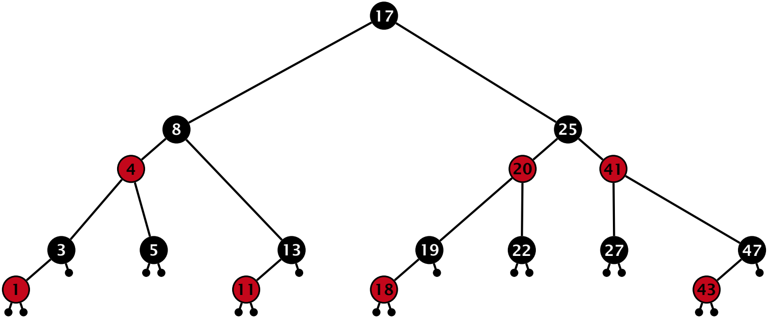

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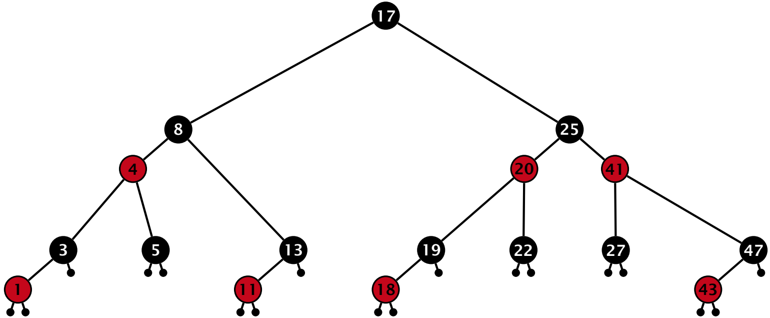

Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same $(2,4)$-tree.

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- time for search $\Theta(n)$
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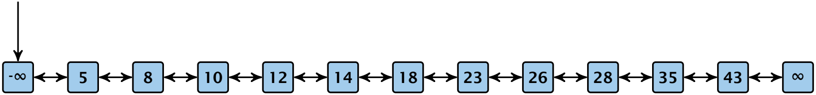

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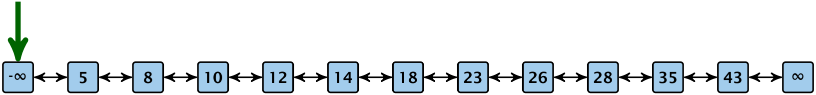

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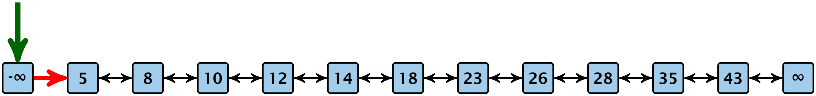

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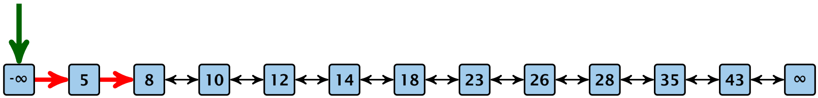

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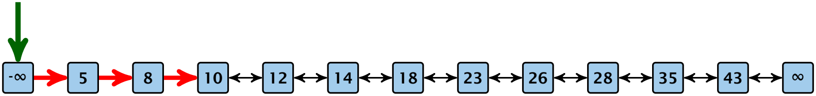

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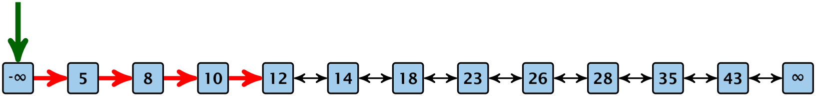

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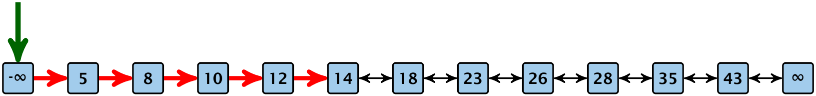

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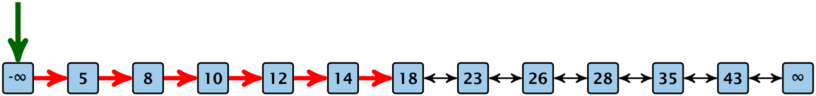

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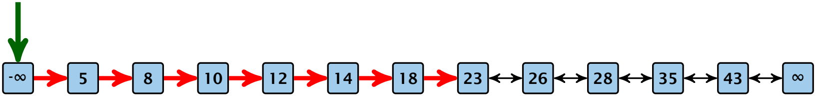

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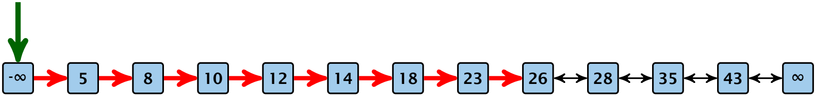

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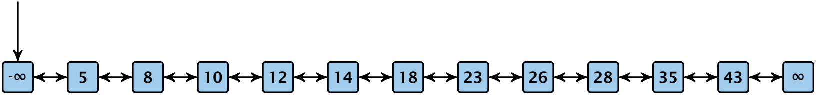

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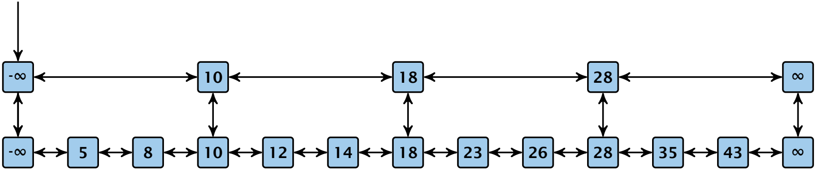

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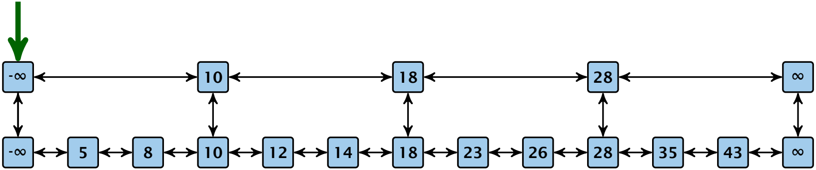

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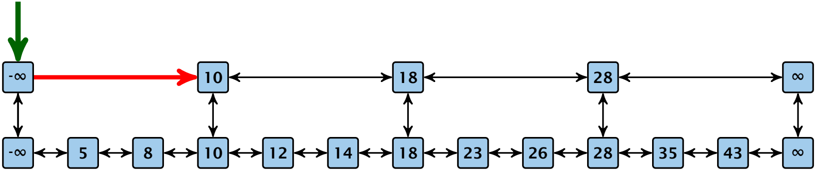

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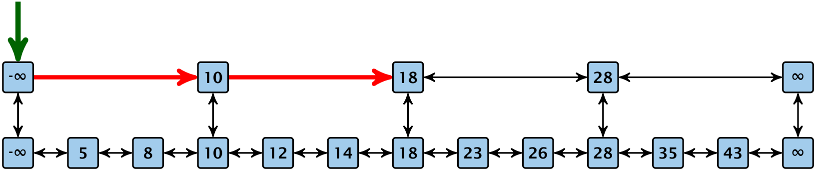

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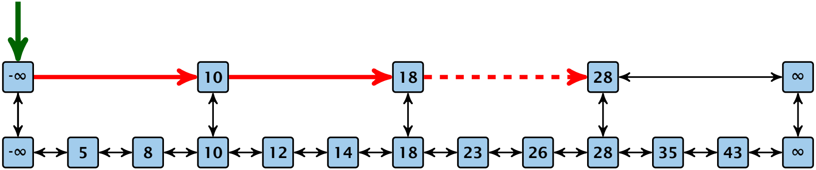

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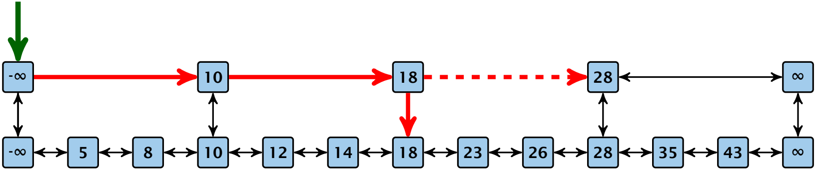

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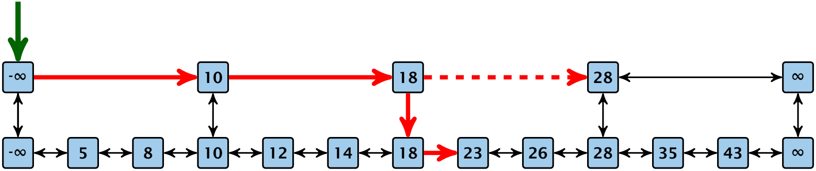

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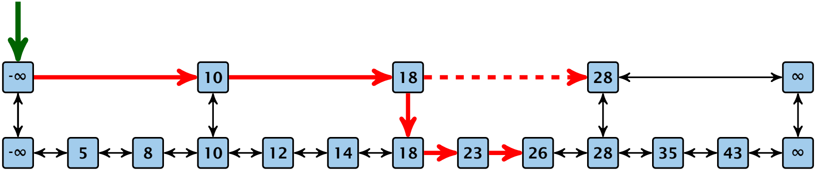

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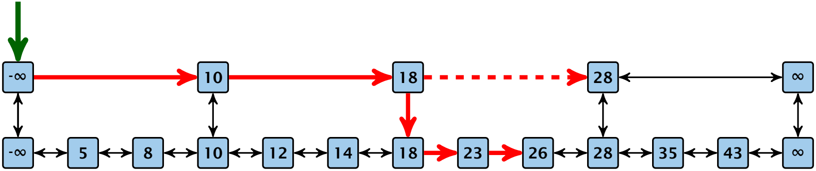

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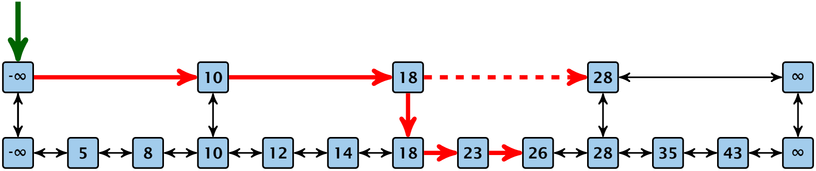

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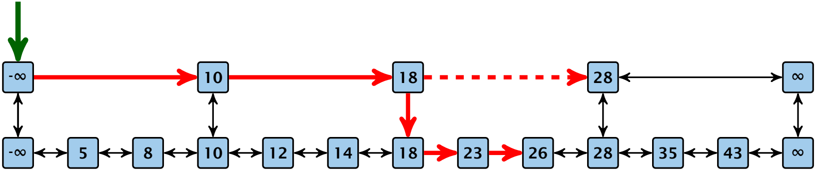

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k+1)$ steps.

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Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.
Choose $r = n^{\frac{1}{k+1}}$. Then

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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How to do insert and delete?

- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

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Insert:

- A search operation gives you the insert position for element x in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- Insert x into lists $L_0, \dots, L_{t-1}$.

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- You get all predecessors via backward pointers.
- Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

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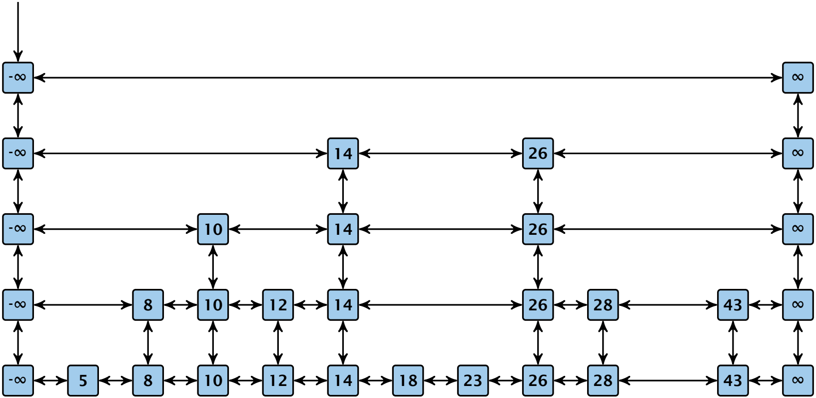
Insert (35):



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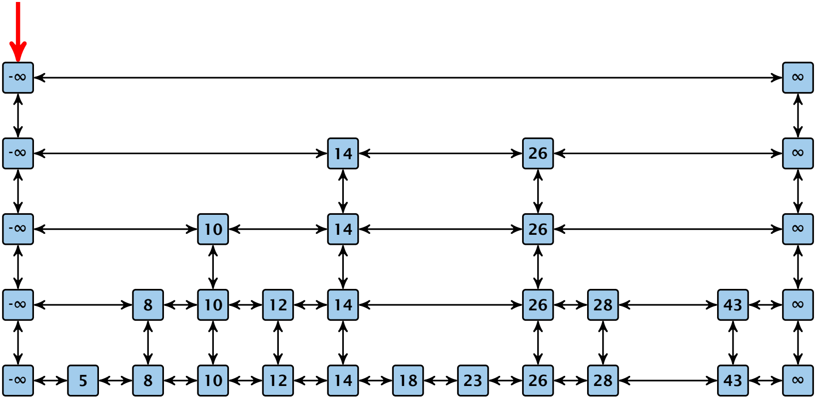

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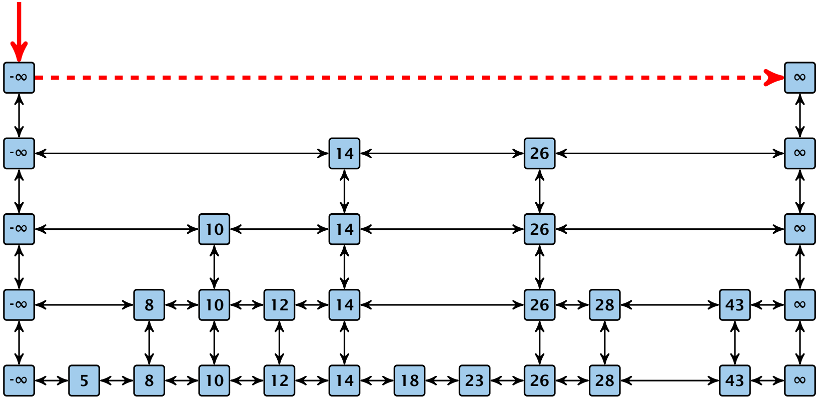

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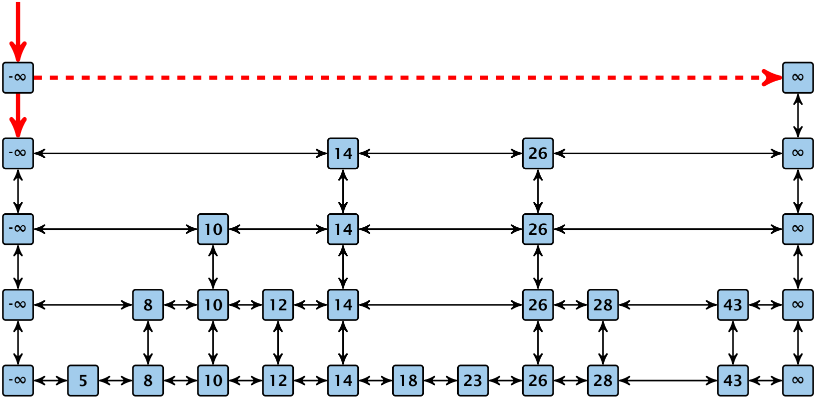

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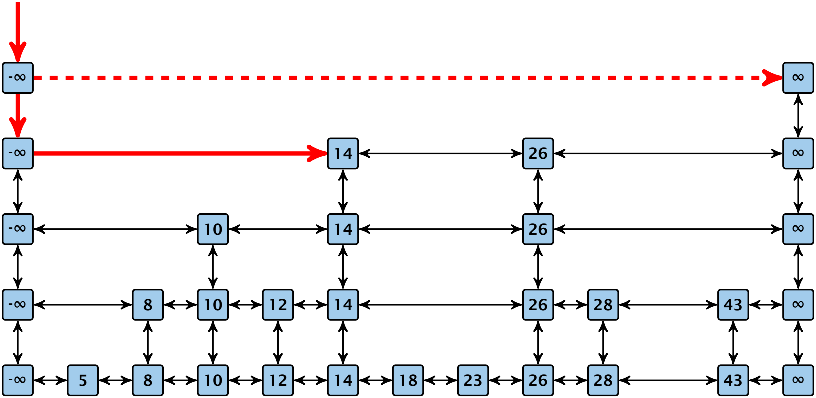

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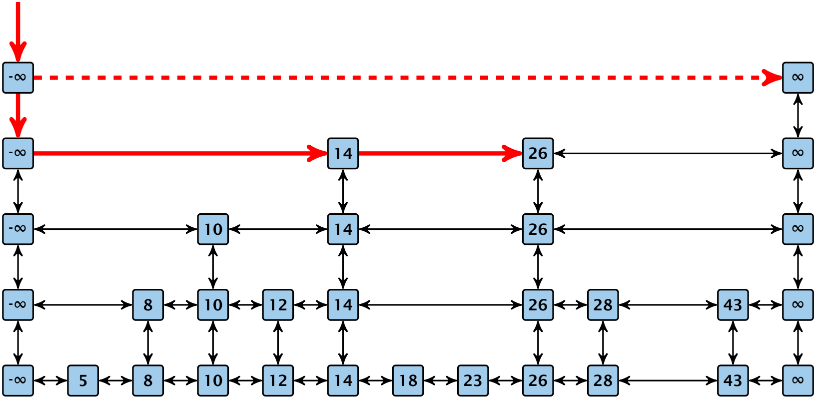

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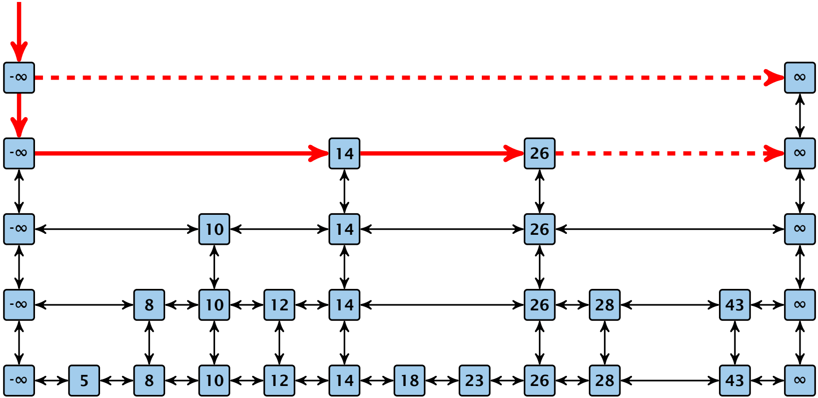

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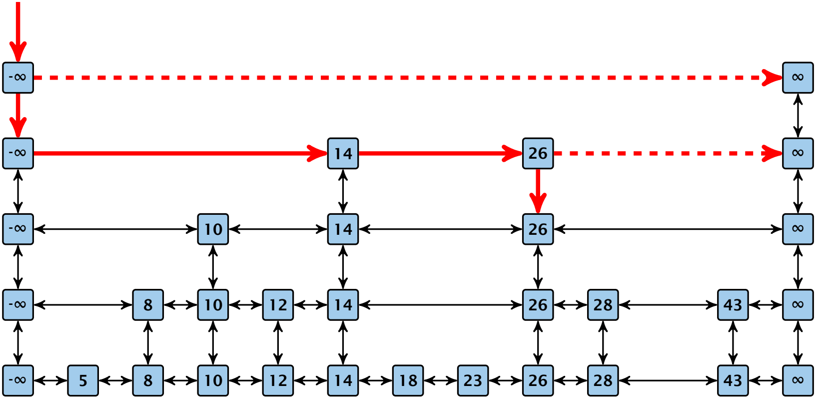

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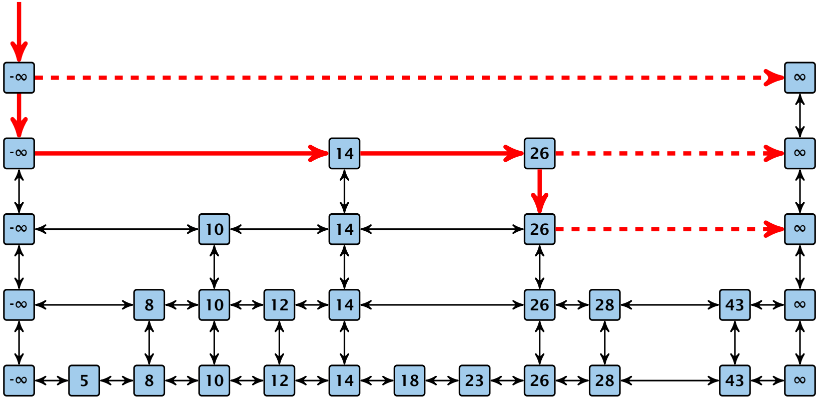

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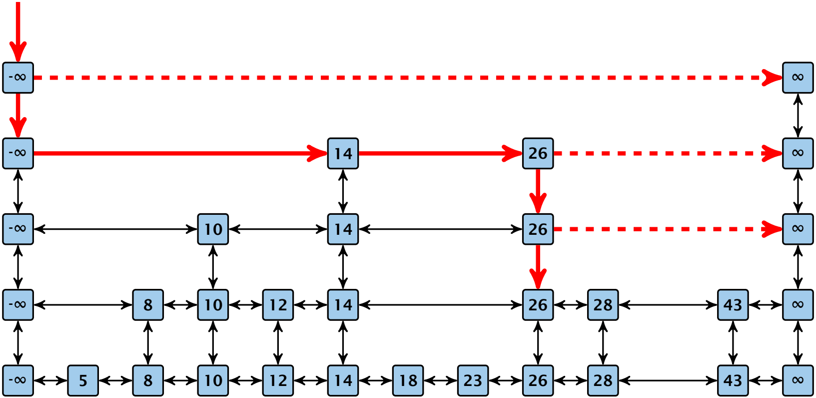

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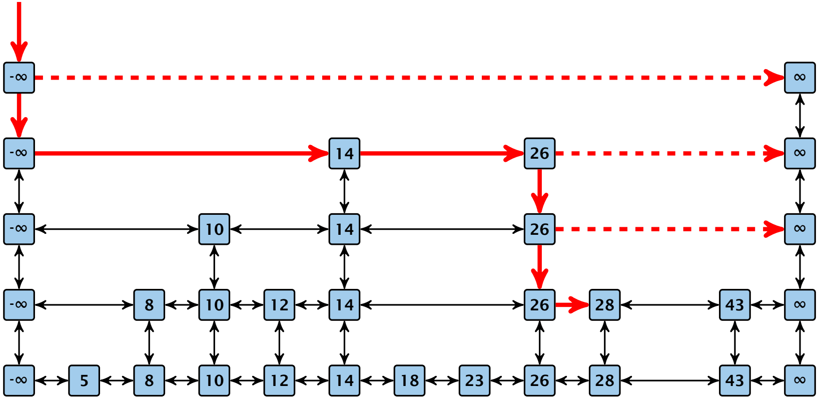

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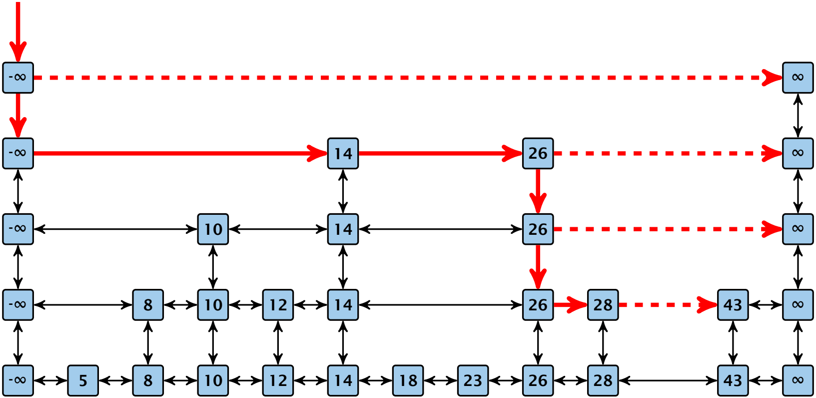

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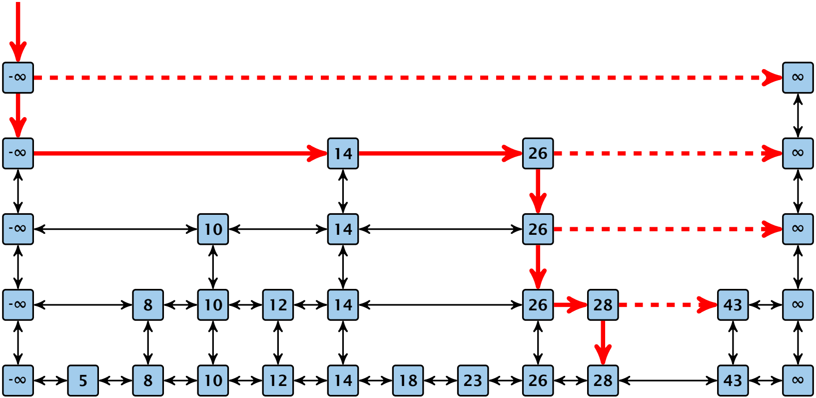

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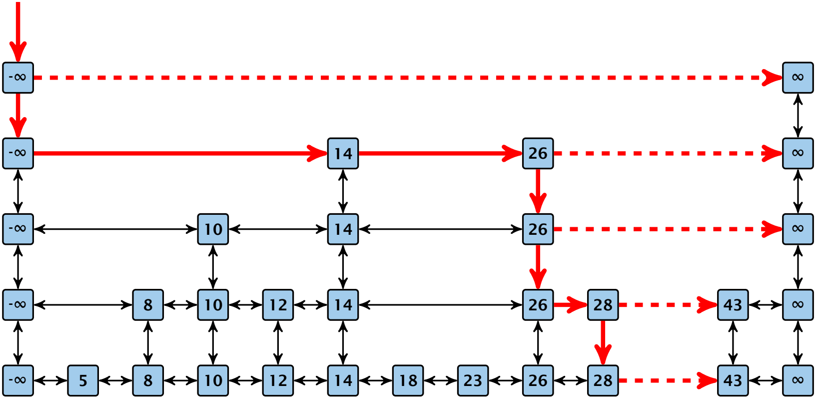

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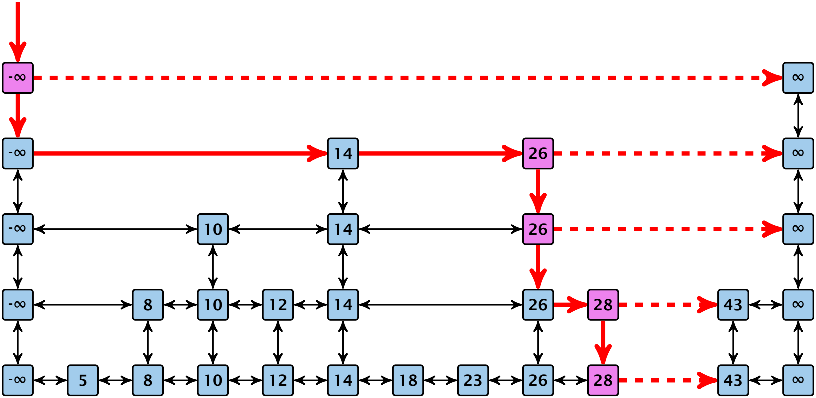

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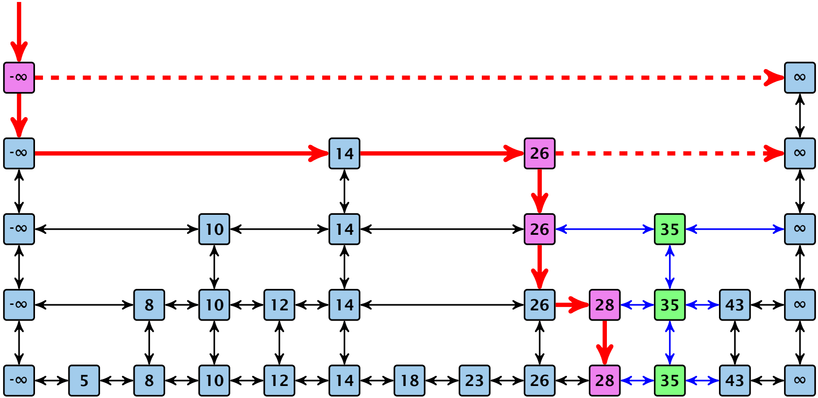

High Probability

Definition 20 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

7.6 Skip Lists

Lemma 21

A search (and, hence, also insert and delete) in a skip list with n *elements takes time* $\mathcal{O}(\log n)$ *with high probability (w. h. p.).*

7.6 Skip Lists

Backward analysis:

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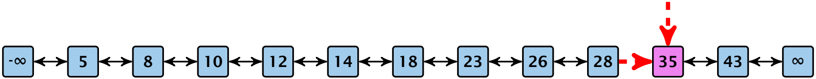

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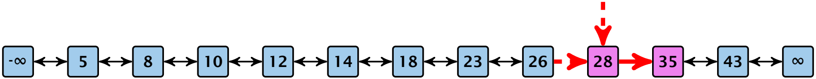

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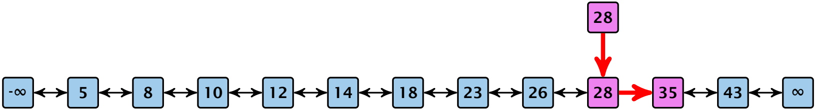

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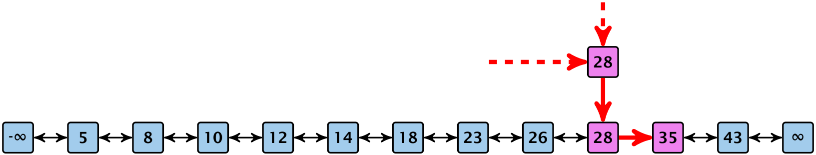

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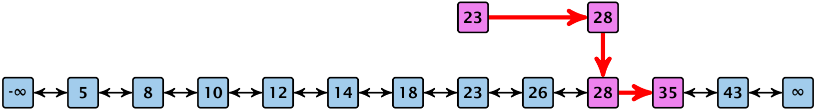

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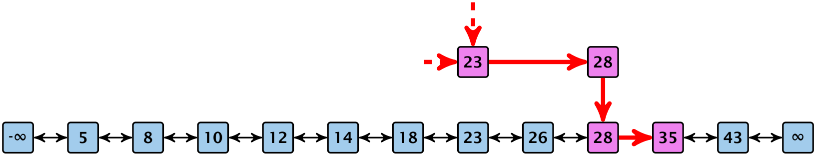

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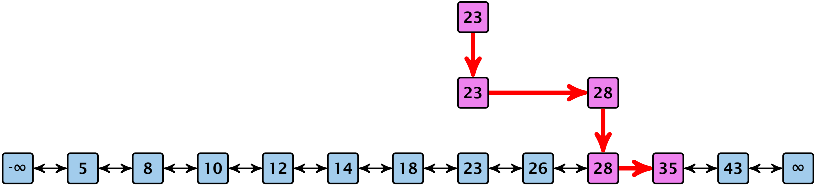

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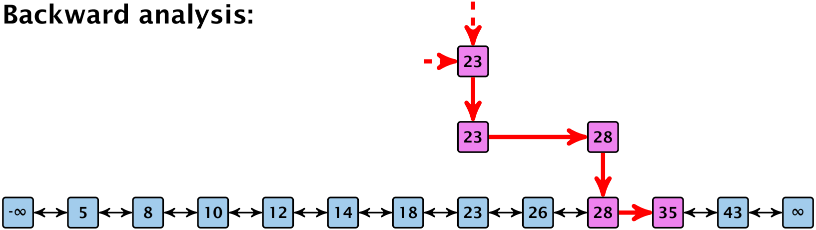

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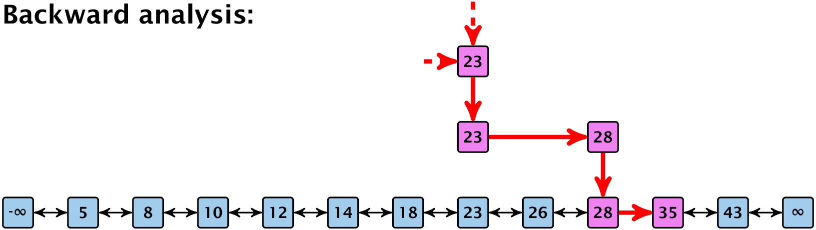

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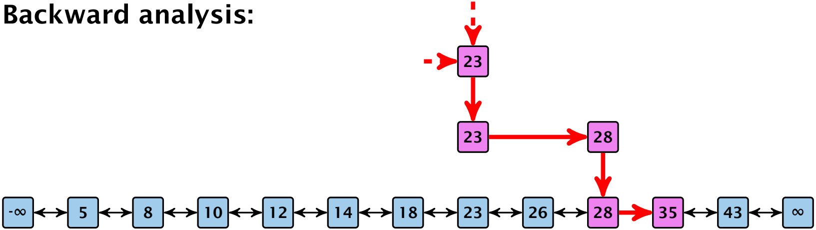

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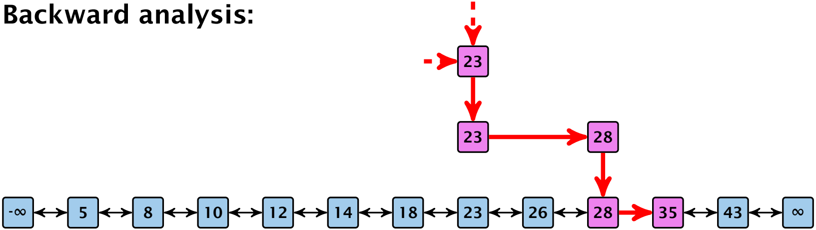

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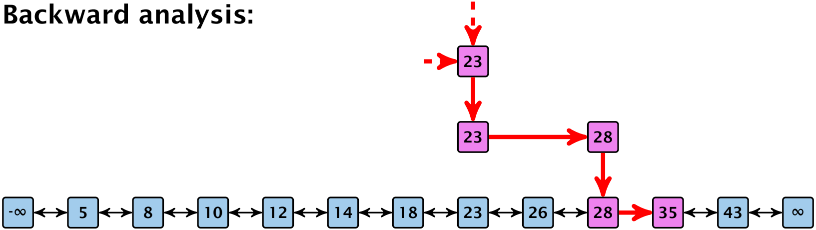

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From this it follows that w.h.p. there are no long paths.

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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This means, the search requires at most z steps, w. h. p.

7.7 Hashing

Dictionary:

- $\boldsymbol{S.\,\mathrm{insert}(x)}$: Insert an element x.
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- $\boldsymbol{S.\,\mathrm{search}(k)}$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

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Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- Array $T[0, \ldots, n-1]$ hash-table.
- Hash function $h : U \to [0, \ldots, n-1]$.

The hash-function h should fulfill:

- Fast to evaluate.
- Small storage requirement.
- Good distribution of elements over the whole table.

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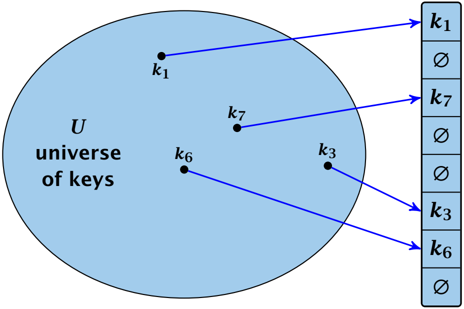
Direct Addressing

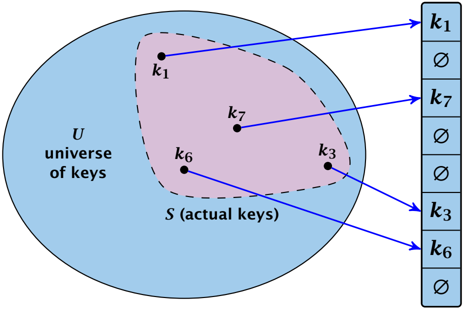
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n.

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

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Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 22

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \ .$$

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Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

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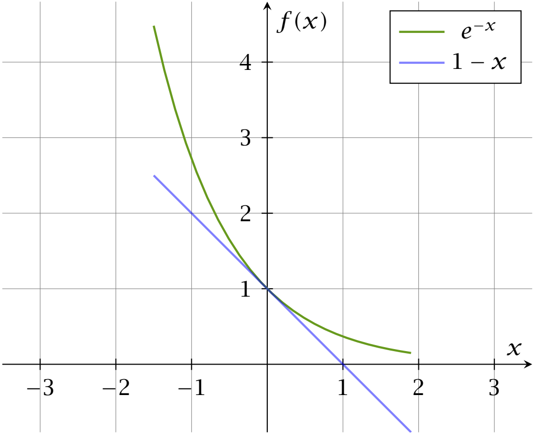

The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

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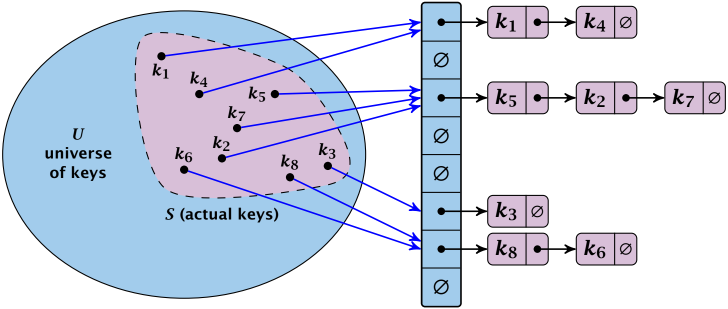
Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\text{key}[x]$.
- Insert: insert at the front of the list.



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Let A denote a strategy for resolving collisions. We use the following notation:

- A^+ denotes the average time for a **successful** search when using A;
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$$A^- = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

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The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1 + \sum_{j=i+1}^{m} X_{ij}\right)\right]$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

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$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\underbrace{\left(1+\sum_{j=i+1}^{m}X_{ij}\right)}_{\text{cost for key } k_i}\right]$$

Hashing with Chaining

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Hashing with Chaining

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right]=\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right)$$

Hashing with Chaining

$$
\begin{aligned}
\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right] &= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right)\\
&= \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\frac{1}{n}\right)
\end{aligned}
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$$=1+\frac{1}{mn}\sum_{i=1}^{m}(m-i)$$

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\end{aligned}
$$

Hence, the expected cost for a successful search is $A^{+} \le 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k,j)$ that determines the table-position to be examined in the j-th step. The values $h(k,0)$,. . . ,$h(k,n-1)$ must form a permutation of $0, \ldots, n-1$.

Search(k): Try position $h(k,0)$; if it is empty your search fails; otw. continue with $h(k,1)$, $h(k,2)$, ...

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k,n-1)$, and this slot is non-empty then your table is full.

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Open Addressing

Choices for $h(k, j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).
- Quadratic probing:
  $h(k,i) = h(k) + c_1 i + c_2 i^2 \mod n$.
- Double hashing:
  $h(k,i) = h_1(k) + i h_2(k) \mod n$.

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.
- Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 23

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

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- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

Lemma 24

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

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Double Hashing

- Any probe into the hash-table usually creates a cache-miss.

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Let A be the method of double hashing for resolving collisions:

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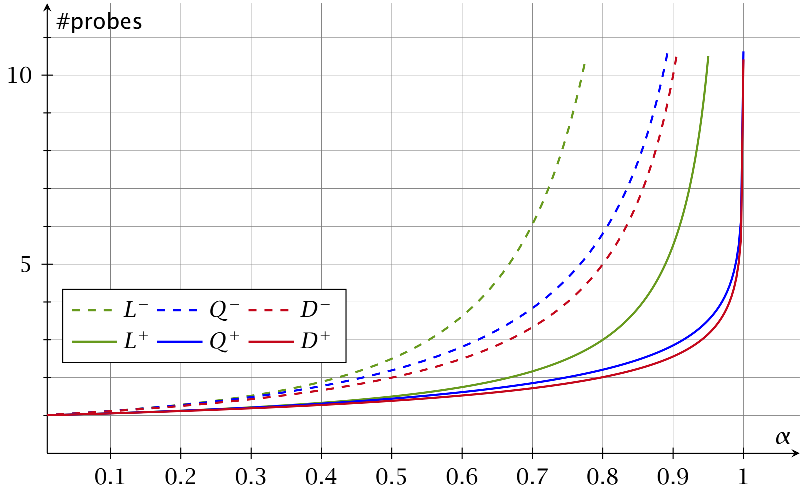
Open Addressing

Some values:

α	*Linear Probing*		*Quadratic Probing*		*Double Hashing*	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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$\mathrm{E}[X]$

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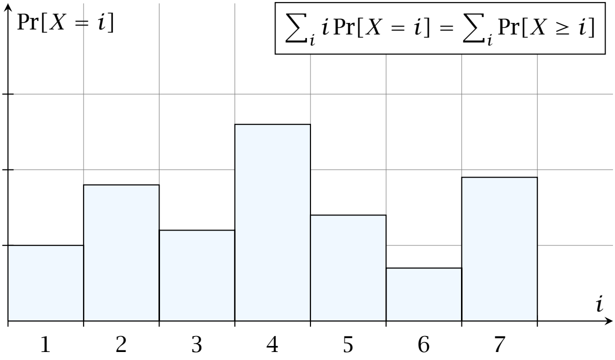
Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^{i} = \frac{1}{1-\alpha} \ .$$

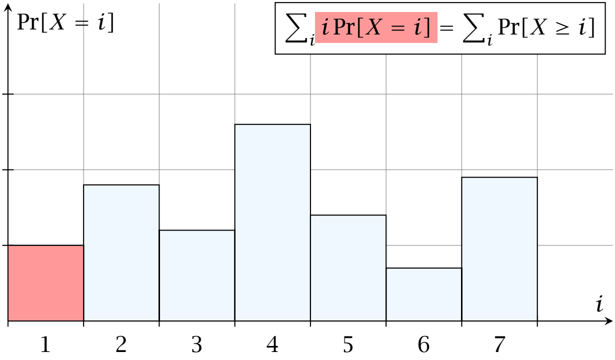
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



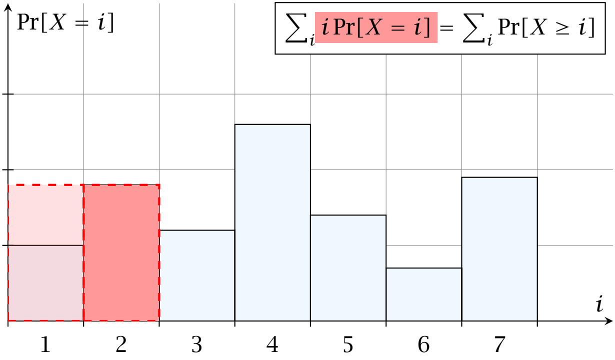
Analysis of Idealized Open Address Hashing

$i = 1$



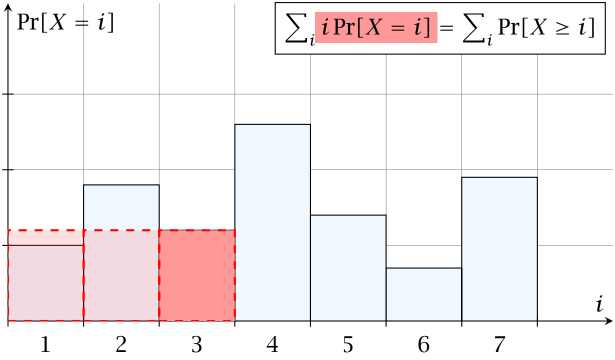
Analysis of Idealized Open Address Hashing

$i = 2$



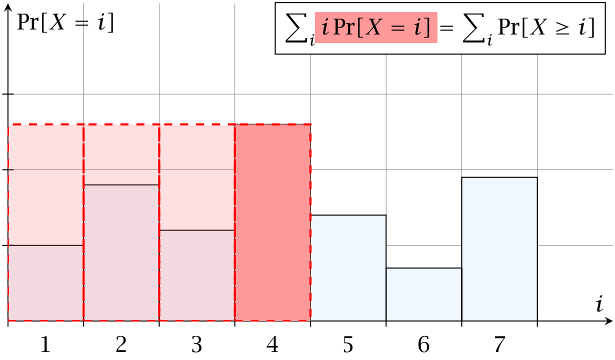
Analysis of Idealized Open Address Hashing

$i = 3$



Analysis of Idealized Open Address Hashing

$i = 4$



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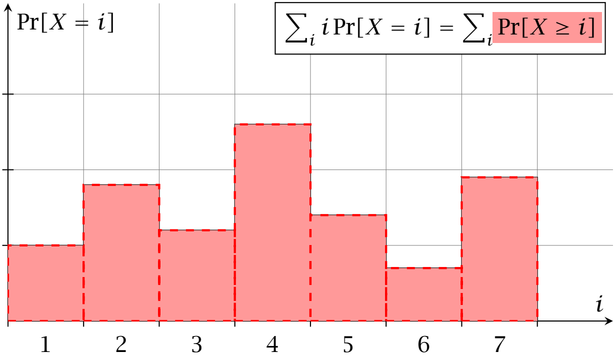

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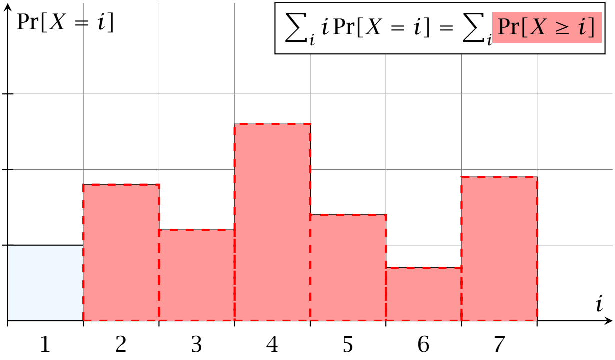
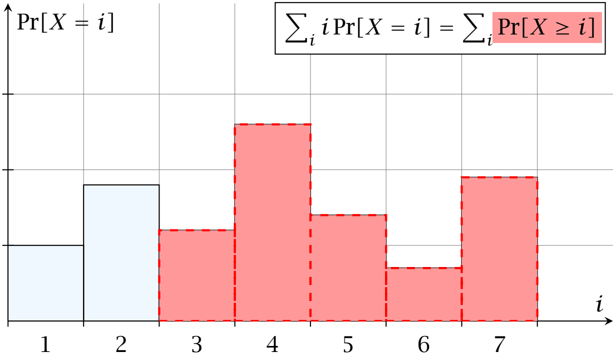


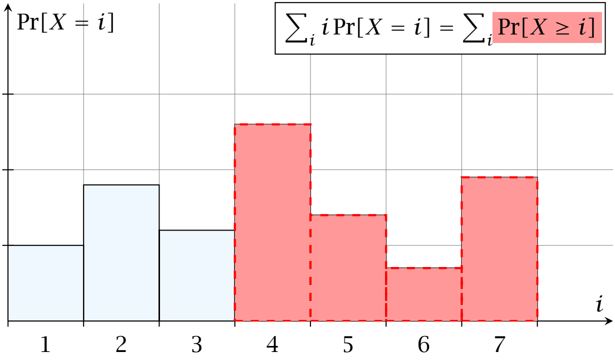
Analysis of Idealized Open Address Hashing

$i = 3$



Analysis of Idealized Open Address Hashing

$i = 4$



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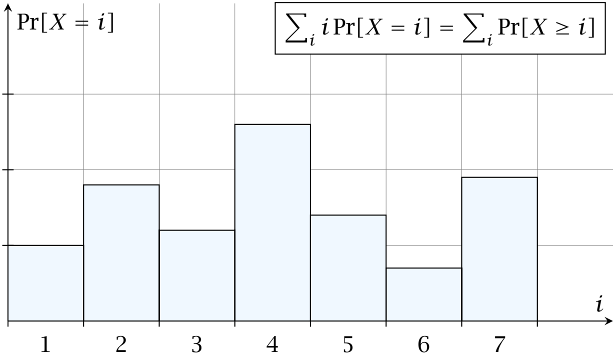

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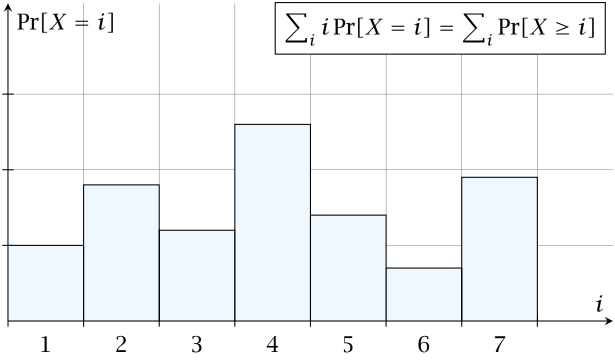

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

Analysis of Idealized Open Address Hashing

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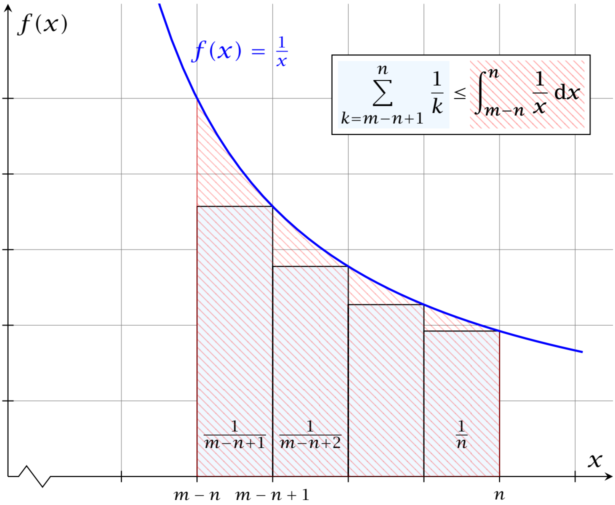

Deletions in Hashtables

How do we delete in a hash-table?

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- For open addressing this is difficult.

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Deletions in Hashtables

- Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- One can delete an element by replacing it with a deleted-marker.
  - During an insertion if a deleted-marker is encountered an element can be inserted there.
  - During a search a deleted-marker must not be used to terminate the probe sequence.
- The table could fill up with deleted-markers leading to bad performance.
- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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Deletions for Linear Probing

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Deletions for Linear Probing

Algorithm 12 delete(p)

$T[p] \leftarrow \text{null}$
$p \leftarrow \text{succ}(p)$
while $T[p] \neq \text{null}$ **do**
    $y \leftarrow T[p]$
    $T[p] \leftarrow \text{null}$
    $p \leftarrow \text{succ}(p)$
    $\text{insert}(y)$

p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

Deletions for Linear Probing

Algorithm 12 delete(p)

```
T[p] ← null
p ← succ(p)
while T[p] ≠ null do
    y ← T[p]
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```

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Pointers into the hash-table become invalid.

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

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Universal Hashing

Definition 26
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called k-independent if for any choice of $\ell \le k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \dots, t_\ell$:

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Universal Hashing

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k)-independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \dots, t_\ell$:

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Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 30

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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  If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

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  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

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$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$
$$b \equiv t_y - ay \pmod{p}$$

Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a,b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\bmod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p-1$ possible values for choosing t_y.

From the range $0, \dots, p-1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

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This happens with probability at most $\frac{1}{n}$.

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 31

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i \Big) \bmod q$$

The polynomial is defined by $d+1$ distinct points.

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Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i$ for all $i \in \{1, \ldots, \ell\}\}$
Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
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Now, we choose $d-\ell+1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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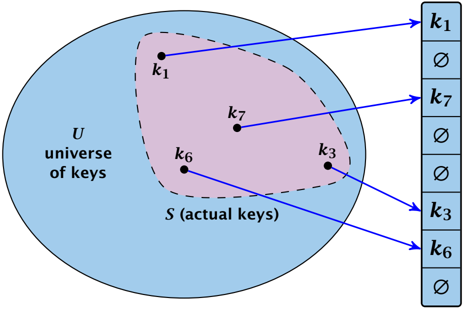
This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$\mathrm{E}[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

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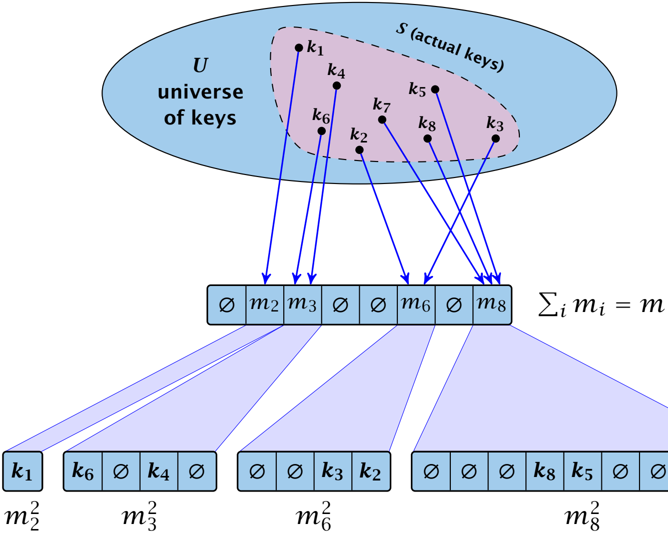

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$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- Two hash-tables $T_1[0, \dots, n-1]$ and $T_2[0, \dots, n-1]$, with hash-functions h_1, and h_2.
- An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- A search clearly takes constant time if the above constraint is met.

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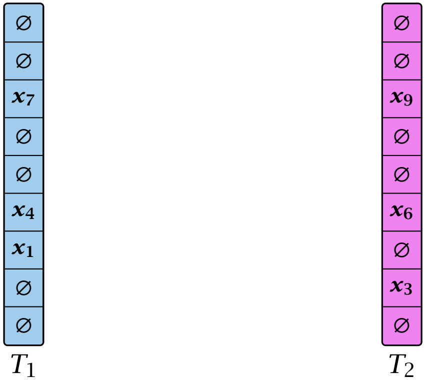

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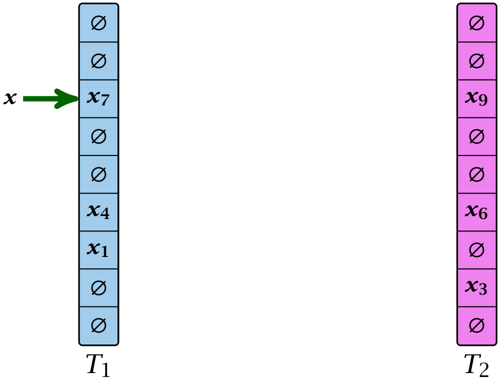

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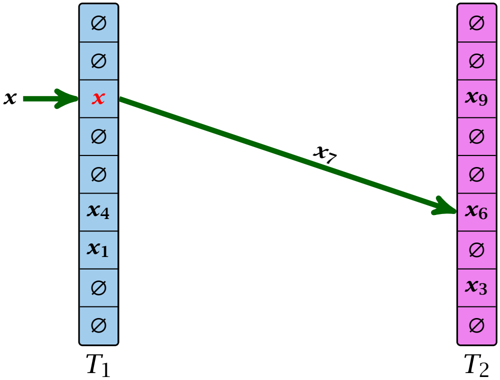

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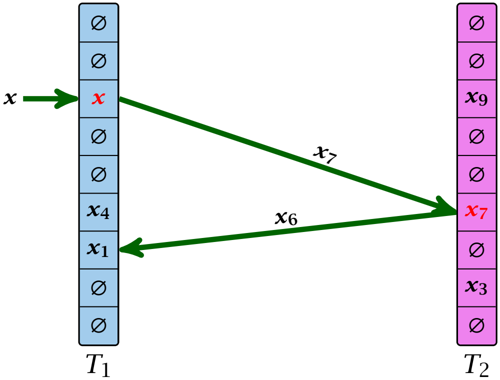

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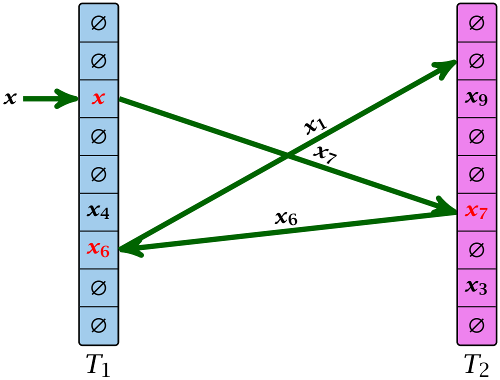

Cuckoo Hashing

Algorithm 13 Cuckoo-Insert(x)

if $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$ **then return**
$\text{steps} \leftarrow 1$
while $\text{steps} \le \text{maxsteps}$ **do**
    exchange x and $T_1[h_1(x)]$
    if $x = \text{null}$ **then return**
    exchange x and $T_2[h_2(x)]$
    if $x = \text{null}$ **then return**
    $\text{steps} \leftarrow \text{steps} + 1$
rehash() // change hash-functions; rehash everything
Cuckoo-Insert(x)

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.
- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = \text{null}$.

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Cuckoo Hashing

What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after $\mathrm{maxsteps}$ steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing

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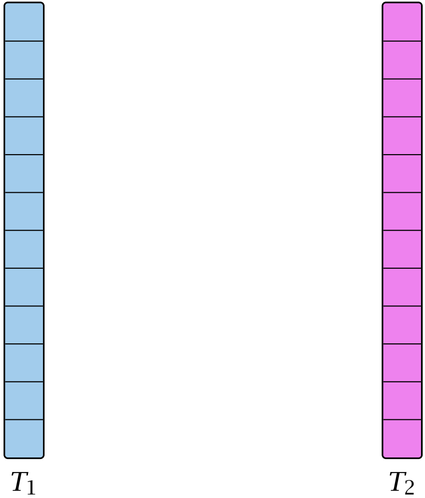

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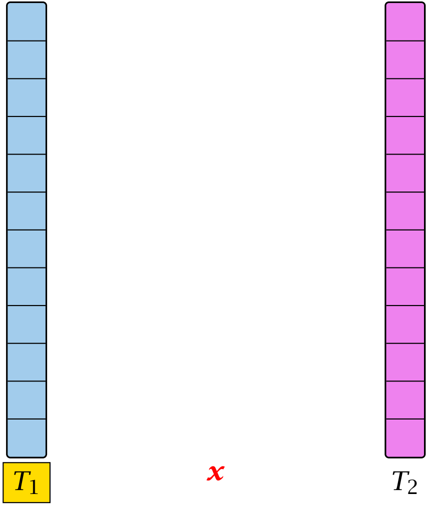

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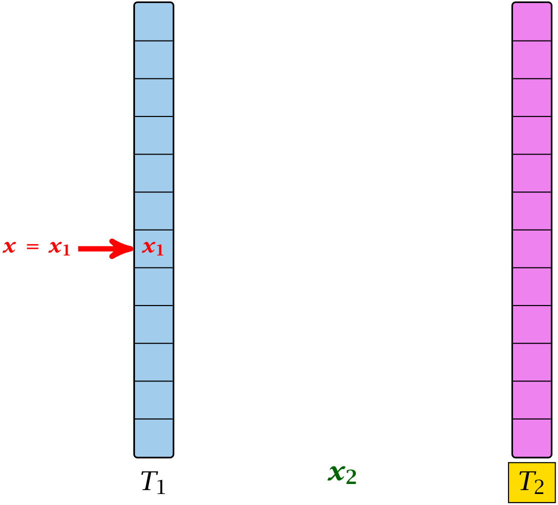

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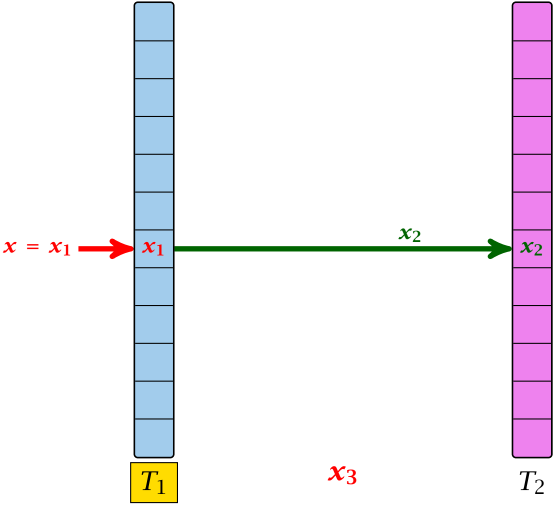

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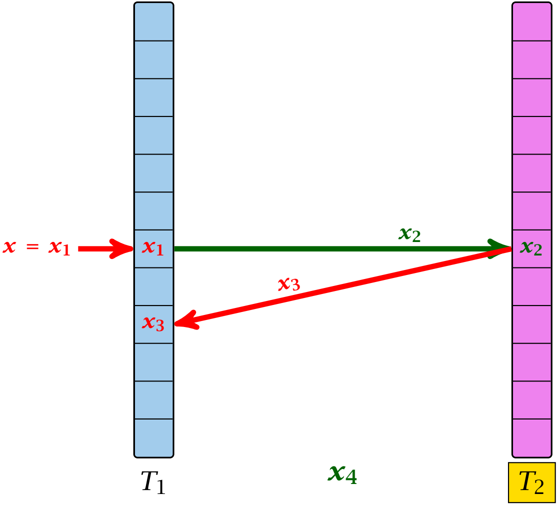

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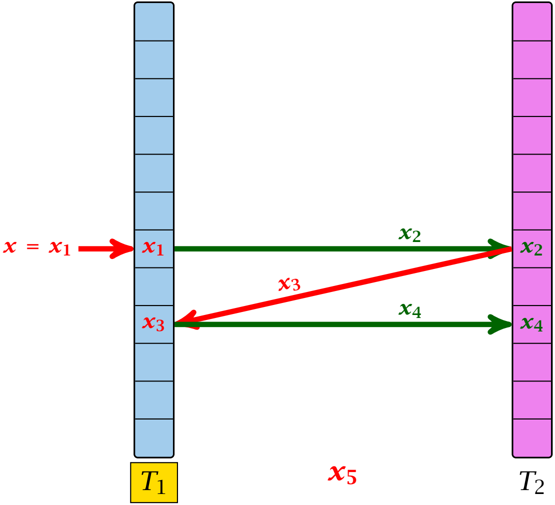

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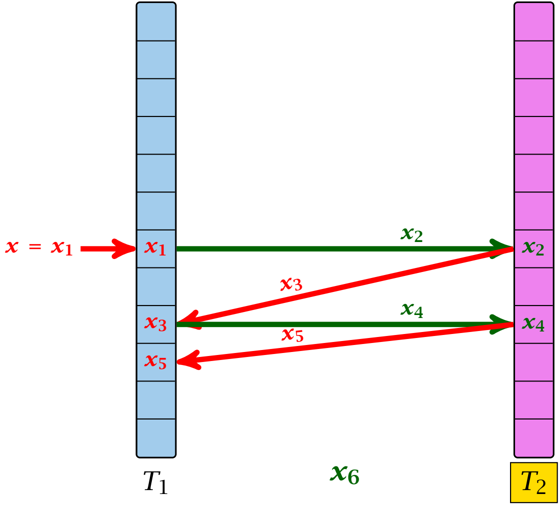

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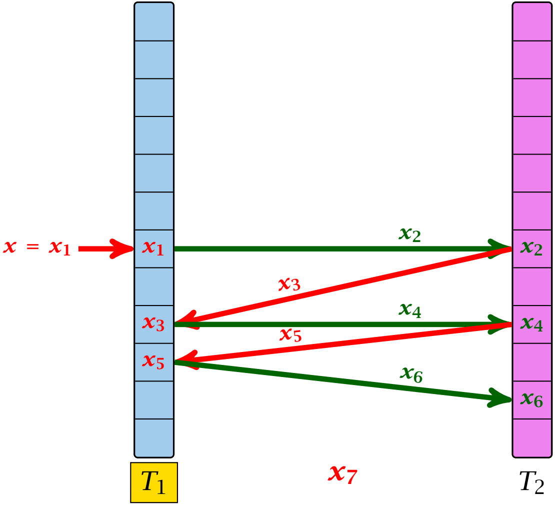

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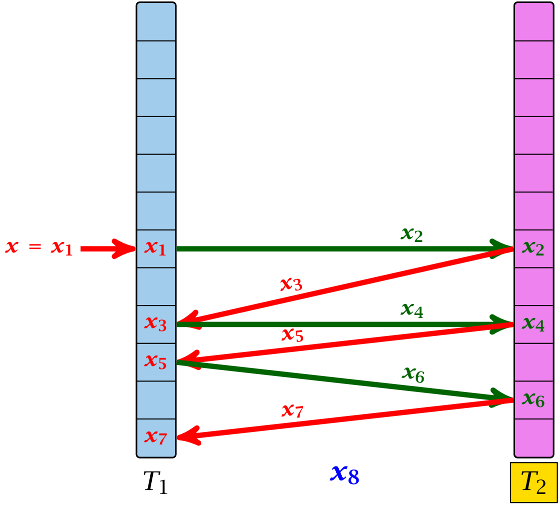

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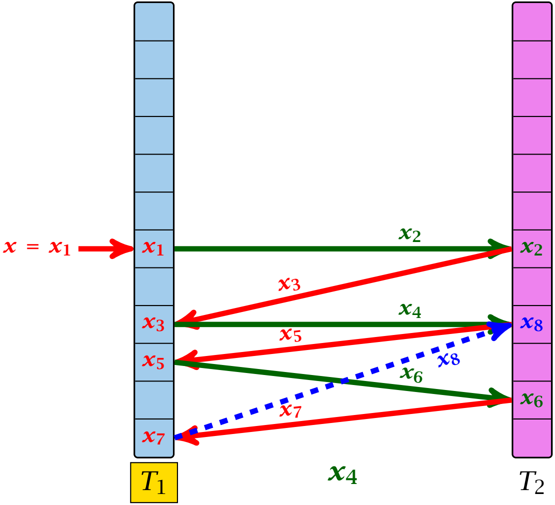

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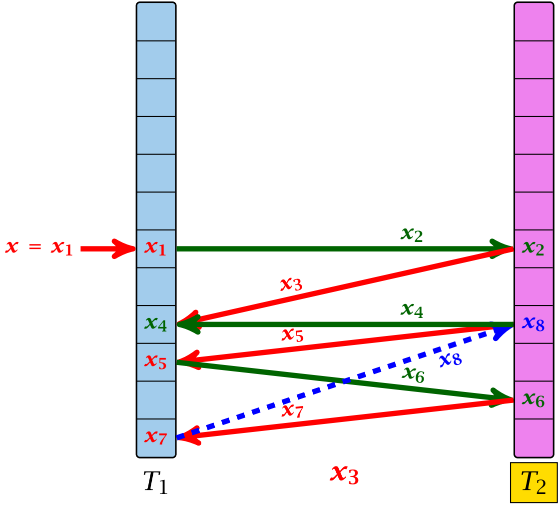

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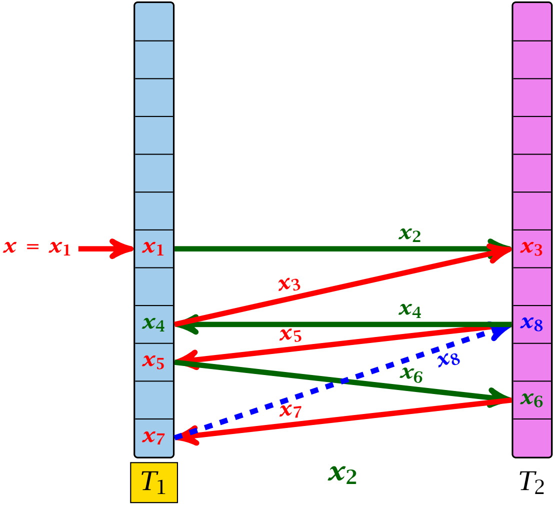

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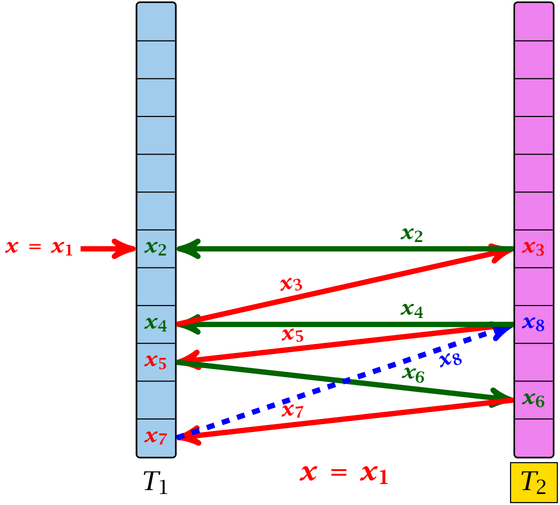

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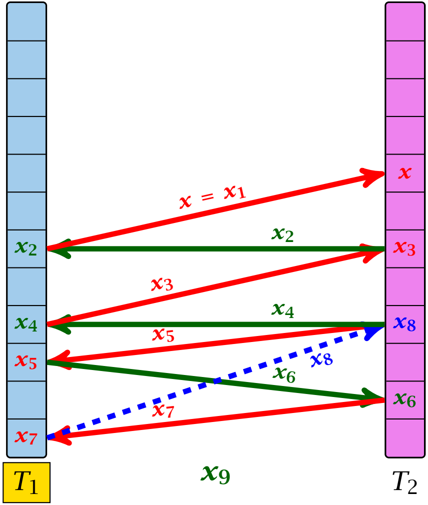

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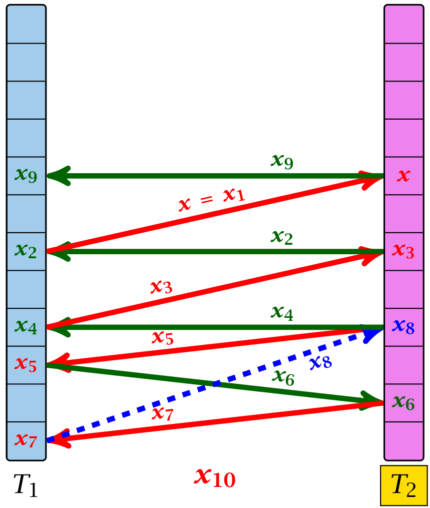

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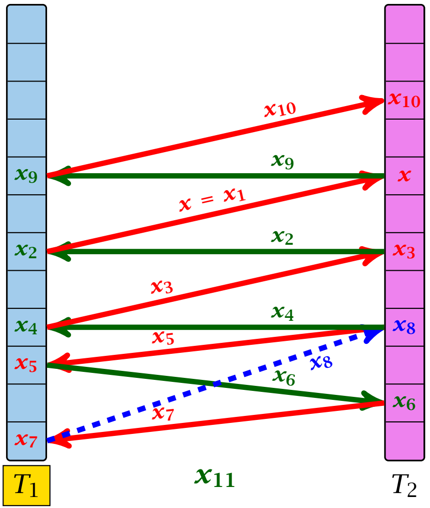

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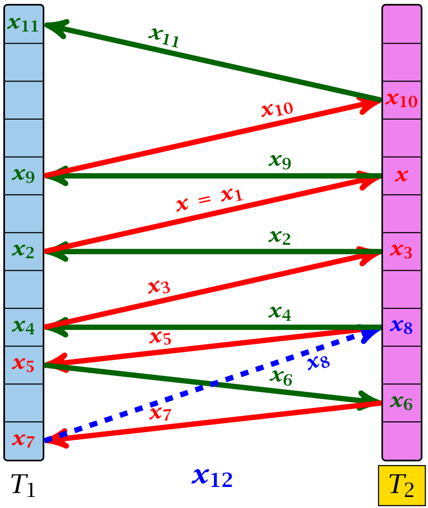

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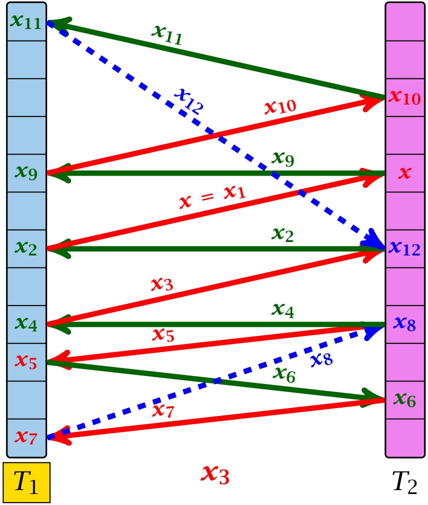

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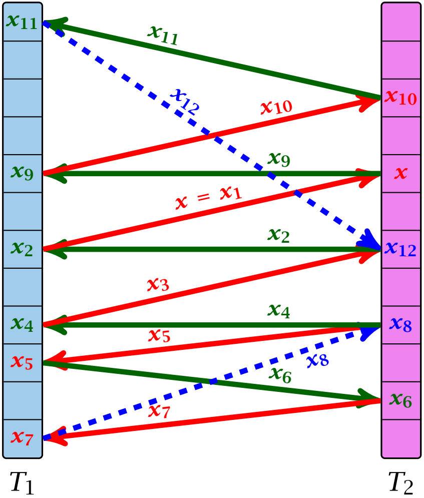

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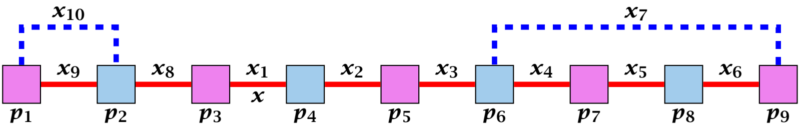

A cycle-structure of size s is defined by

- $s - 1$ different cells (alternating btw. cells from T_1 and T_2).
- s distinct keys $x = x_1, x_2, \dots, x_s$, linking the cells.
- The leftmost cell is "linked forward" to some cell on the right.
- The rightmost cell is "linked backward" to a cell on the left.
- One link represents key x; this is where the counting starts.

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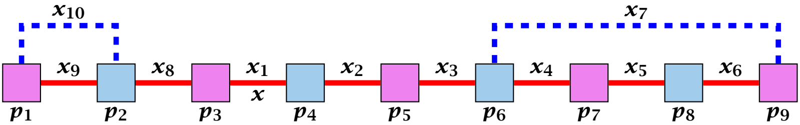

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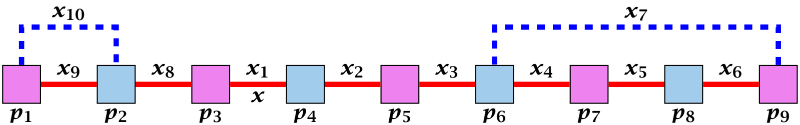

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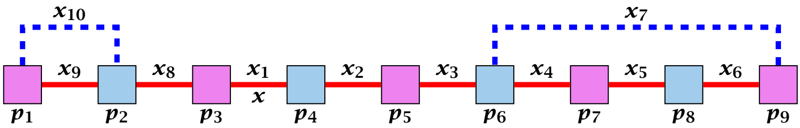

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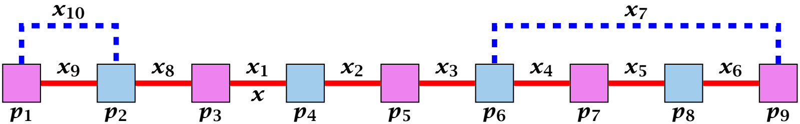

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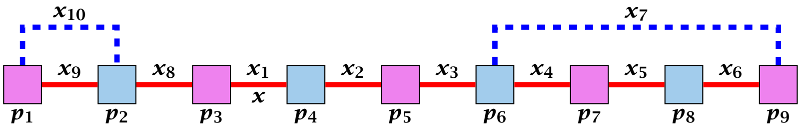

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Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

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Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s?

Cuckoo Hashing

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What is the probability that there exists an active cycle structure of size s?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

- There are at most s^2 possibilities where to attach the forward and backward links.
- There are at most s possibilities to choose where to place key x.
- There are m^{s-1} possibilities to choose the keys apart from x.
- There are n^{s-1} possibilities to choose the cells.

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Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

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$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s$$

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Here we used the fact that $(1+\epsilon)m \le n$.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

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$$\le \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \le \mathcal{O}\left(\frac{1}{m^2}\right) \ .$$

Here we used the fact that $(1+\epsilon)m \le n$.

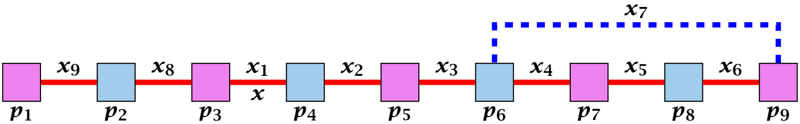
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \ .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 32

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

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Lemma 32

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 \ .$$

Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. □

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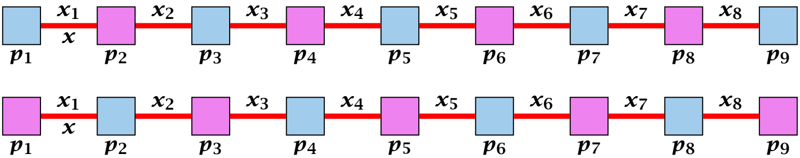
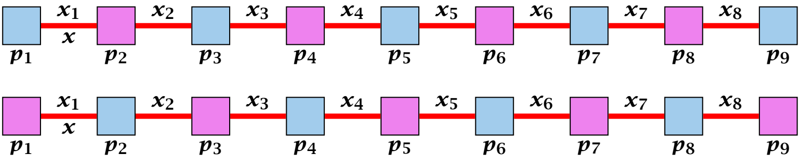


A path-structure of size s is defined by

- $s + 1$ different cells (alternating btw. cells from T_1 and T_2).
- s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.
- The leftmost cell is either from T_1 or T_2.

Cuckoo Hashing



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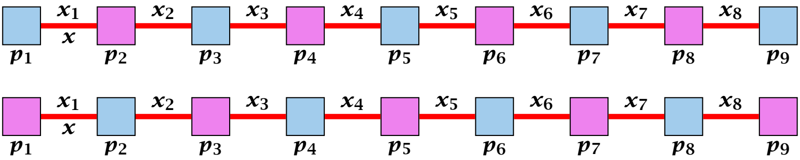

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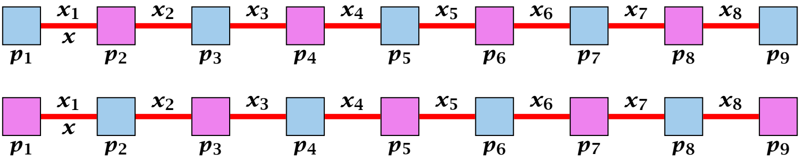

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Cuckoo Hashing

A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t+2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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This gives $\text{maxsteps} = \Theta(\log m)$.

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

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for a suitable constant $c > 0$.

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching $\mathrm{maxsteps}$ without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is $\sum_{i\geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

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A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is $\sum_{i\geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Formal Proof

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The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

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Cuckoo Hashing

What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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How do we make sure that $n \geq (1+\epsilon)m$?

- Let $\alpha := 1/(1+\epsilon)$.
- Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).
- Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).
- Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
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Lemma 33
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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

- **S. build($x_1, \ldots, x_n$)**: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.
- **S. insert(x)**: Adds element x to the data-structure.
- **element S. minimum()**: Returns an element $x \in S$ with minimum key-value key[x].
- **element S. delete-min()**: Deletes the element with minimum key-value from S and returns it.
- **boolean S. is-empty()**: Returns true if the data-structure is empty and false otherwise.

Sometimes we also have

- **S. merge(S')**: $S := S \cup S'$; $S' := \emptyset$.

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An addressable Priority Queue also supports:

- **handle S. insert(x)**: Adds element x to the data-structure, and returns a handle to the object for future reference.
- **S. delete(h)**: Deletes element specified through handle h.
- **S. decrease-key(h, k)**: Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

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Dijkstra's Shortest Path Algorithm

Algorithm 14 Shortest-Path$(G = (V, E, d), s \in V)$

Input: weighted graph $G = (V, E, d)$; start vertex s;
Output: key-field of every node contains distance from s;
S.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ **do**
    $v.\text{key} \leftarrow \infty$;
    $h_v \leftarrow S$.insert(v);
$s.\text{key} \leftarrow 0$; S.insert(s);
while S.is-empty$() = \text{false}$ **do**
    $v \leftarrow S$.delete-min$()$;
    for all $x \in V$ s.t. $(v, x) \in E$ **do**
        if $x.\text{key} > v.\text{key} + d(v, x)$ **then**
            S.decrease-key$(h_x, v.\text{key} + d(v, x))$;
            $x.\text{key} \leftarrow v.\text{key} + d(v, x)$;

Prim's Minimum Spanning Tree Algorithm

Algorithm 15 Prim-MST($G = (V, E, d), s \in V$)

Input: weighted graph $G = (V, E, d)$; start vertex s;
Output: pred-fields encode MST;
S.build(); // build empty priority queue
for all $v \in V \setminus \{s\}$ **do**
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        if $x.\text{key} > d(v, x)$ **then**
            S.decrease-key(h_x, $d(v, x)$);
            $x.\text{key} \leftarrow d(v, x)$;
            $x.\text{pred} \leftarrow v$;

Analysis of Dijkstra and Prim

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
- $|V|$ delete-min() operations
- $|V|$ is-empty() operations
- $|E|$ decrease-key() operations

How good a running time can we obtain?

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8 Priority Queues

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap**
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$**	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an amortized guarantee.

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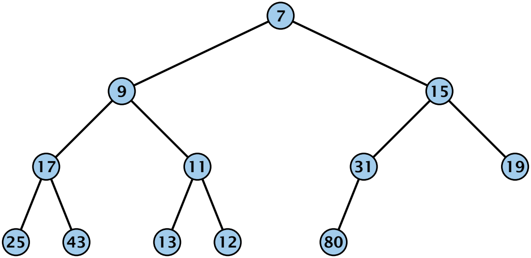
8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

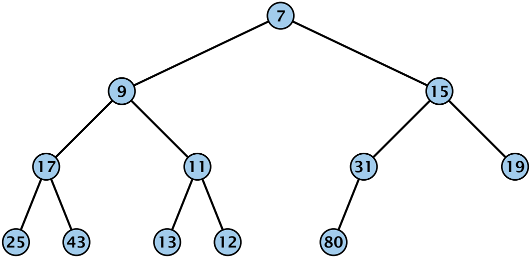
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



8.1 Binary Heaps

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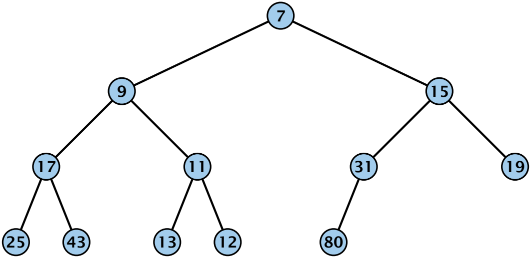
- Heap property: A node's key is not larger than the key of one of its children.



Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.

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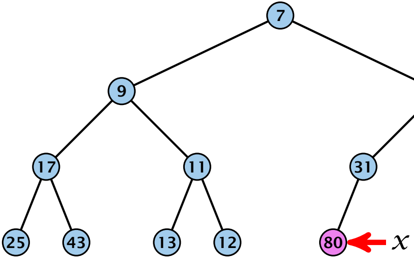
8.1 Binary Heaps

Maintain a pointer to the last element x.

- We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.

  go up until the last edge used was a right edge.
  go left; go right until you reach a leaf

  if you hit the root on the way up, go to the rightmost element



8.1 Binary Heaps

Maintain a pointer to the last element x.

- We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.

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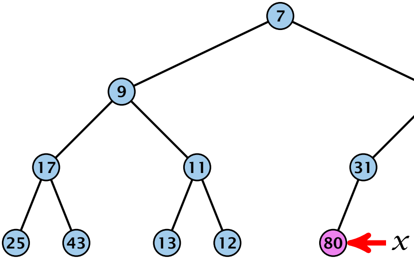

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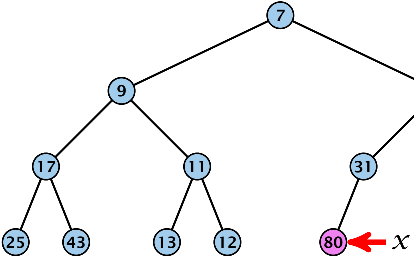

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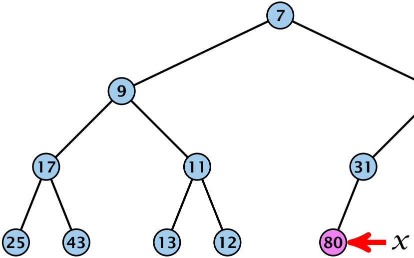

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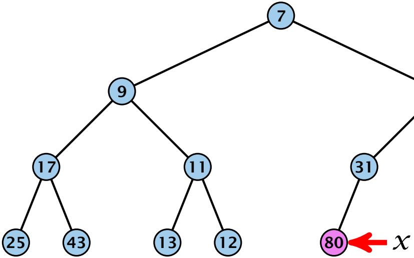
  if you hit the root on the way up, go to the leftmost element; insert a new element as a left child;



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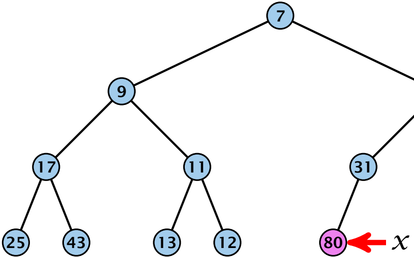

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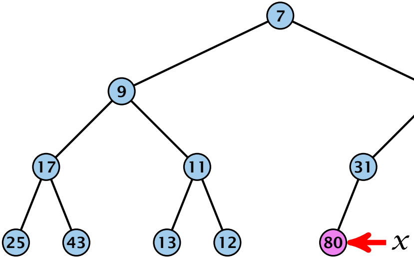

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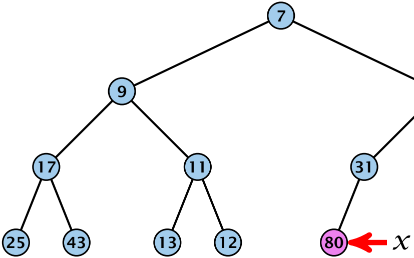

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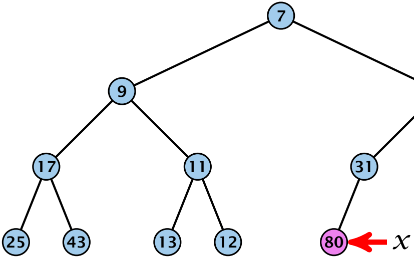

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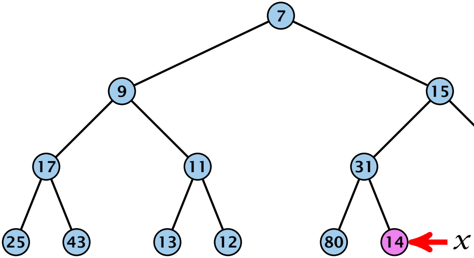

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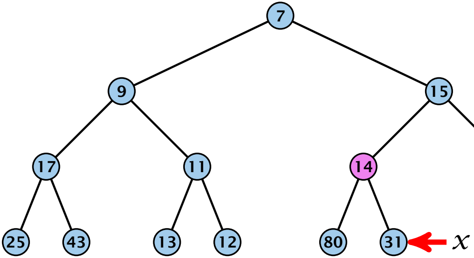

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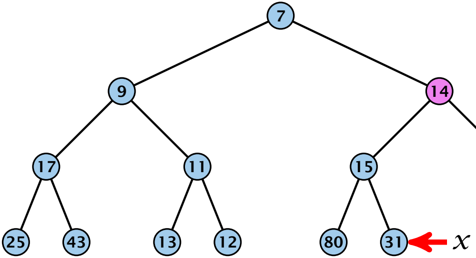

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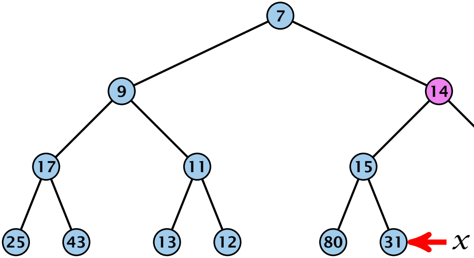

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Delete

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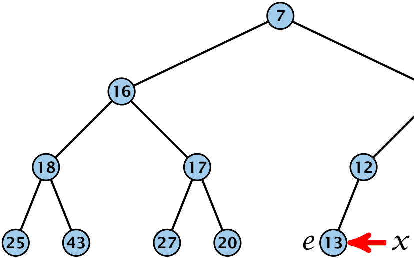

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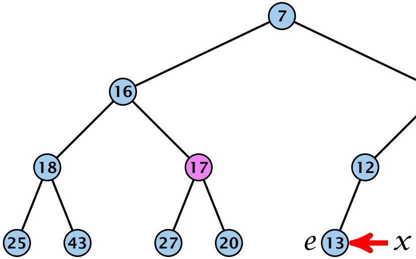

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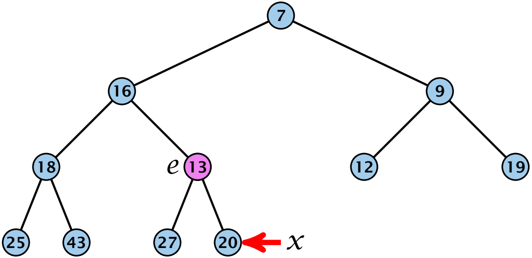

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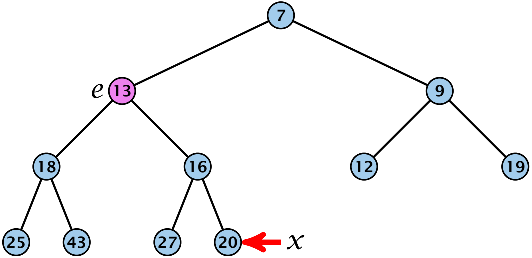

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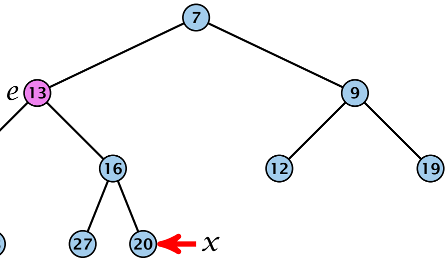

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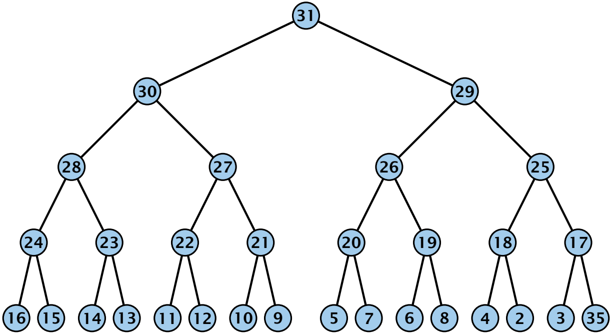
Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

Build Heap

We can build a heap in linear time:



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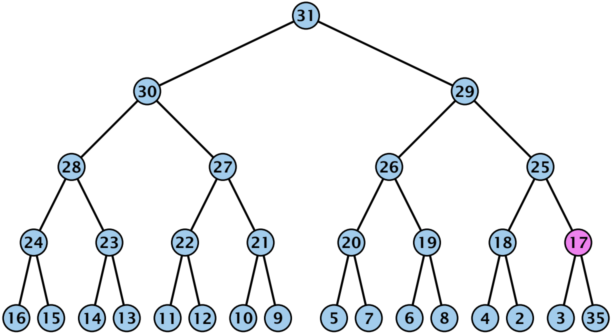

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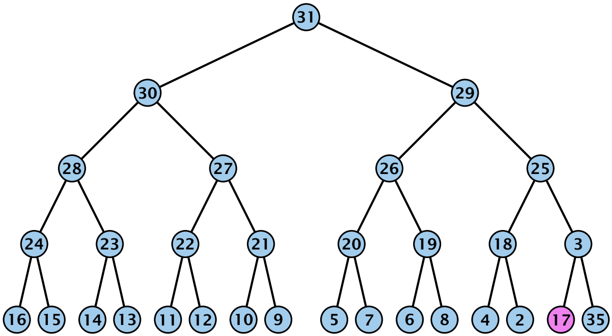

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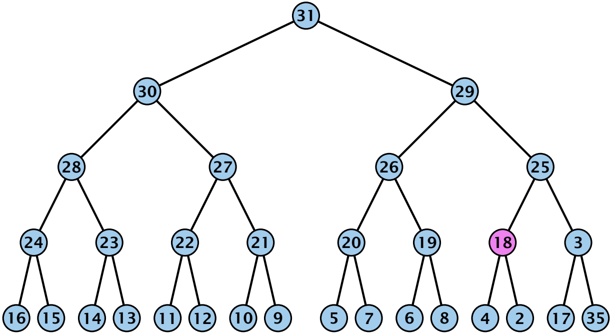

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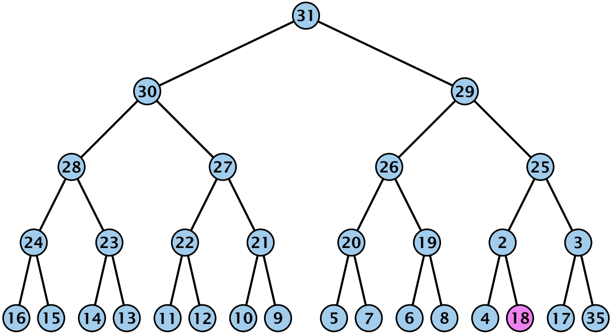

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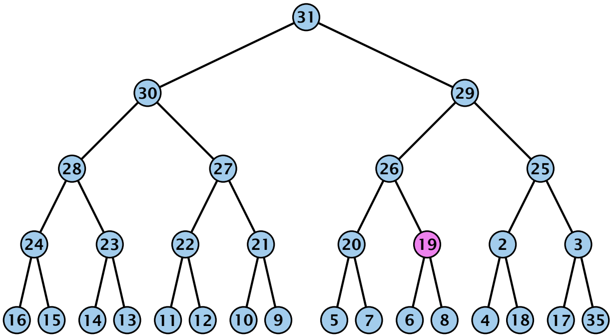

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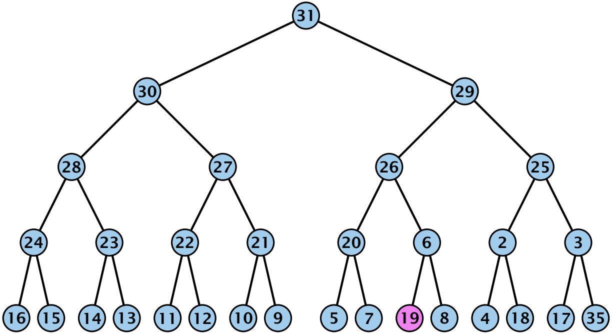

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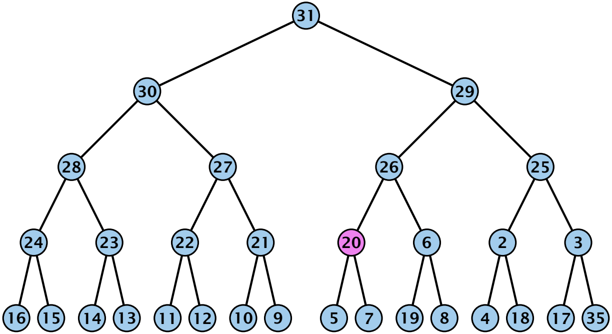

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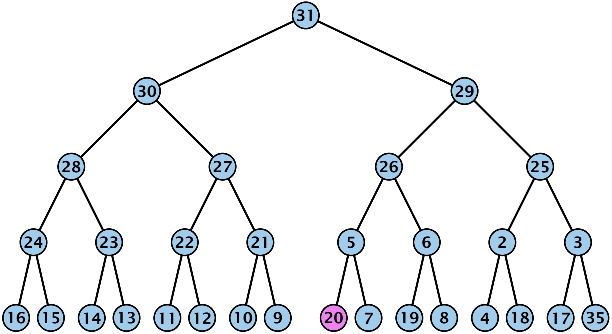

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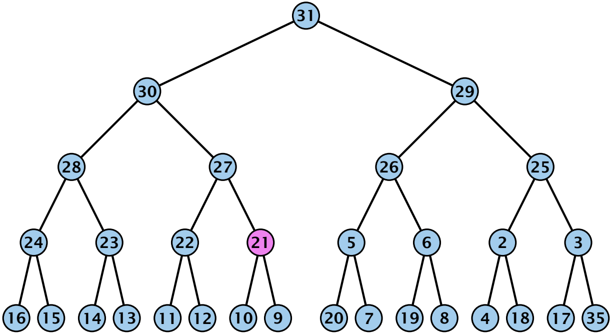

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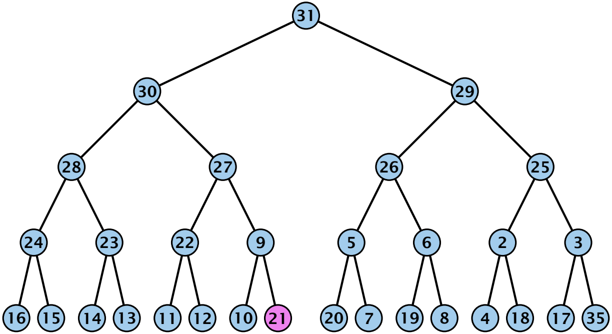

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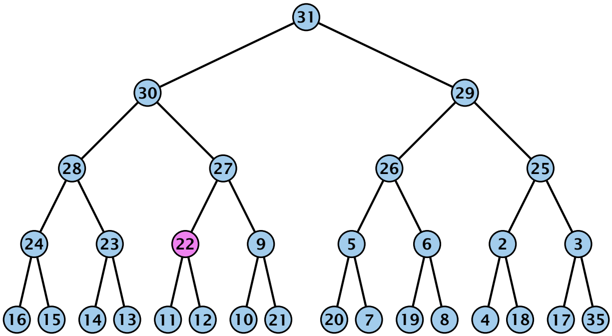

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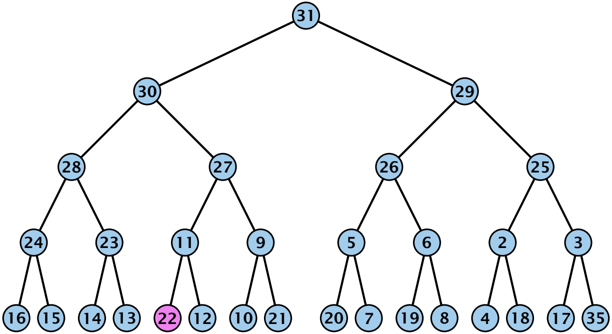

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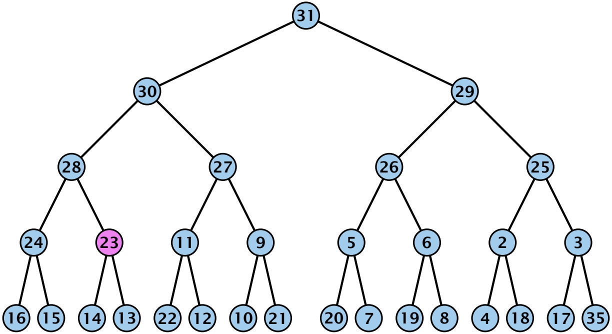

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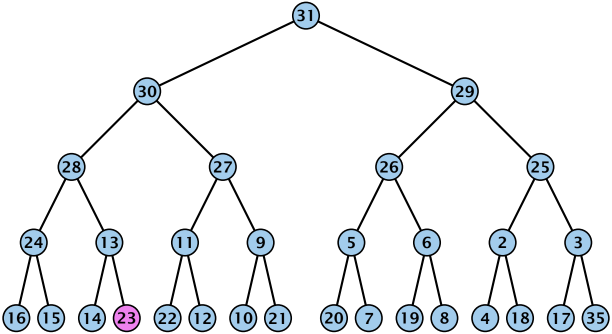

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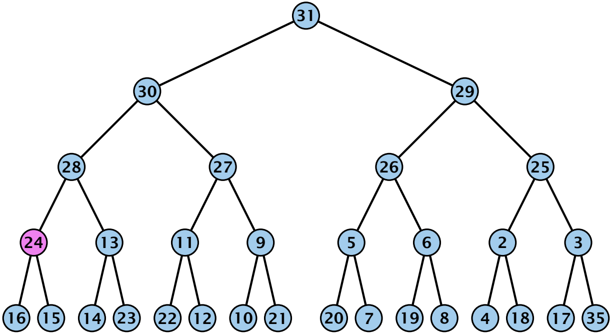

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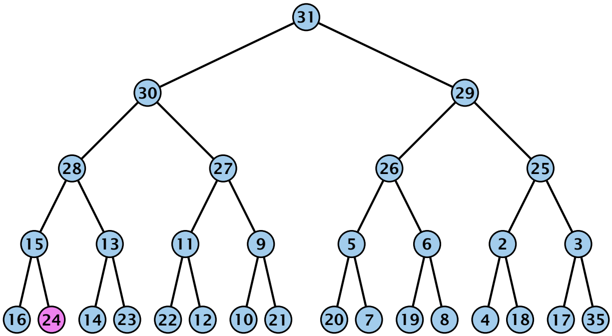

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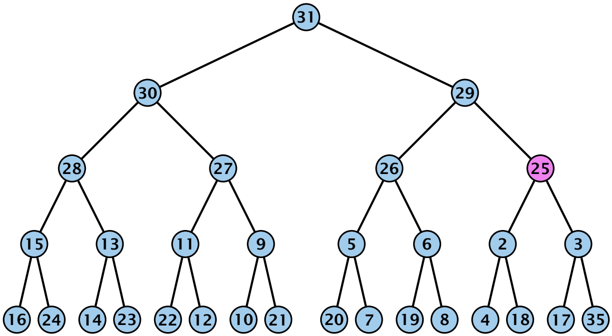

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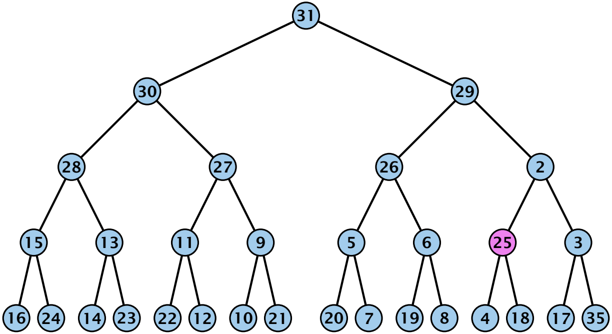

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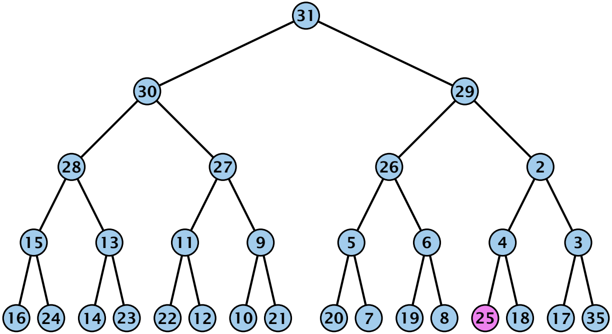

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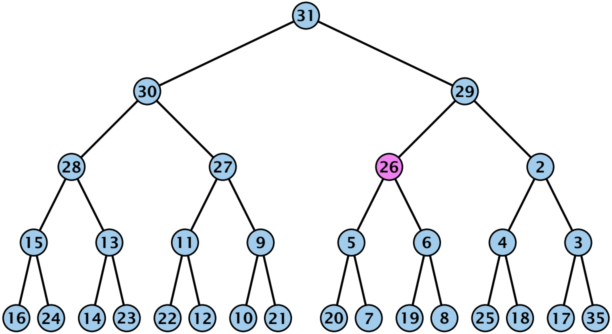

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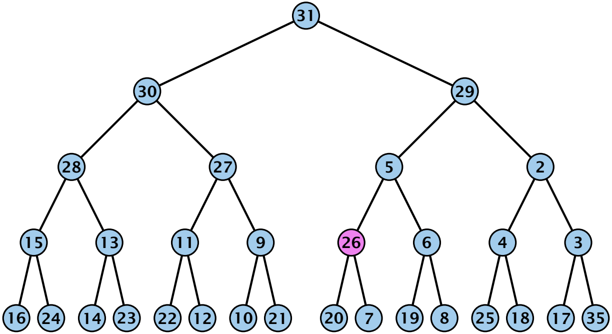

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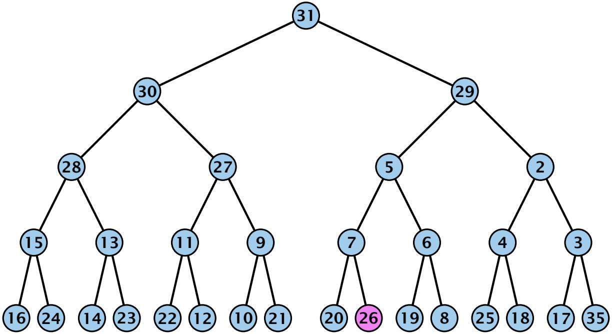

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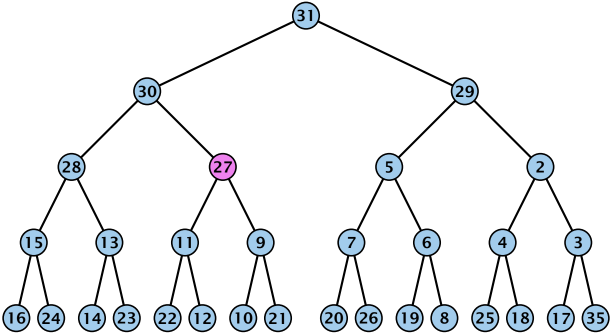

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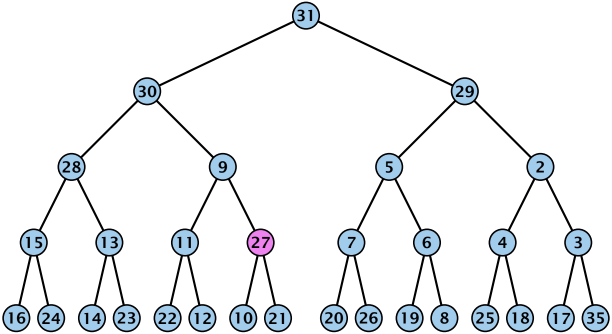

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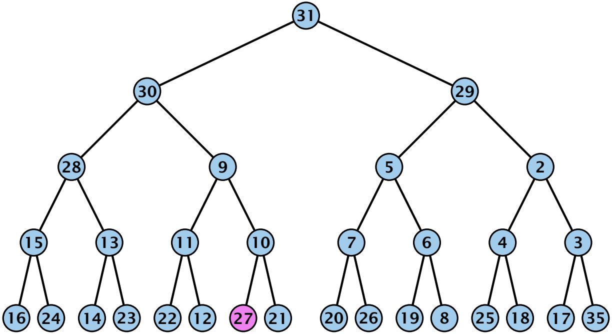

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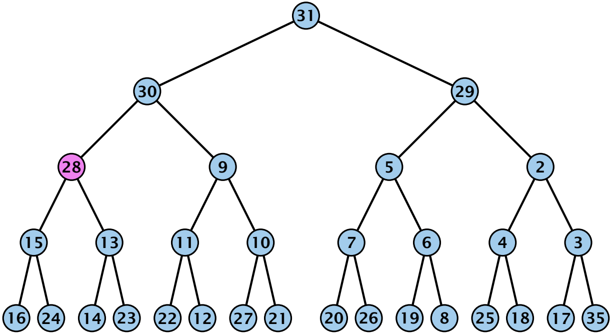

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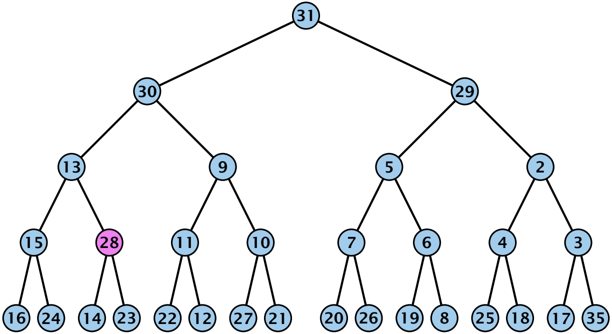

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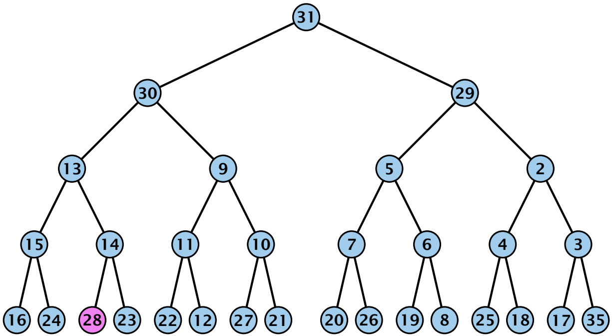

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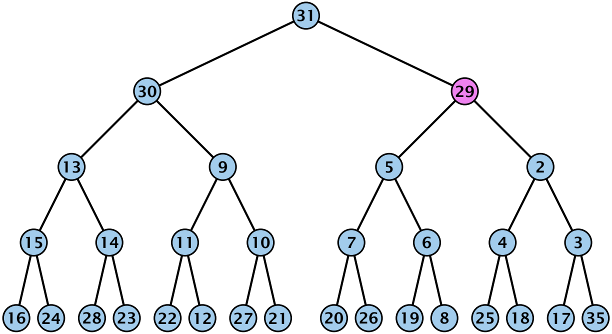

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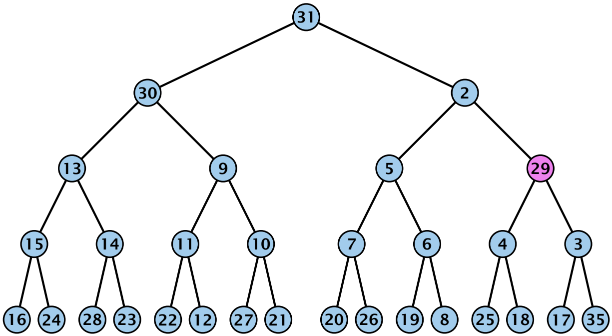

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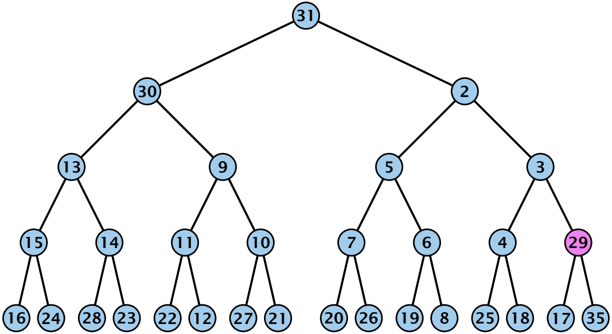

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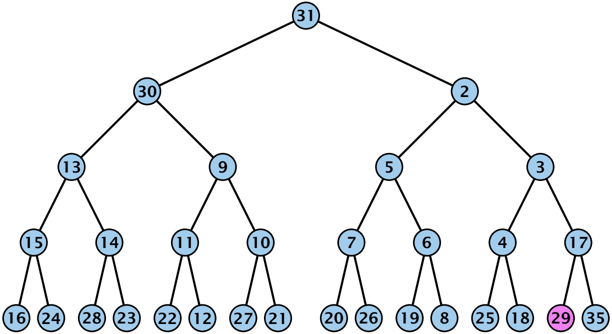

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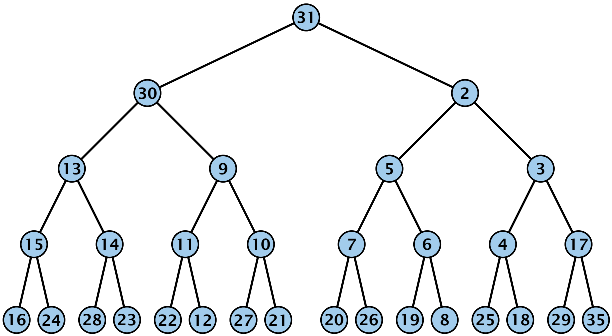

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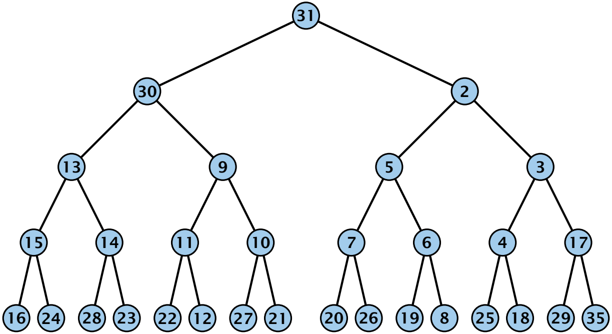

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Binary Heaps

Operations:

- **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \dots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- The left child of i-th element is at position $2i + 1$.
- The right child of i-th element is at position $2i + 2$.

Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

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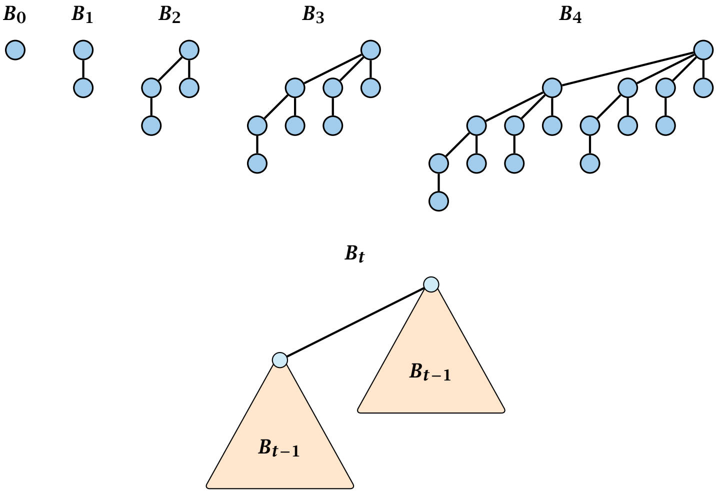
Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap* *
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$ **	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees



Binomial Trees

Properties of Binomial Trees

- B_k has 2^k nodes.
- B_k has height k.
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

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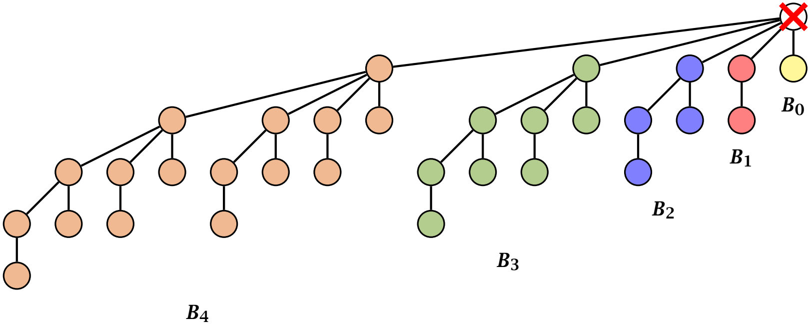
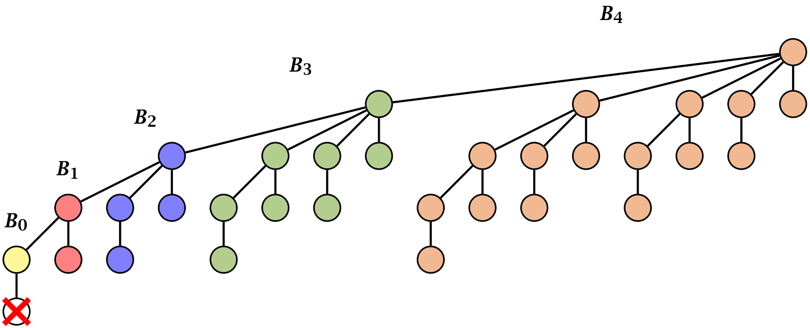


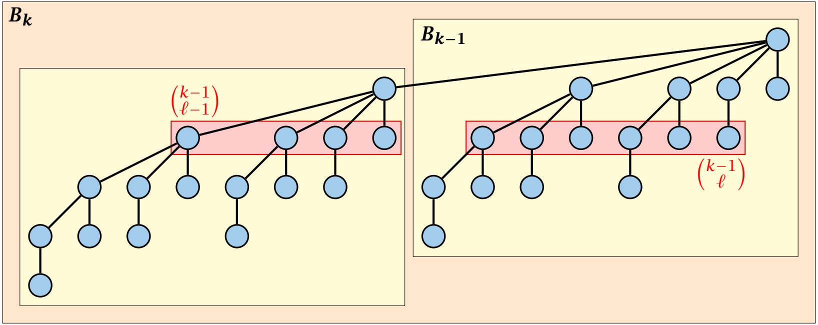
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

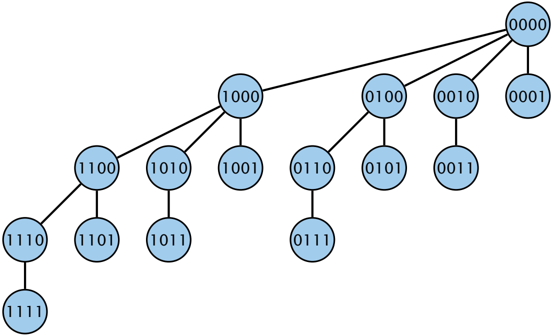
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

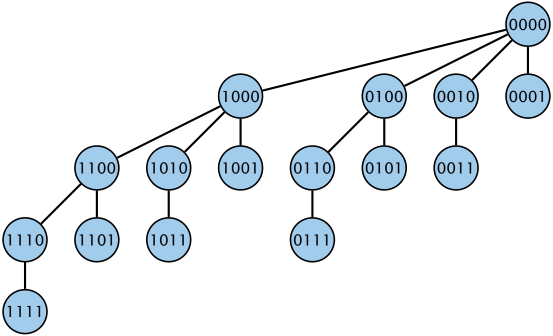


The binomial tree B_k is a sub-graph of the hypercube H_k.

The parent of a node with label $b_n, \dots, b_1, b_0$ is obtained by setting the least significant 1-bit to 0.

The ℓ-th level contains nodes that have ℓ 1's in their label.

Binomial Trees



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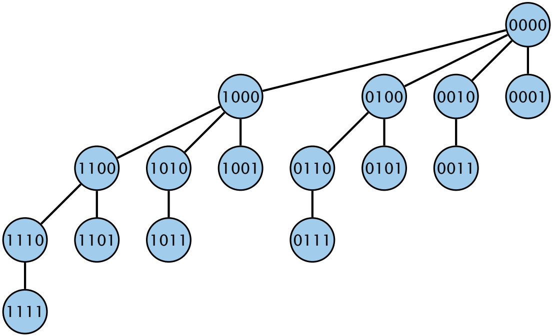

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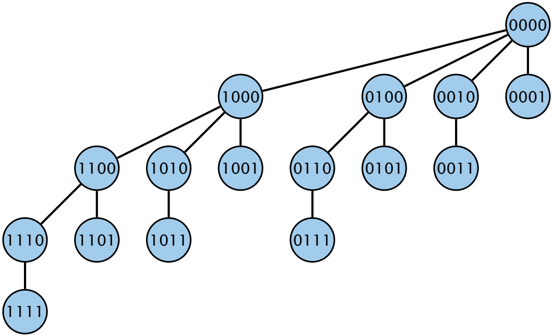

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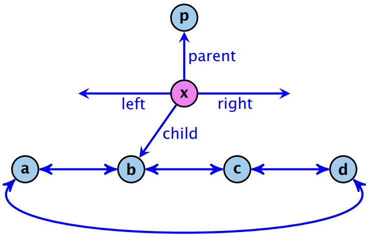
8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.
- A child-pointer points to an arbitrary node within the list.
- A parent-pointer points to the parent node.
- Pointers $x.\mathrm{left}$ and $x.\mathrm{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\mathrm{left} = x.\mathrm{right} = x$).



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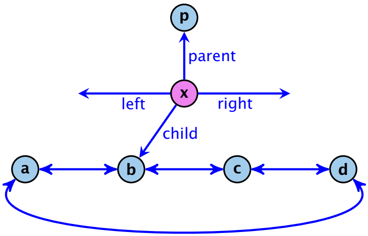

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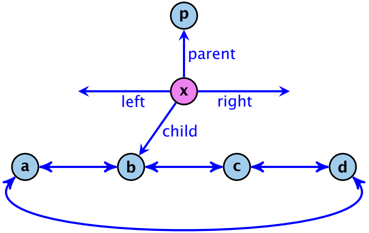

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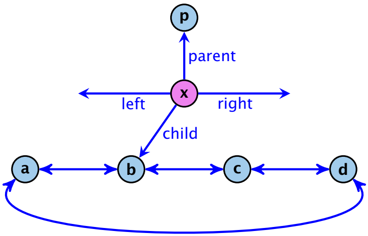
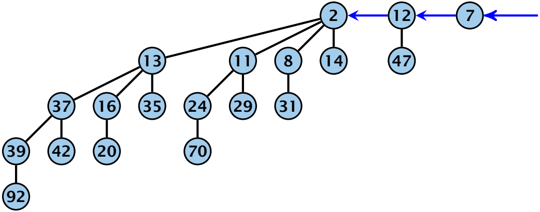


8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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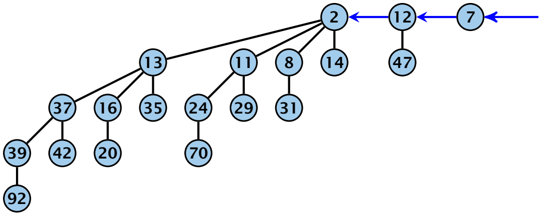

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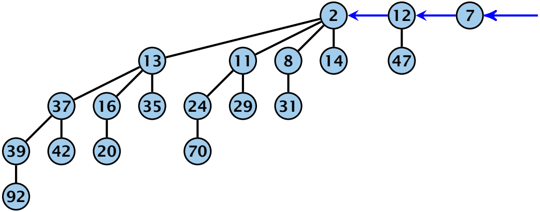

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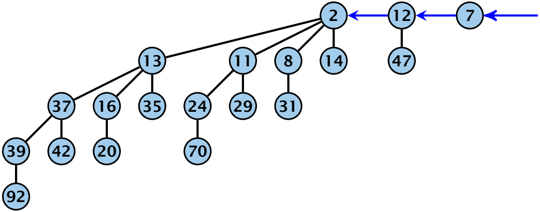

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Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

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Binomial Heap

Properties of a heap with n keys:

- Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.
- The heap contains tree B_i iff $b_i = 1$.
- Hence, at most $\lfloor \log n \rfloor + 1$ trees.
- The minimum must be contained in one of the roots.
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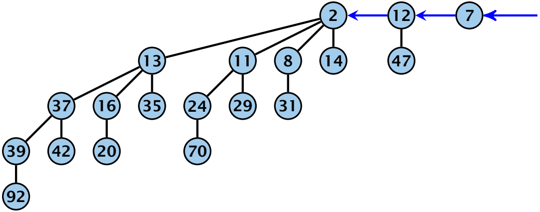
- The trees are stored in a single-linked list; ordered by dimension/size.



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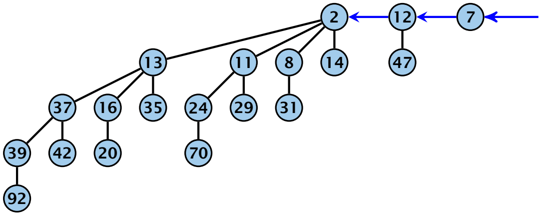

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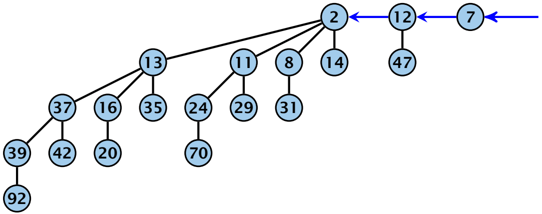

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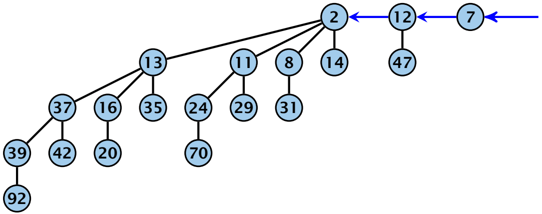

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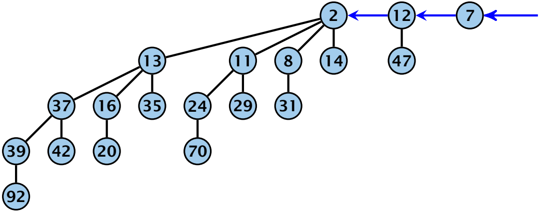

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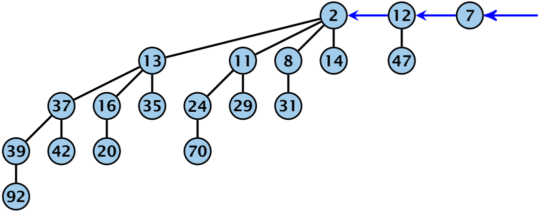

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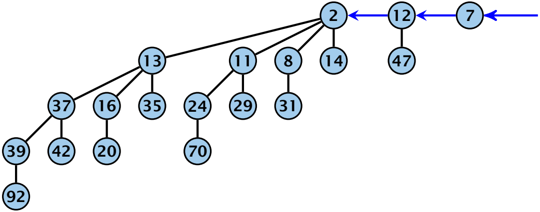

Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.

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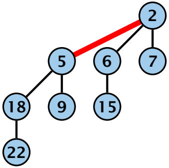

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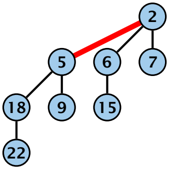

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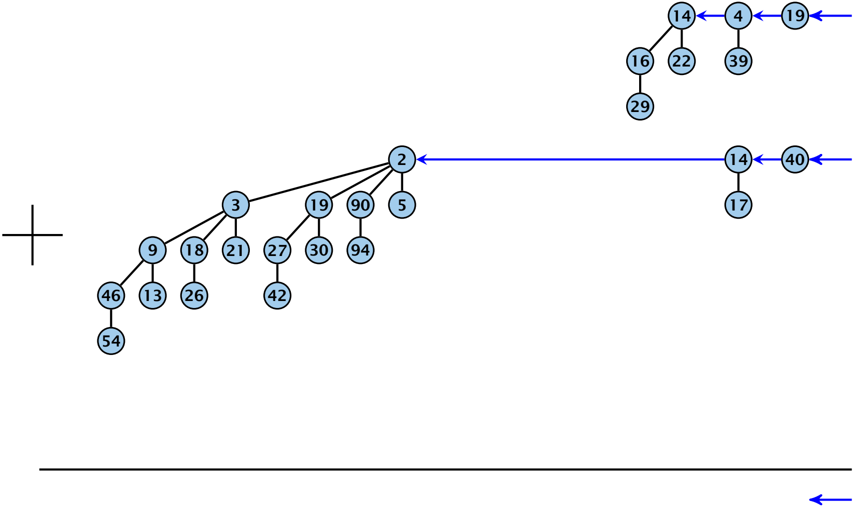
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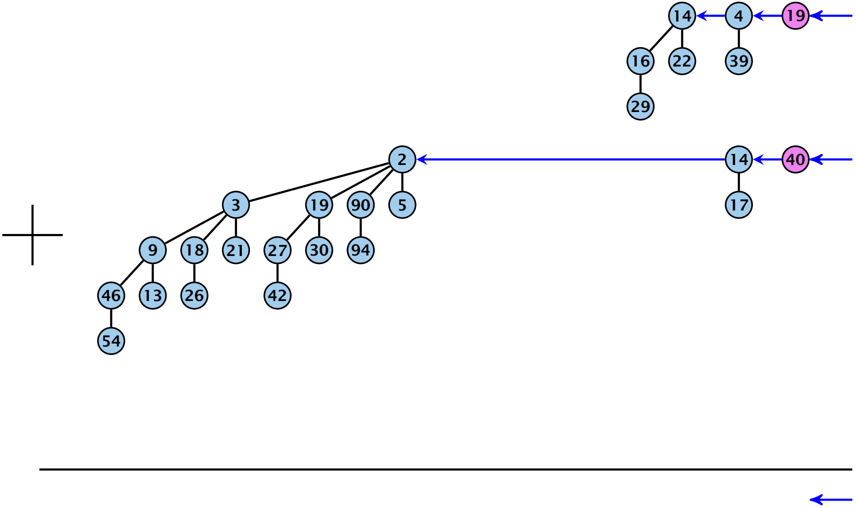
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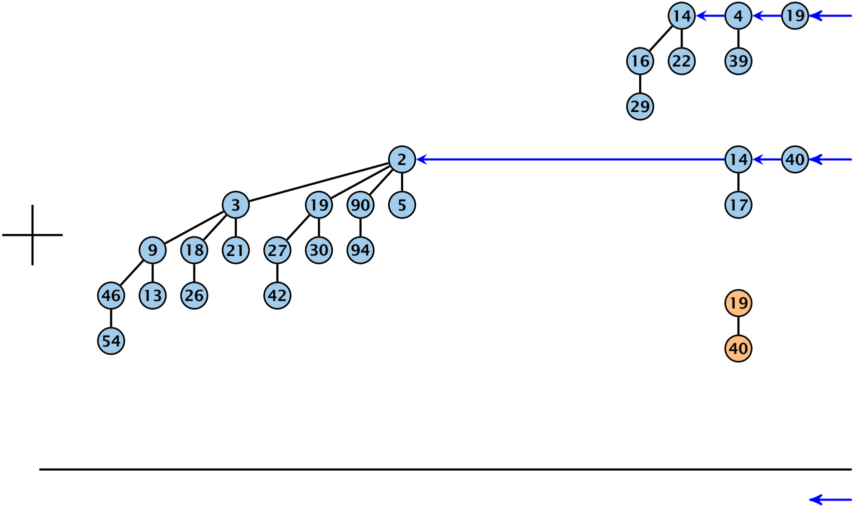
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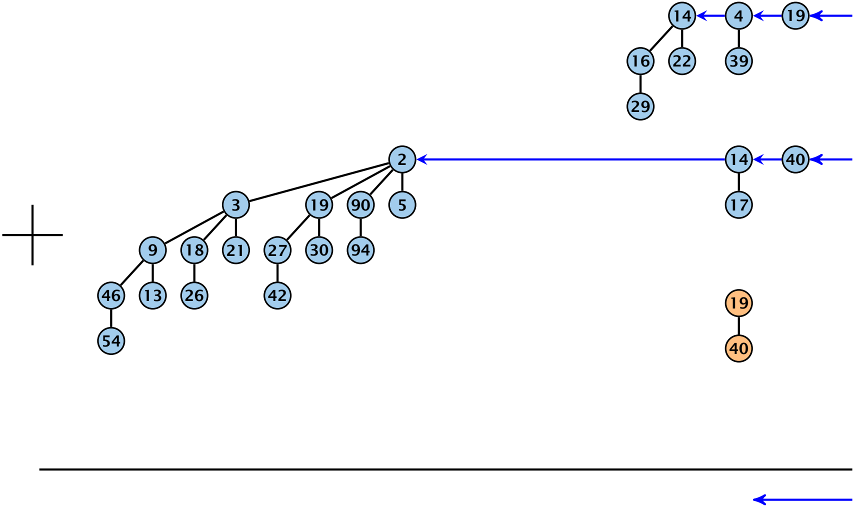

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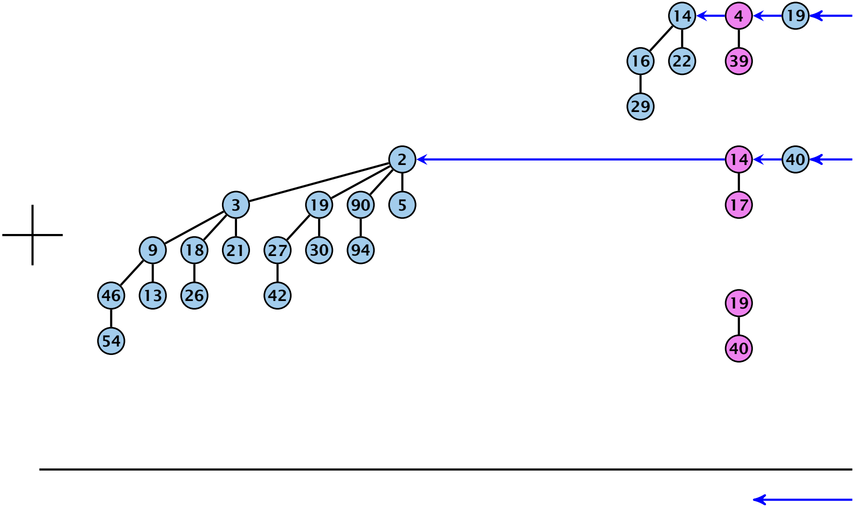
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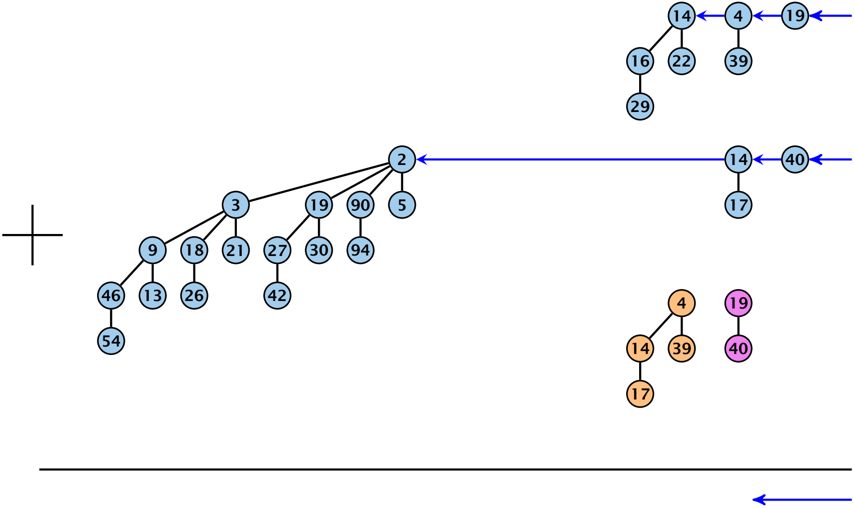

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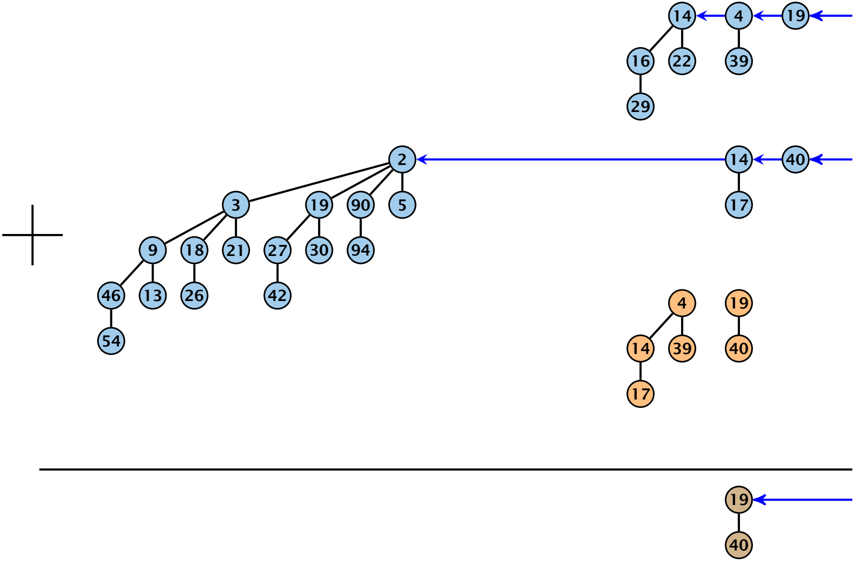
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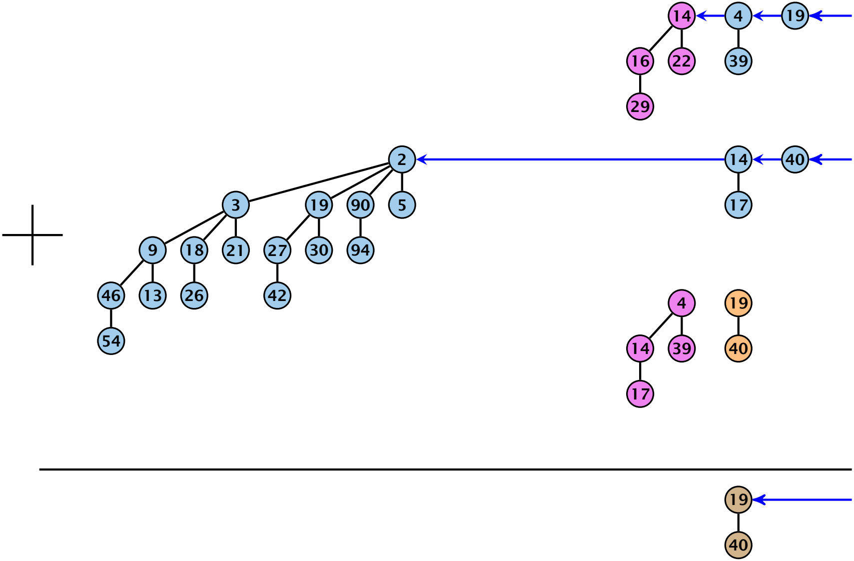
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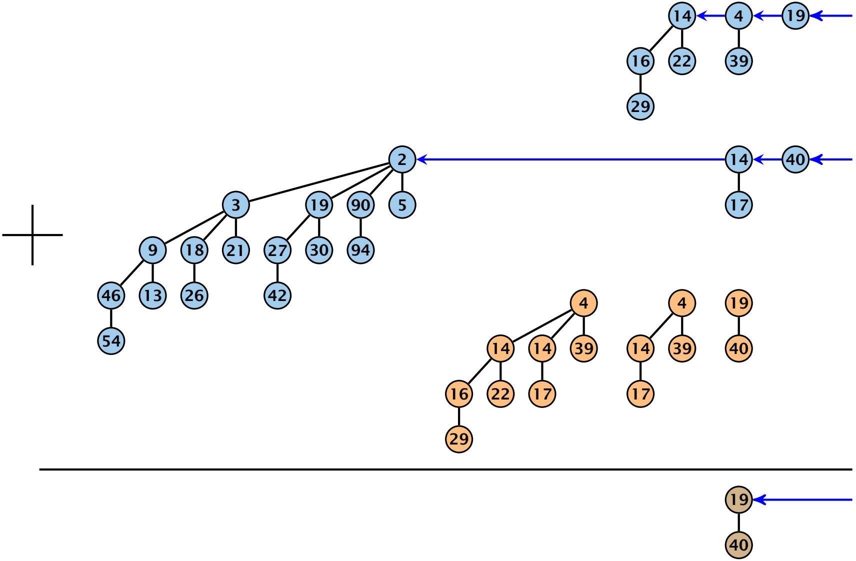
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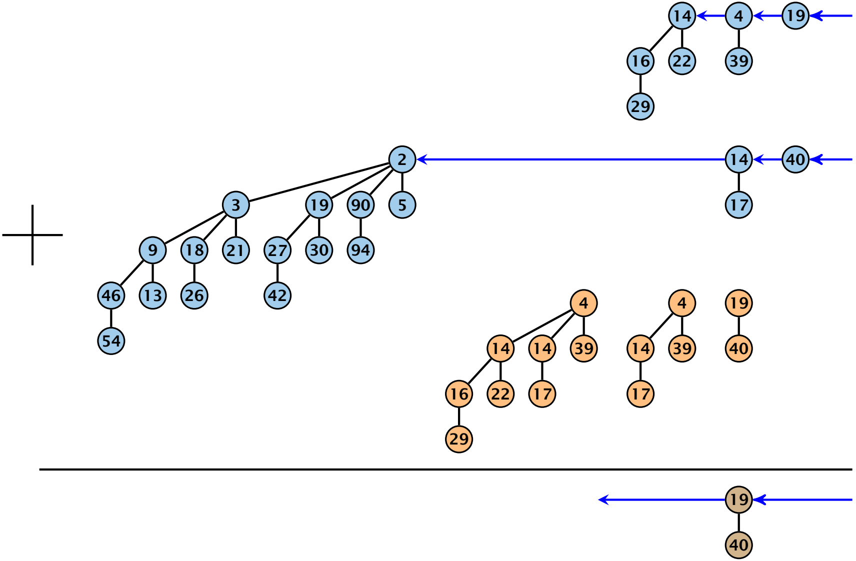
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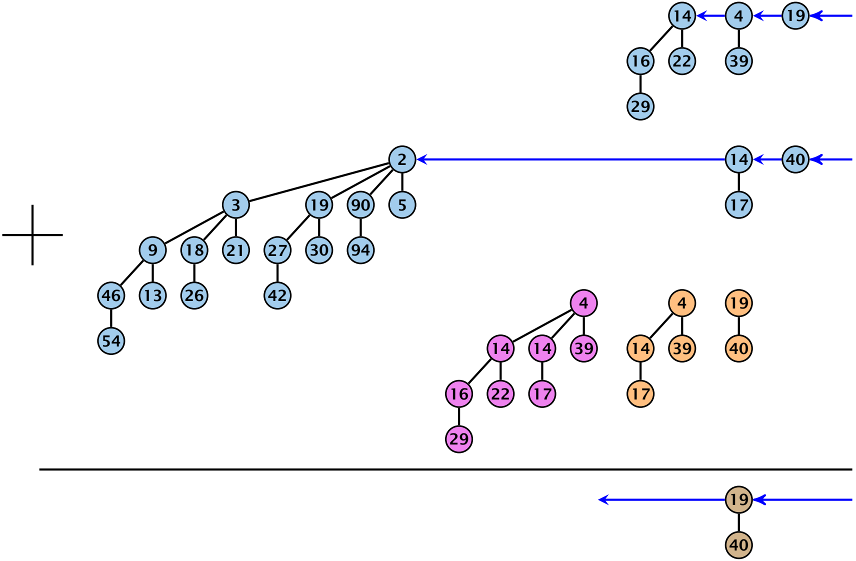
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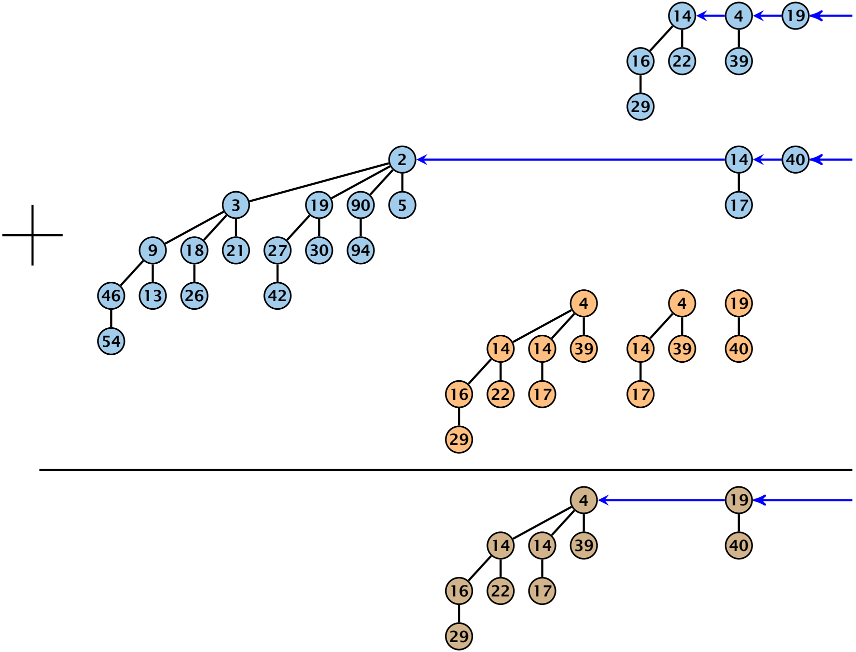

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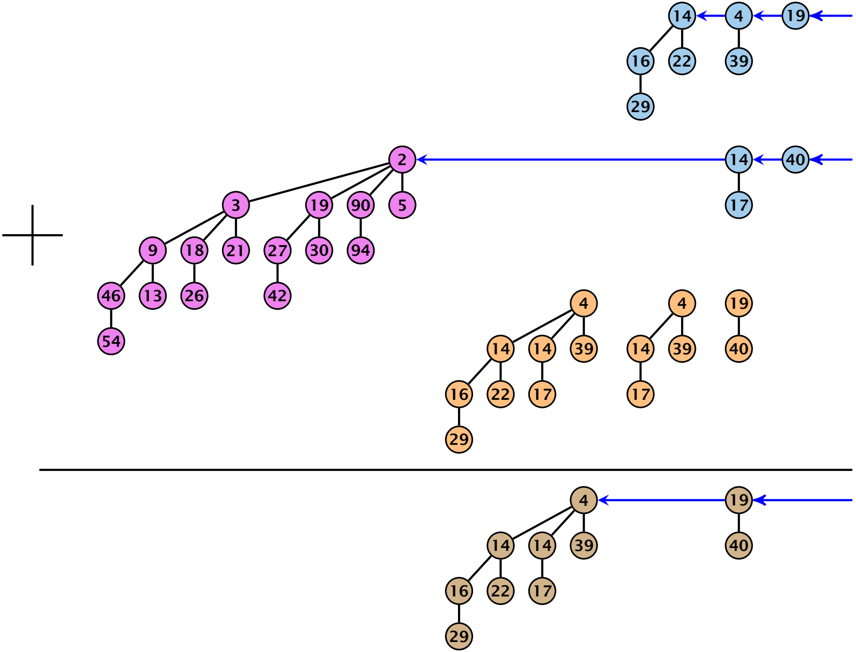
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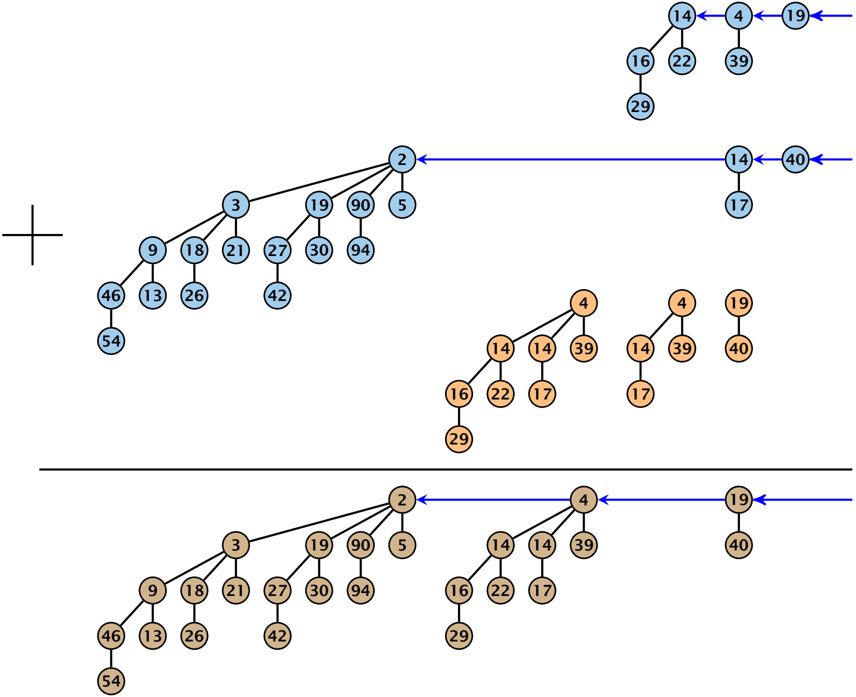

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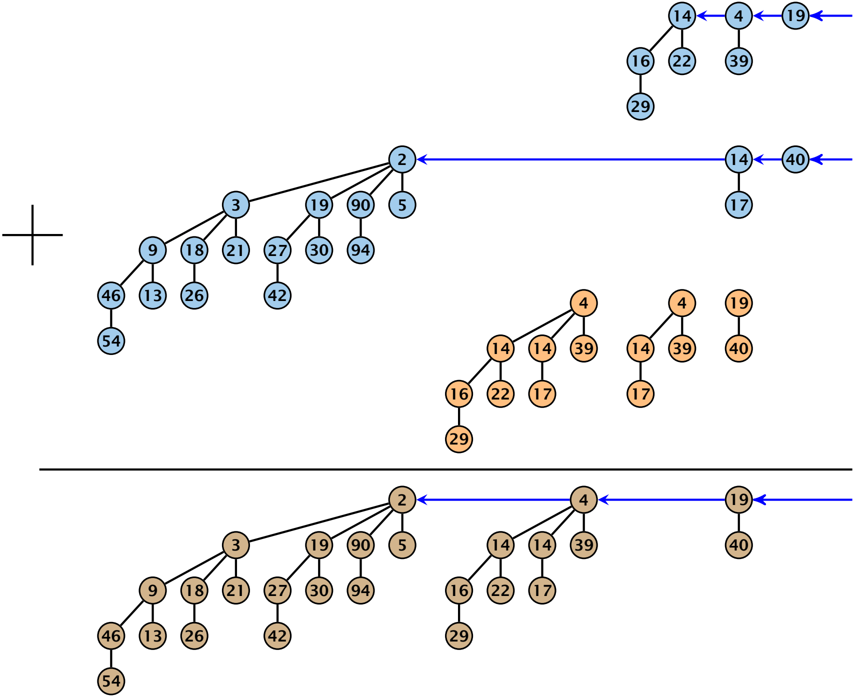
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8.2 Binomial Heaps

$S_1.\mathbf{merge}(S_2)$:

- Analogous to binary addition.
- Time is proportional to the number of trees in both heaps.
- Time: $\mathcal{O}(\log n)$.

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All other operations can be reduced to merge().

S.insert(x):

- Create a new heap S' that contains just the element x.
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S. minimum():

- Find the minimum key-value among all roots.
- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ delete-min():

- Find the minimum key-value among all roots.
- Remove the corresponding tree $T_{\min}$ from the heap.
- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
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S. decrease-key(handle h):

- Decrease the key of the element pointed to by h.
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S. delete(handle h):

- Execute S.decrease-key$(h, -\infty)$.
- Execute S.delete-min().
- Time: $\mathcal{O}(\log n)$.

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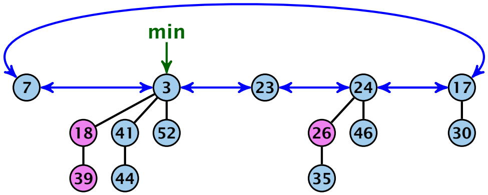
8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

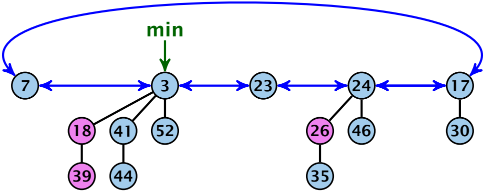
Additional implementation details:

- Every node x stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

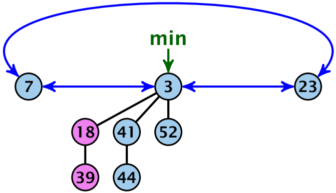
$S.$ **minimum()**

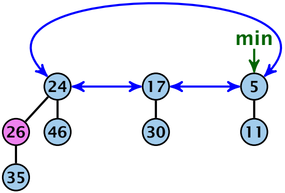
- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\mathbf{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer





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$S.\textbf{merge}(S')$

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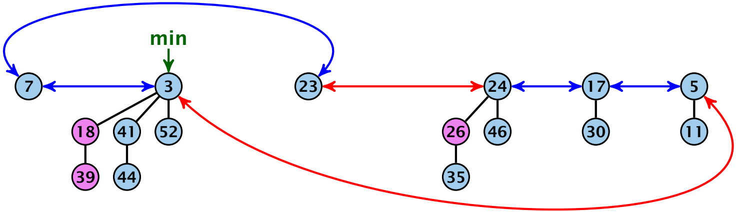

Running time:

- Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. **merge**(*S'*)

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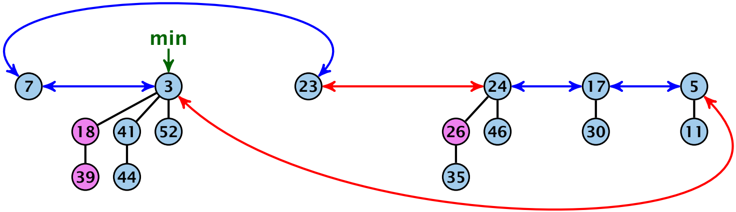
- Adjust the min-pointer



Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.

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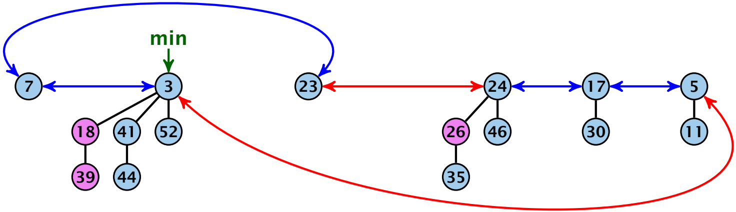


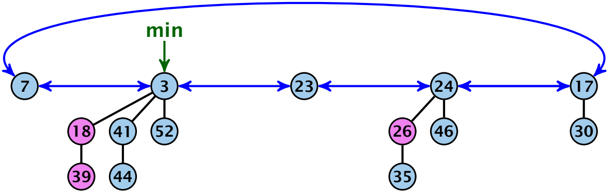
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\,\mathbf{insert}(x)$

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.



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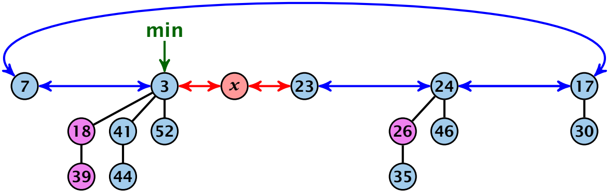

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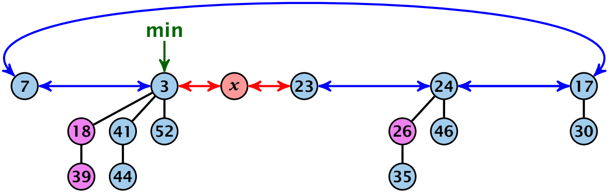
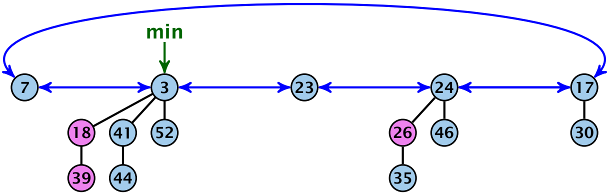


Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

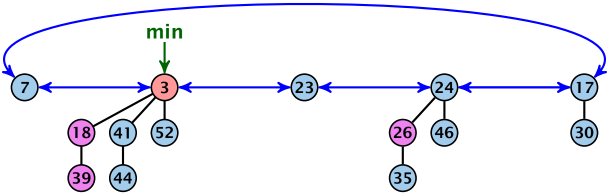
S. delete-min(x)



8.3 Fibonacci Heaps

$S.$ delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.



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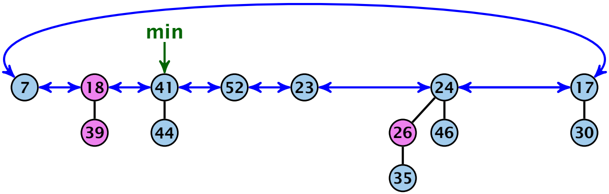
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



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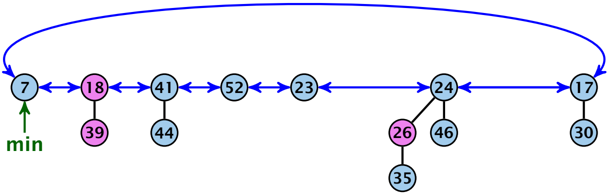
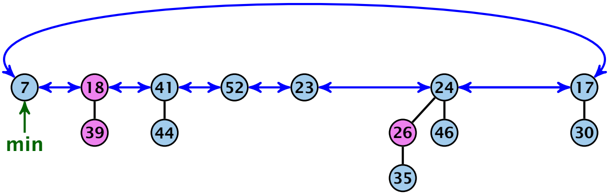


8.3 Fibonacci Heaps

$S.$ **delete-min**(x)

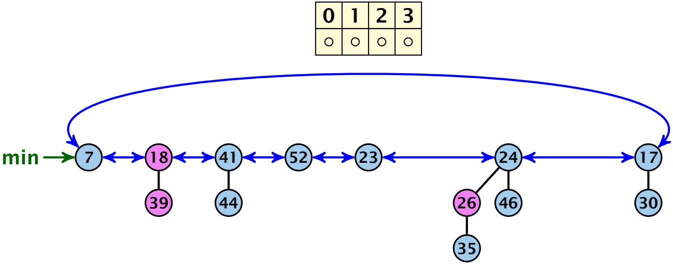
- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:



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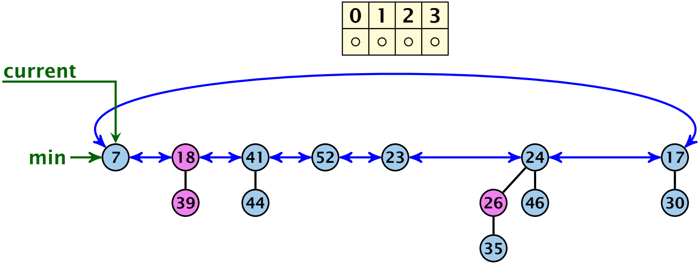

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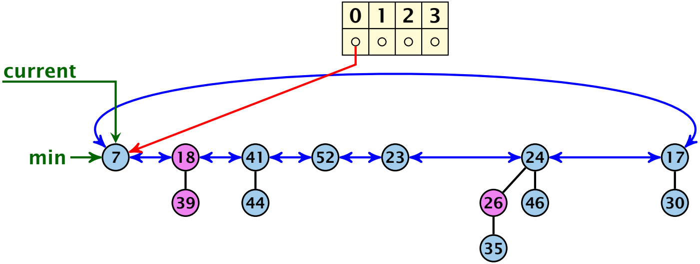

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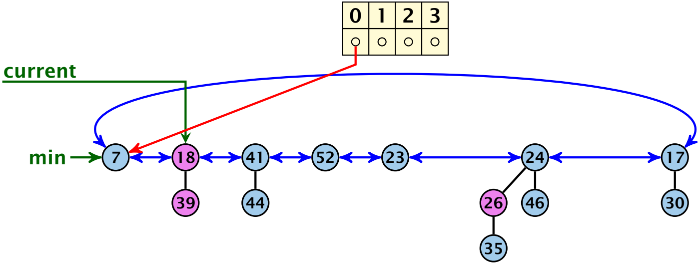

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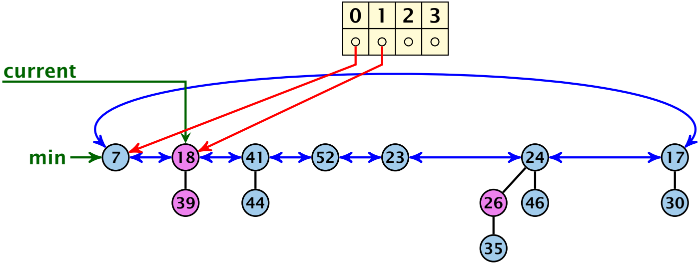

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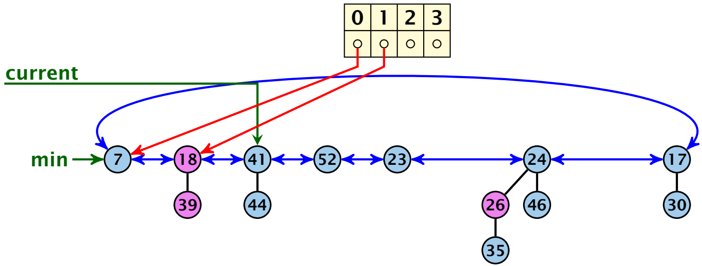

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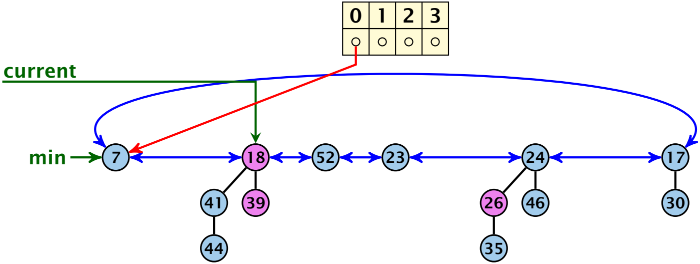

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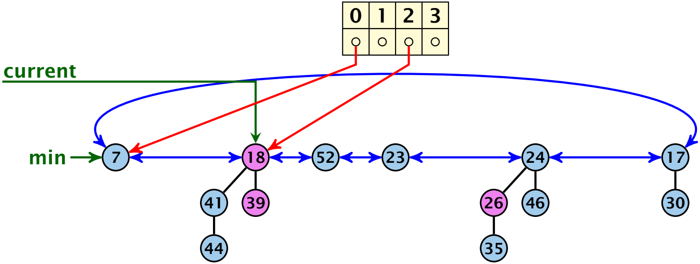

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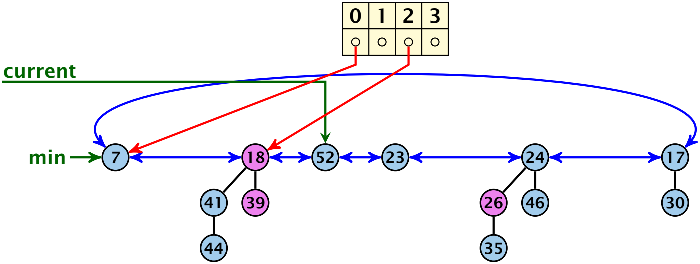

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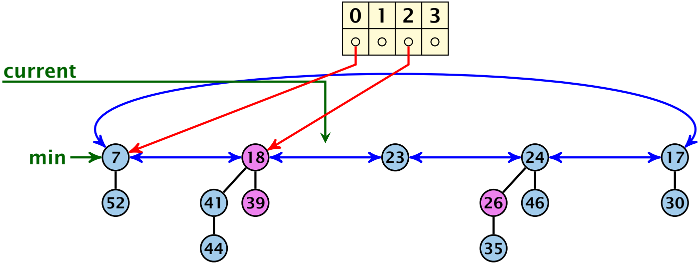

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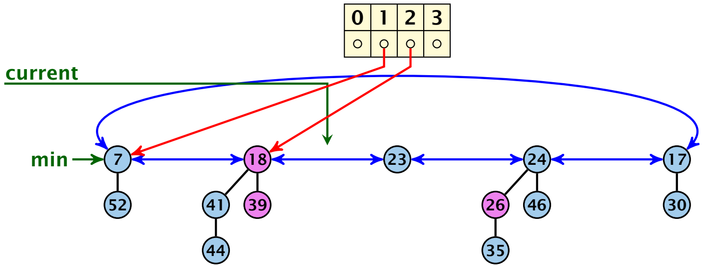

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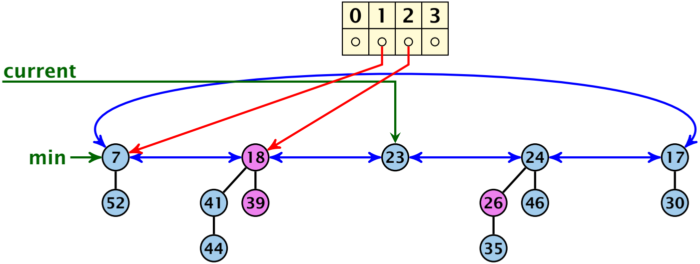

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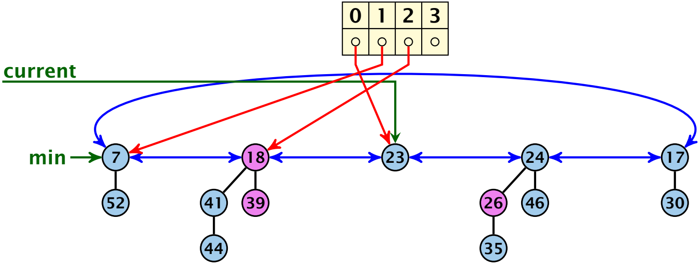

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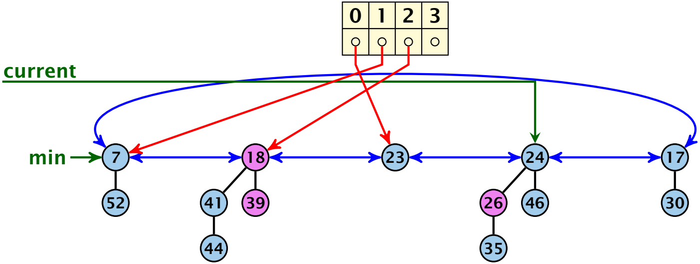

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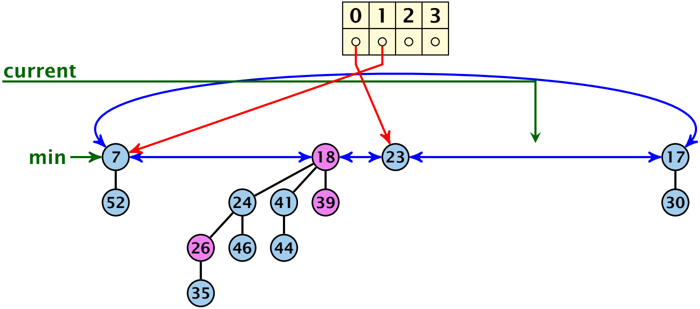

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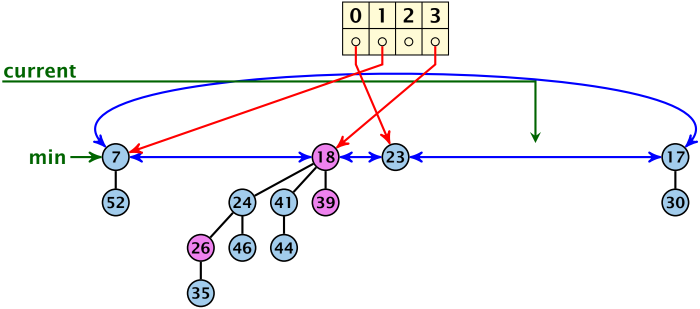

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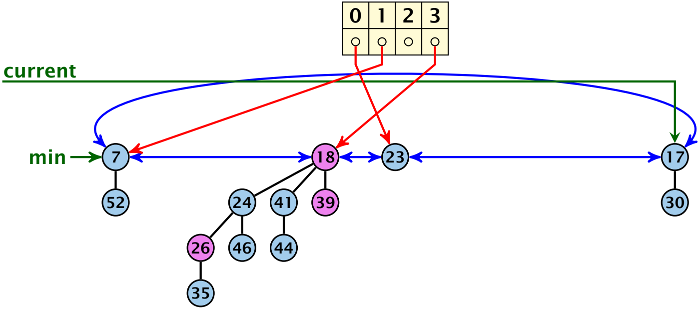

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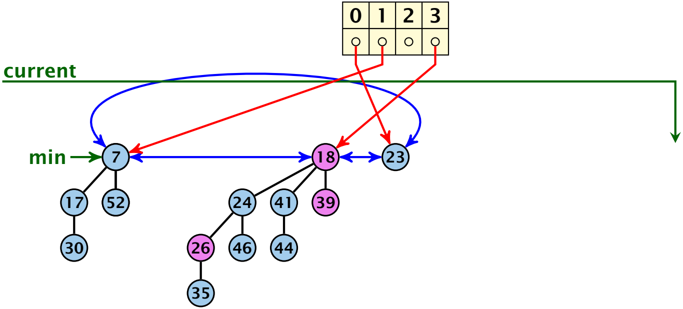

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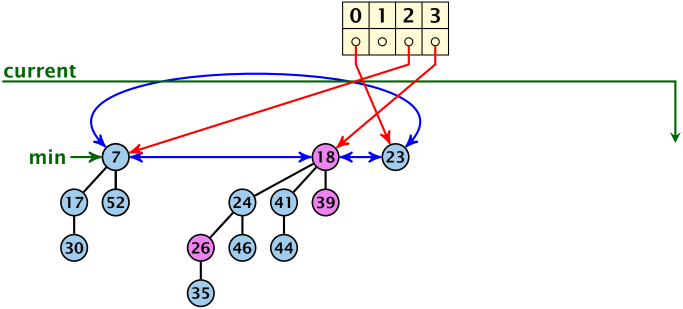

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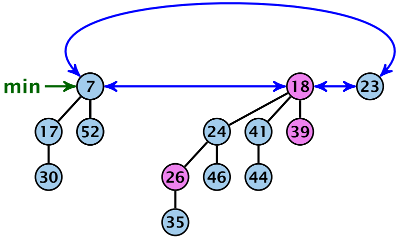

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$$
\begin{aligned}
c_1 \cdot (D_n + t) &- c \cdot (t - D_n - 1) \\
&\le (c_1 + c)D_n + (c_1 - c)t + c
\end{aligned}
$$

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$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
$$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1)$$

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for $c \ge c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

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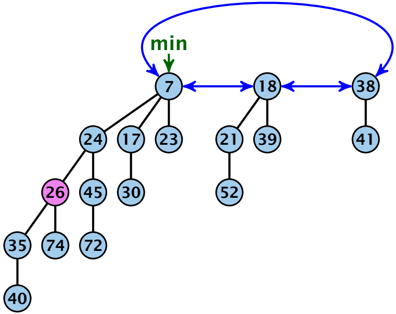
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by h. Nothing else to do.

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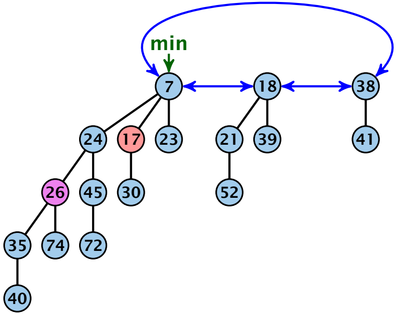

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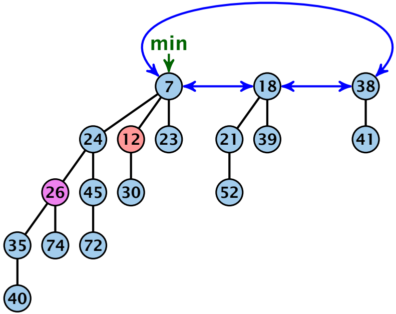

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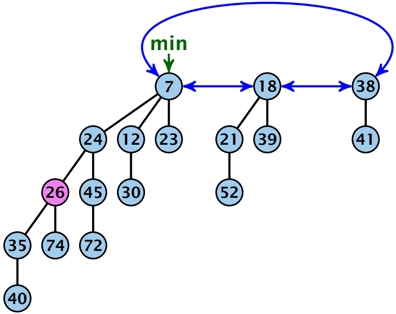

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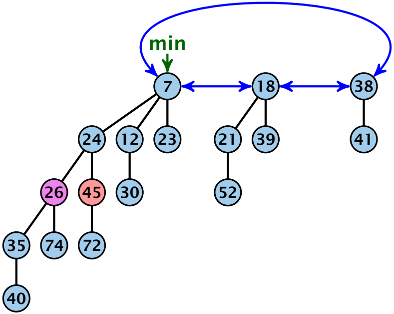

Case 2: heap-property is violated, but parent is not marked

- Decrease key-value of element x reference by h.
- If the heap-property is violated, cut the parent edge of x, and make x into a root.
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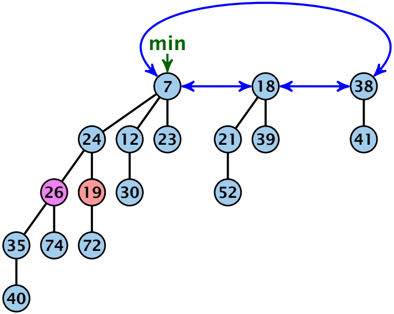

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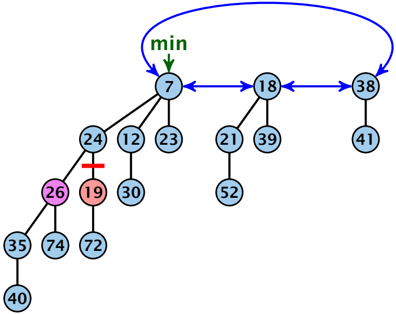

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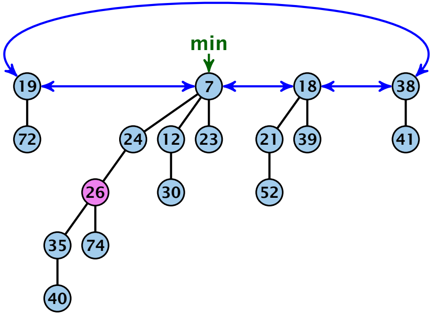

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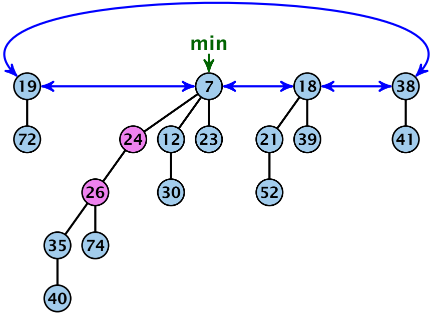

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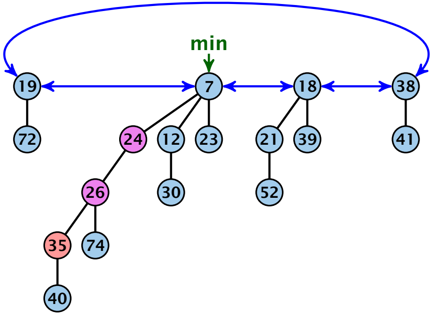

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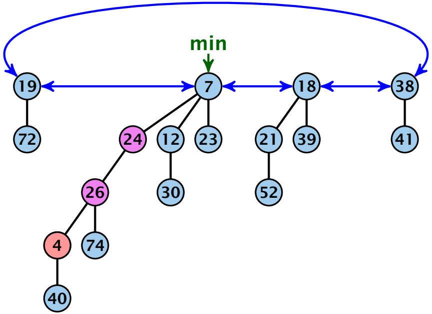

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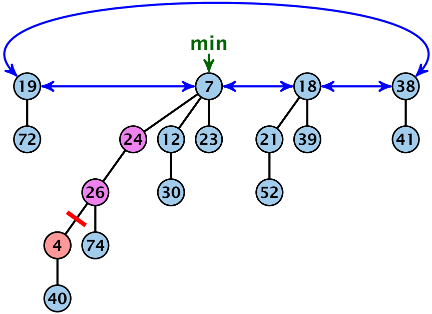

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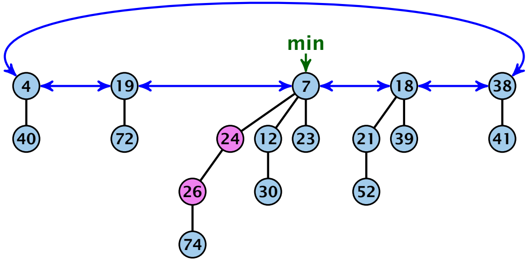

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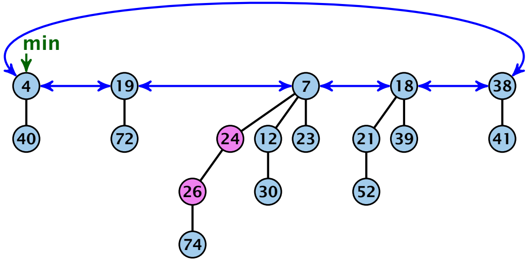

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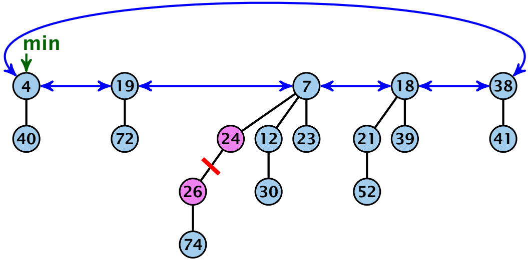

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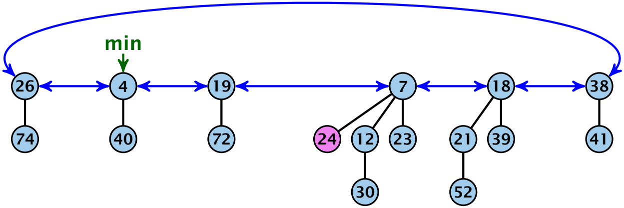

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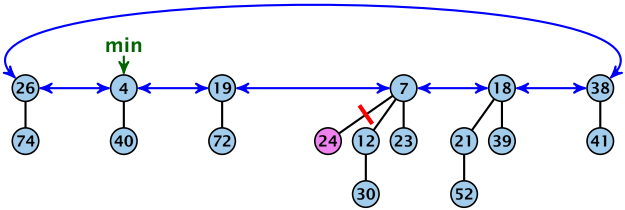

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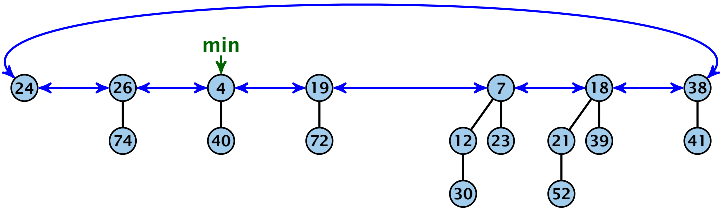

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- Decrease key-value of element x reference by h.
- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

- Decrease key-value of element x reference by h.
- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

```
p ← parent[x];
while (p is marked)
    pp ← parent[p];
    cut of p; make it into a root; unmark it;
    p ← pp;
if p is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- Constant cost for decreasing the value.
- Constant cost for each of ℓ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- $t' = t + \ell$, as every cut creates one new root.
- $m' \le m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta\Phi \le \ell + 2(-\ell + 2) = 4 - \ell$
- Amortized cost is at most

if $c \ge c_2$.

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Delete node

$\boldsymbol{H}.\,\mathbf{delete}(\boldsymbol{x})$:

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

Lemma 34

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.
- Hence, at this time $\mathrm{degree}(x) \geq i - 1$, and therefore also $\mathrm{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- Since, then y_i has lost at most one child.
- Therefore, $\mathrm{degree}(y_i) \geq i - 2$.

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$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^{k} \mathrm{size}(y_i) \\ &\geq 2 + \sum_{i=2}^{k} s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

Definition 35

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0: $1 = F_0 \geq \Phi^0 = 1$

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$

k-2,k-1→ k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi + 1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$

k-1→ k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$

9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- **$\mathcal{P}$. makeset(x):** Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.
- **$\mathcal{P}$. find(x):** Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
- **$\mathcal{P}$. union(x, y):** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

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Applications:

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- Kruskals Minimum Spanning Tree Algorithm

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9 Union Find

Algorithm 16 Kruskal-MST($G = (V, E), w$)

```
A ← ∅;
for all v ∈ V do
    v.set ← P.makeset(v.label)
sort edges in non-decreasing order of weight w
for all (u, v) ∈ E in non-decreasing order do
    if P.find(u.set) ≠ P.find(v.set) then
        A ← A ∪ {(u, v)}
        P.union(u.set, v.set)
```

List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.

- makeset(x) can be performed in constant time.
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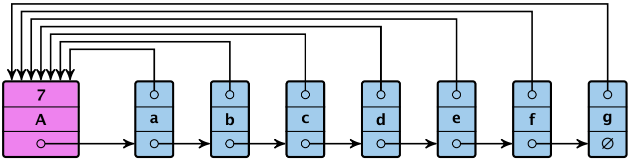

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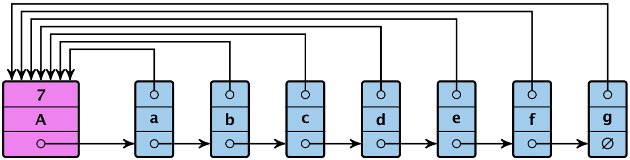

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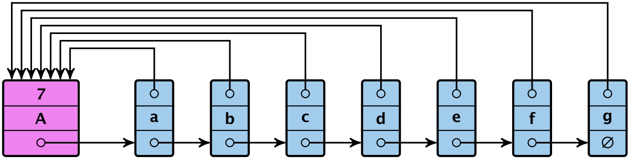

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union(x, y)

- Determine sets S_x and S_y.
- Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- Insert list S_y at the head of S_x.
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- Time: $\min\{|S_x|, |S_y|\}$.

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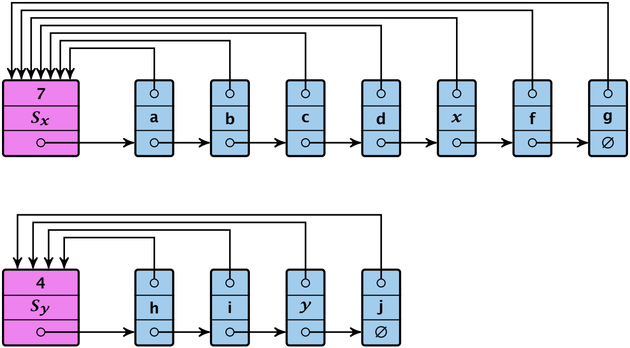

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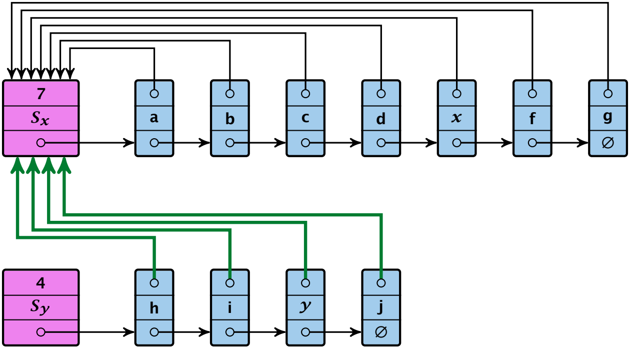

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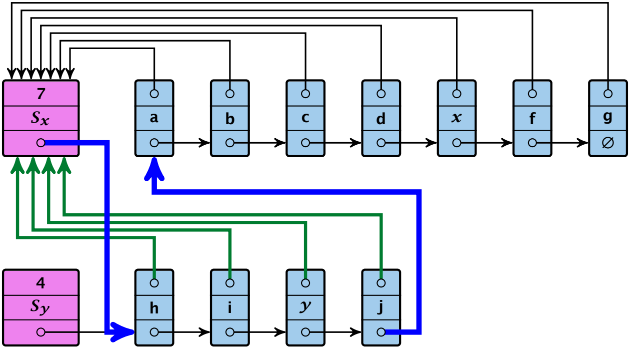

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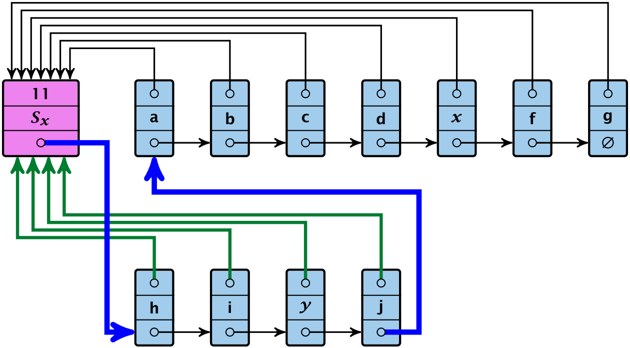

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x,y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 36

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\mathrm{find}(x)$: $\mathcal{O}(1)$.
- $\mathrm{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\mathrm{union}(x, y)$: $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
- Initially the balance on all accounts is zero.
- Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
- Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
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- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
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List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

- If $S_x = S_y$ the cost is constant; no bank accounts change.
- Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.
- Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.
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Lemma 37
An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

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Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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- Maintain nodes of a set in a tree.
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Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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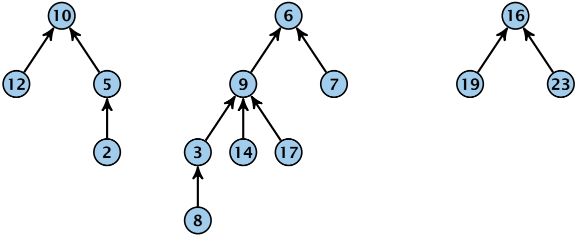

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- Create a singleton tree. Return pointer to the root.
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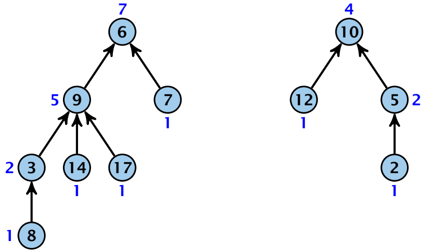

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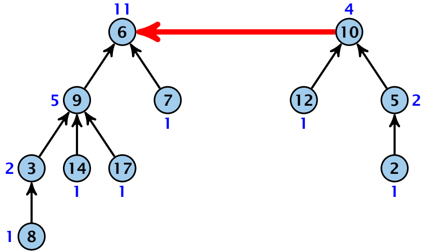

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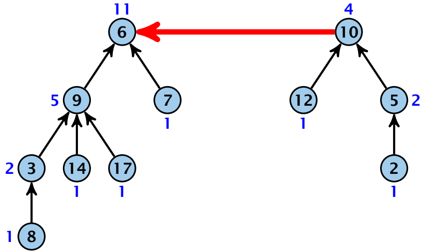

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

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find(x):

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
- Speeds up successive find-operations.

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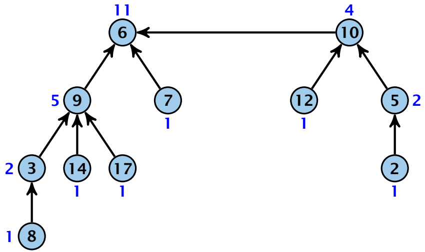

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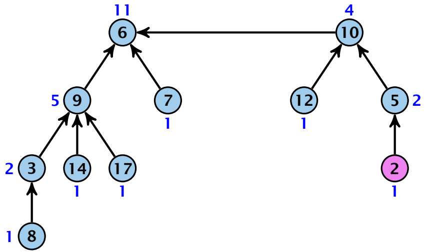

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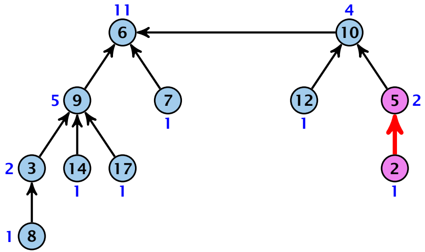

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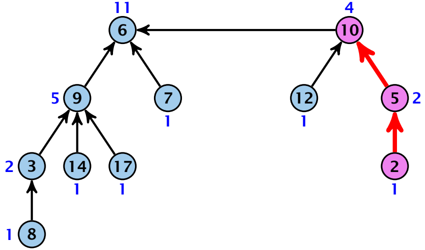

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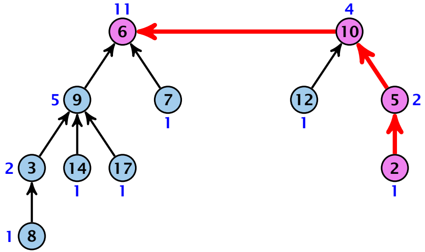

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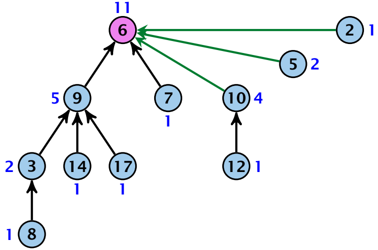

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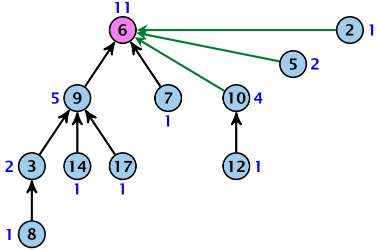

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Definitions:

- $\text{size}(v)$ = the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find operations.
- $\text{rank}(v) = \lfloor \log(\text{size}(v)) \rfloor$.
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There are at most $n/2^s$ *nodes of rank* s.

Proof.

- Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 41

Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
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- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

- A node with rank $\text{rank}(v)$ is in rank group $\log^*(\text{rank}(v))$.
- The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.
- A rank group $g \geq 1$ contains ranks $\text{tow}(g-1)+1, \ldots, \text{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

Amortized Analysis

Accounting Scheme:

- create an account for every find-operation
- create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.
- Otherwise we charge the cost to the find-account.

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Observations:

- A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).
- After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.
- After some charges to v the parent will be in a larger rank-group. $\Rightarrow v$ will never be charged again.
- The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g-1) - 1 \le \text{tow}(g)$.

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What is the total charge made to nodes?

- The total charge is at most

$$\sum_g n(g) \cdot \mathrm{tow}(g) \ ,$$

where $n(g)$ is the number of nodes in group g.

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Hence,

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\sum_g n(g)\operatorname{tow}(g) \leq n(0)\operatorname{tow}(0) + \sum_{g\geq 1} n(g)\operatorname{tow}(g)
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$$

Hence,

$$
\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g\geq 1} n(g)\,\mathrm{tow}(g) \leq n \log^*(n)
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Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

There is also a lower bound of $\Omega(\alpha(m,n))$.

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Amortized Analysis

$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1, A(x,y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- $A(0,y) = y+1$
- $A(1,y) = y+2$
- $A(2,y) = 2y+3$
- $A(3,y) = 2^{y+3}-3$
- $A(4,y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$

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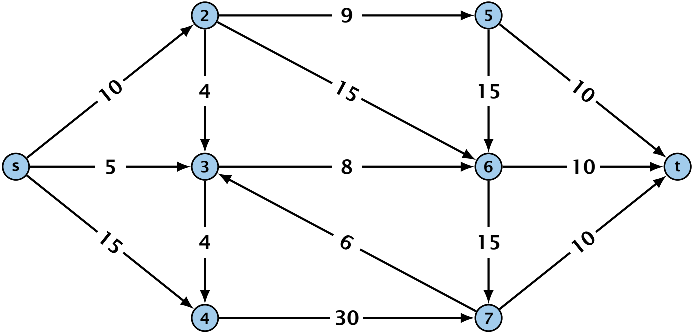
Part IV

Flows and Cuts

The following slides are partially based on slides by Kevin Wayne.

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



10 Introduction

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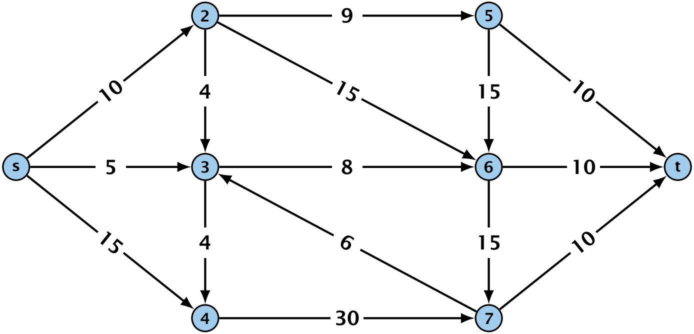

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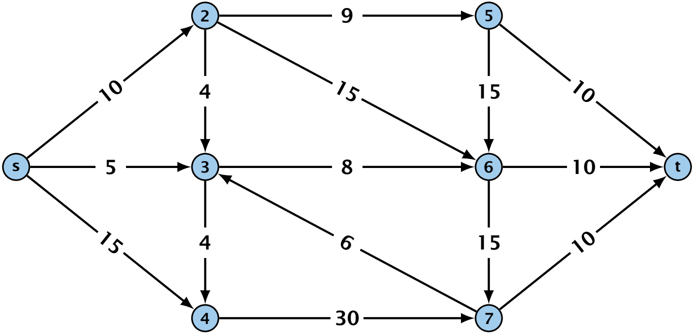

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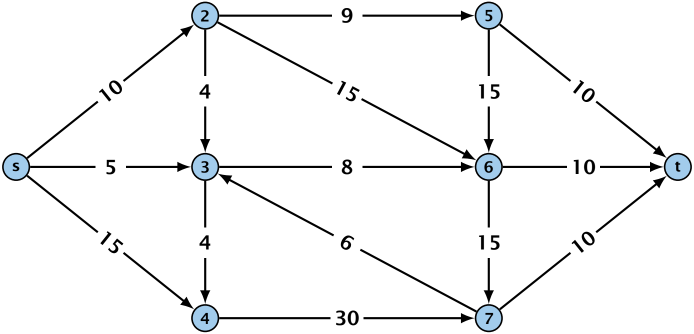

Cuts

Definition 42

An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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The capacity of a cut A is defined as

$$\mathrm{cap}(A, V \setminus A) := \sum_{e \in \mathrm{out}(A)} c(e) \ ,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

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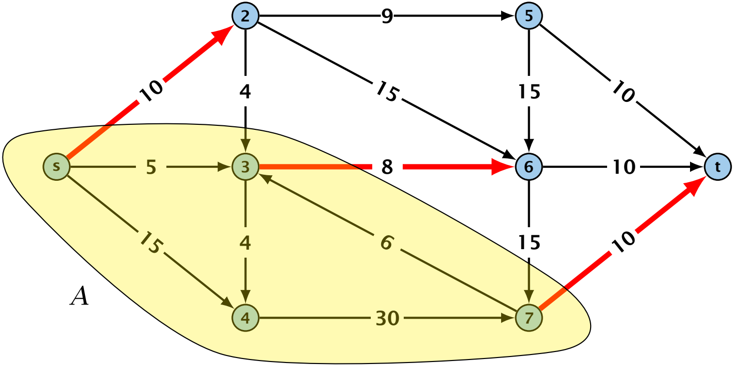
Minimum Cut Problem: Find an (s,t)-cut with minimum capacity.

Cuts

Example 44



The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 45
An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \le f(e) \le c(e) \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$

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Definition 46
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e) \ .$$

Maximum Flow Problem: Find an (s,t)-flow with maximum value.

Flows

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Flows

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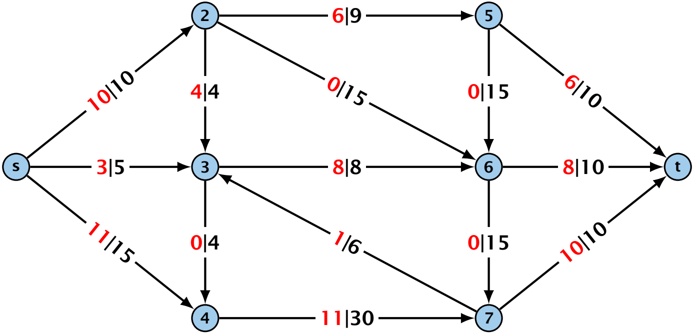

The value of the flow is $\text{val}(f) = 24$.

Flows

Lemma 48 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\operatorname{val}(f) = \sum_{e \in \operatorname{out}(A)} f(e) - \sum_{e \in \operatorname{into}(A)} f(e) \ .$$

Proof.

$\mathrm{val}(f)$

Proof.

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

Proof.

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(s)} f(e) \\ &= \sum_{e\in\mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}} \left(\sum_{e\in\mathrm{out}(v)} f(e) - \sum_{e\in\mathrm{in}(v)} f(e)\right)\end{aligned}$$

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$$\text{val}(f) = \sum_{e\in\text{out}(s)} f(e)$$

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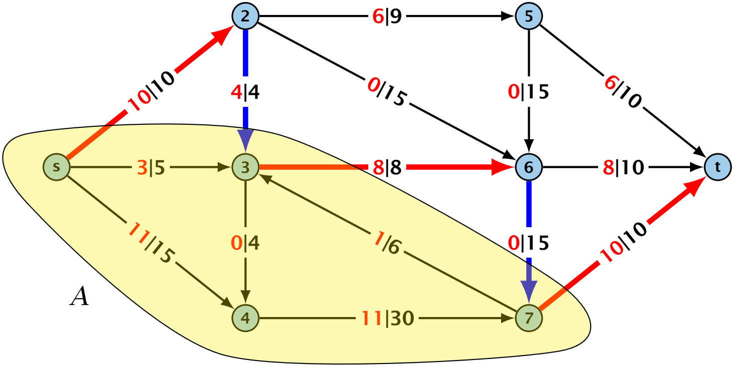
The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 49



2
5
s
3
6
t
4
7
6|9
10|10
4|4
0|15
0|15
6|10
3|5
8|8
8|10
11|15
0|4
1|6
0|15
10|10
A
11|30

Corollary 50

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

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Then f is a maximum flow.

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Suppose that there is a flow f' with larger value. Then

□

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Suppose that there is a flow f' with larger value. Then

$$\mathrm{cap}(A, V \setminus A) < \mathrm{val}(f')$$

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Suppose that there is a flow f' with larger value. Then

$$\begin{aligned}\mathrm{cap}(A, V \setminus A) &< \mathrm{val}(f') \\ &= \sum_{e \in \mathrm{out}(A)} f'(e) - \sum_{e \in \mathrm{into}(A)} f'(e)\end{aligned}$$

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&\leq \sum_{e \in \mathrm{out}(A)} f'(e)
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Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
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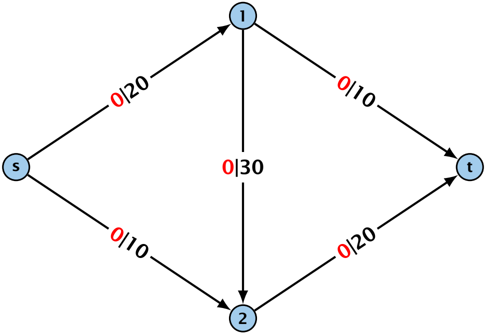

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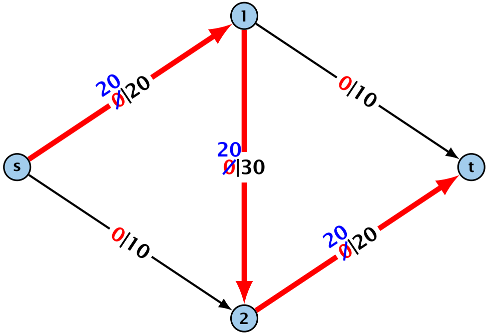

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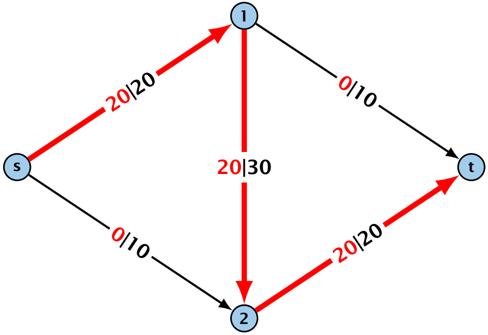

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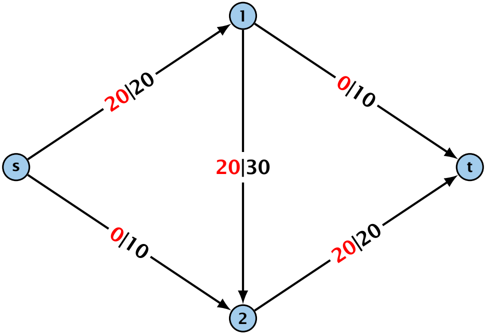

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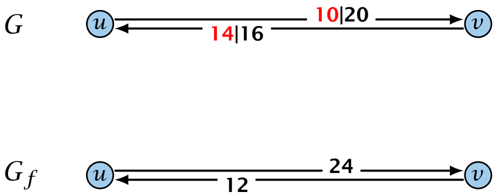

Augmenting Path Algorithm

Definition 51

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 1 FordFulkerson($G = (V, E, c)$)

1. Initialize $f(e) \leftarrow 0$ for all edges.
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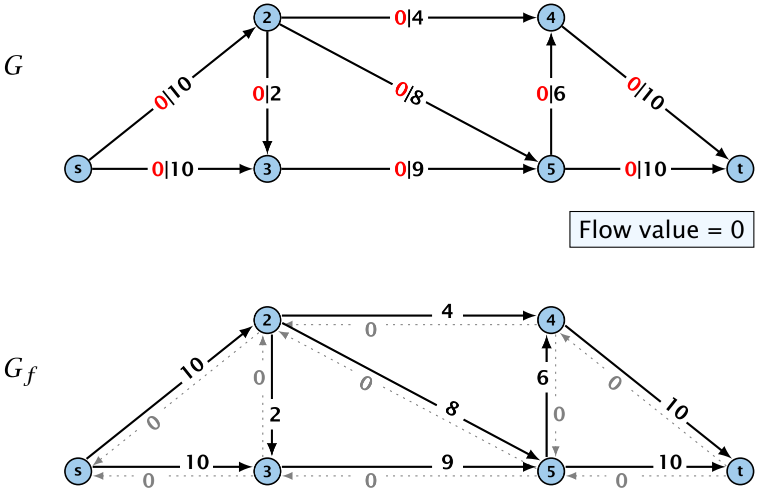

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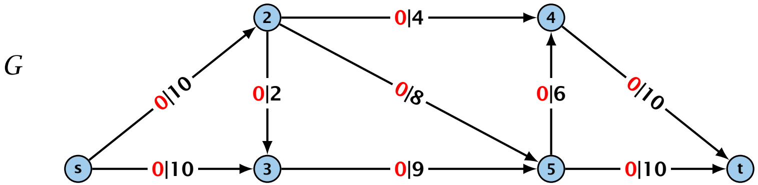
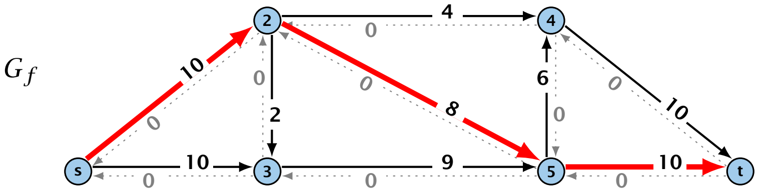
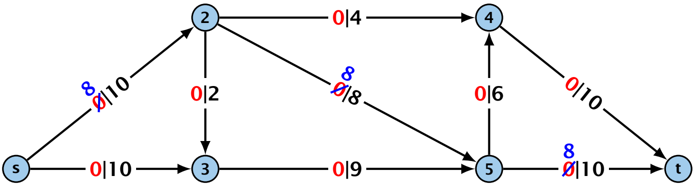


Flow value = 0



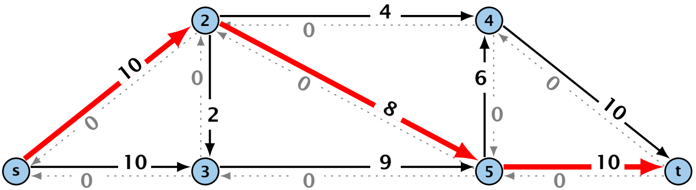
Augmenting Path Algorithm

G



Flow value = 8

G_f



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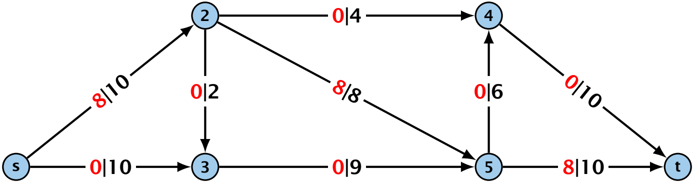

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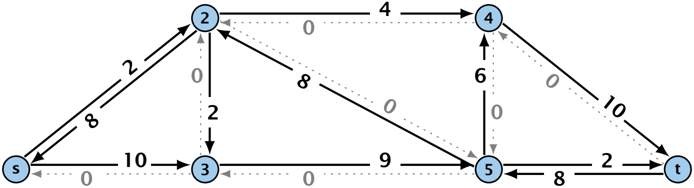

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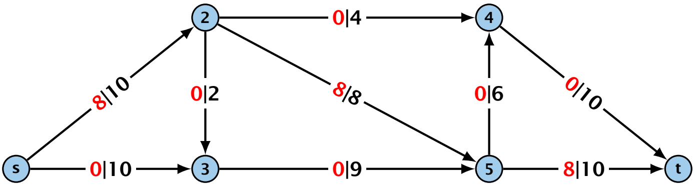

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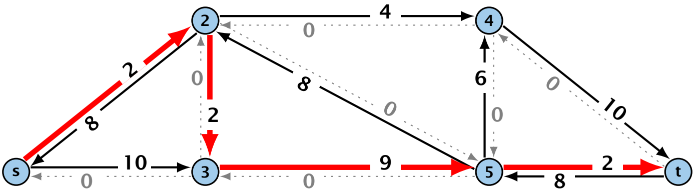

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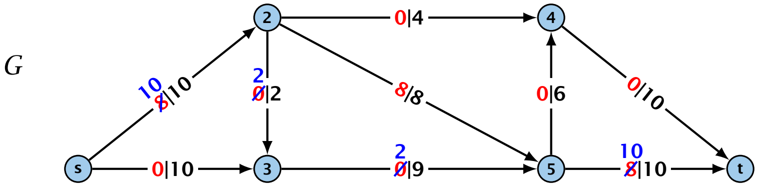
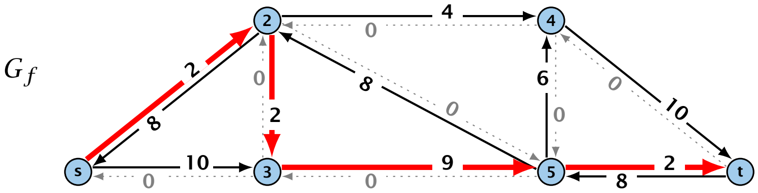


Flow value = 10



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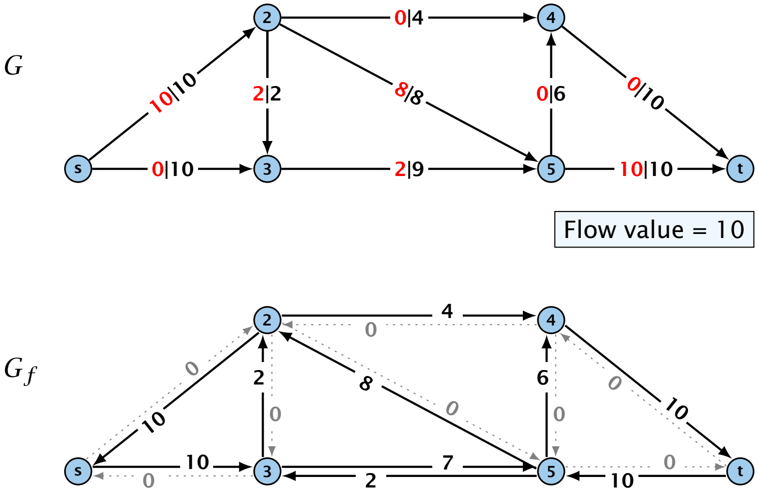

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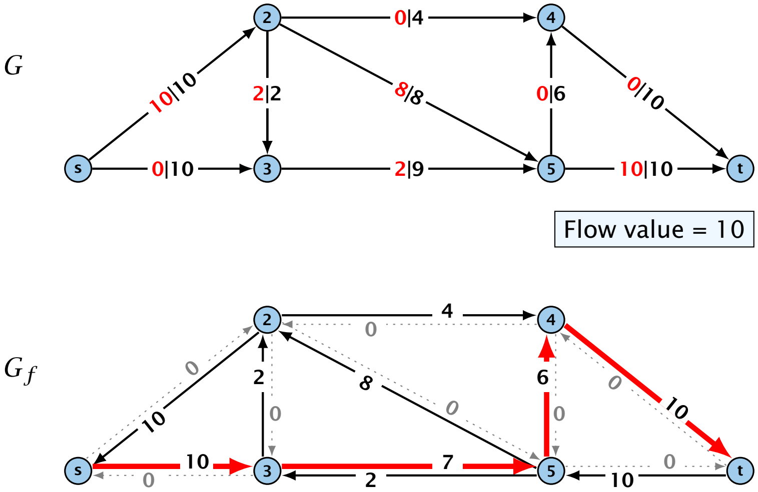

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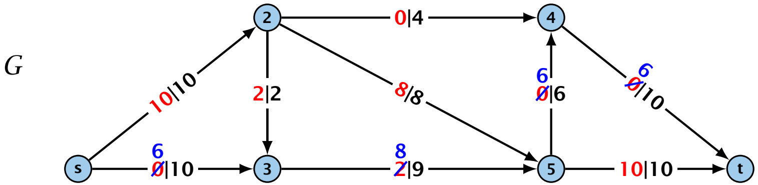
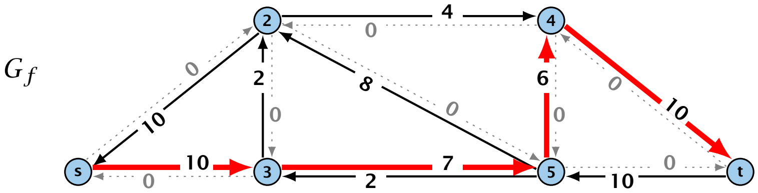


Flow value = 16



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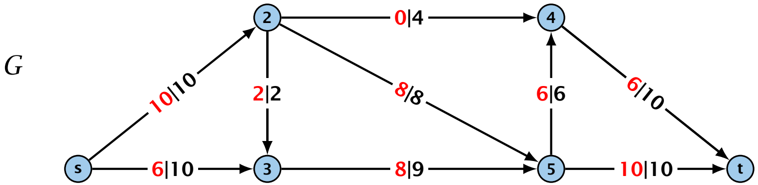

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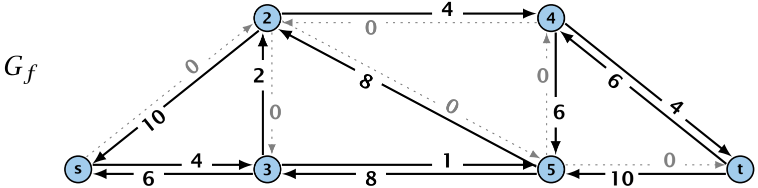

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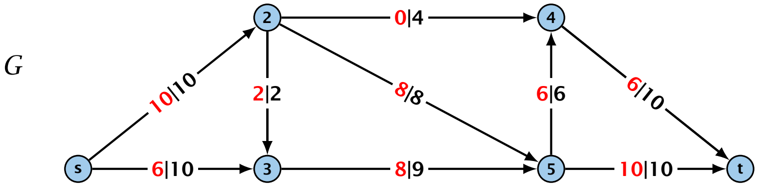

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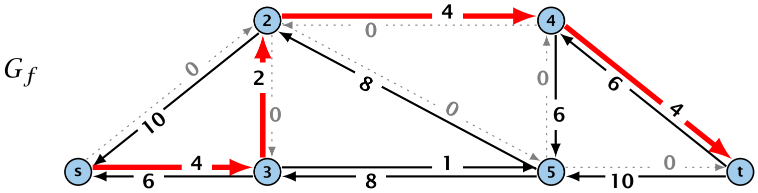

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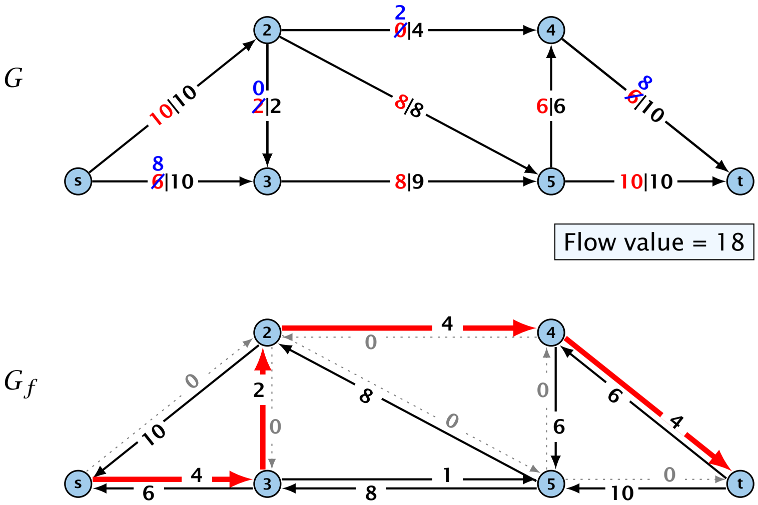

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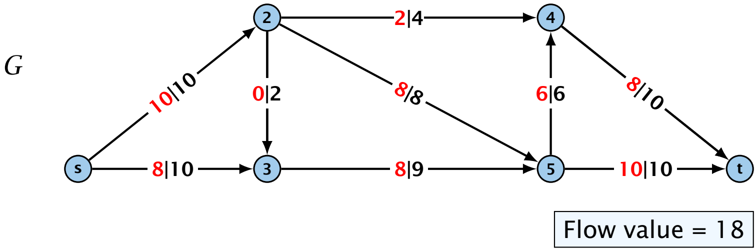

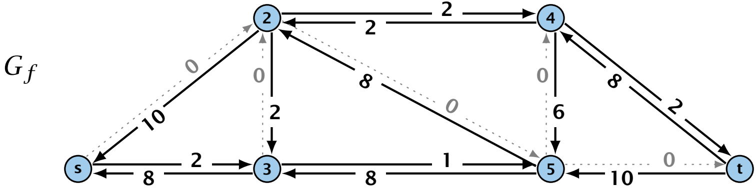

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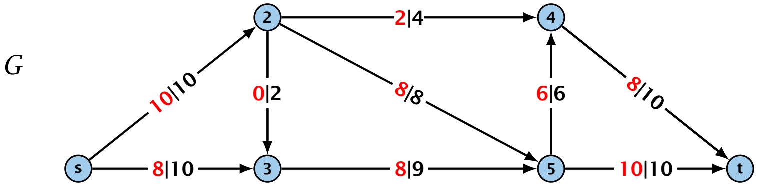
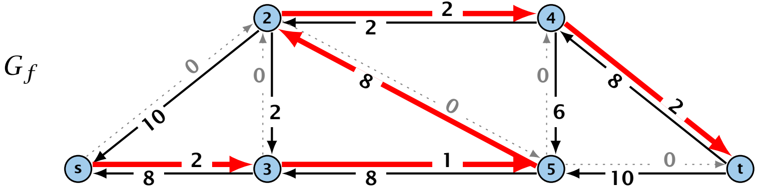


Flow value = 18



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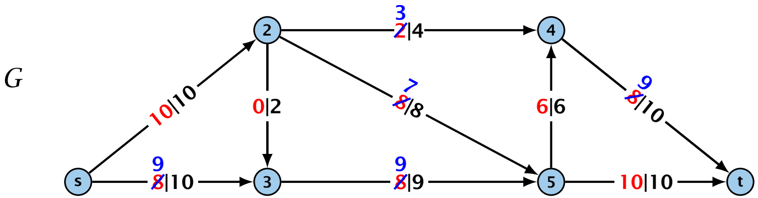

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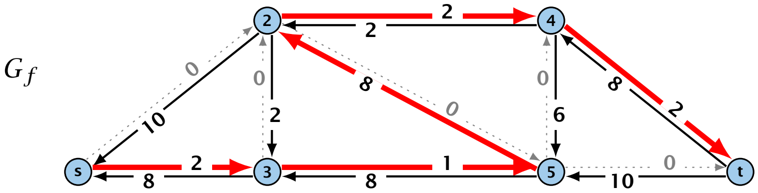

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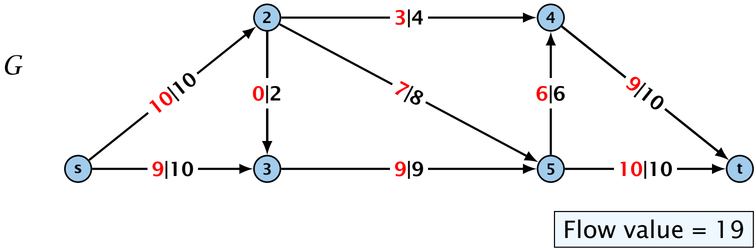

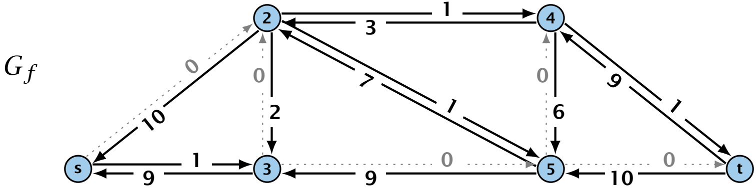

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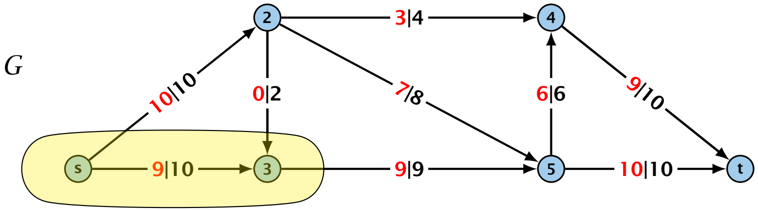

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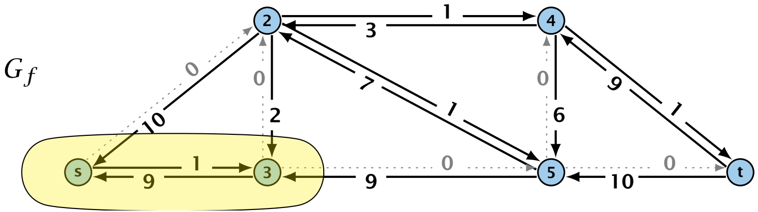

Augmenting Path Algorithm

Theorem 52

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 53

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
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Augmenting Path Algorithm

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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The algorithm terminates in at most $\mathrm{val}(f^*) \le nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$.

Theorem 55

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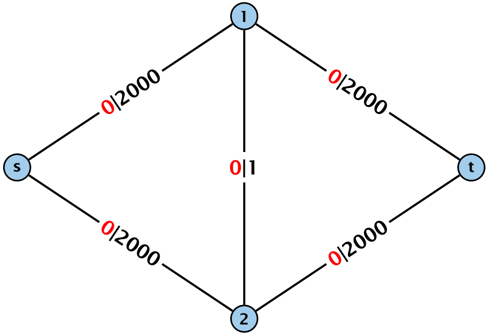
A Bad Input

Problem: The running time may not be polynomial.



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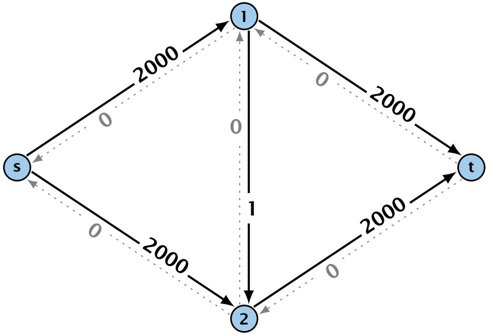

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Can we tweak the algorithm so that the running time is polynomial in the input length?

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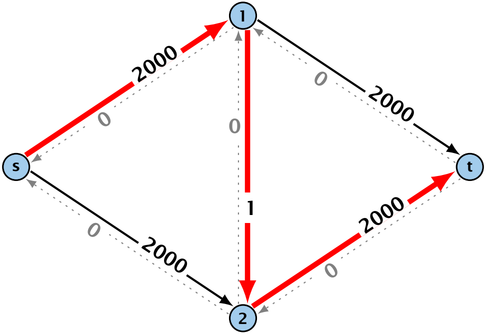

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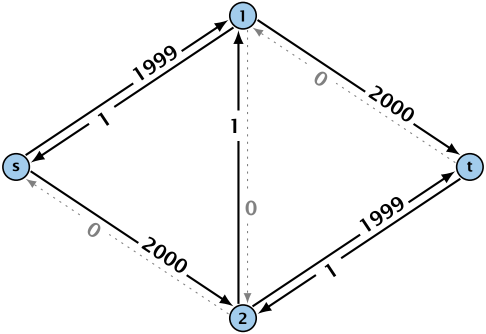

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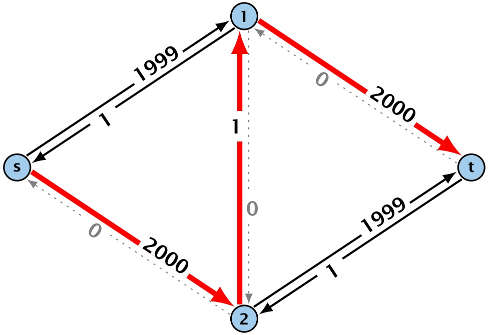

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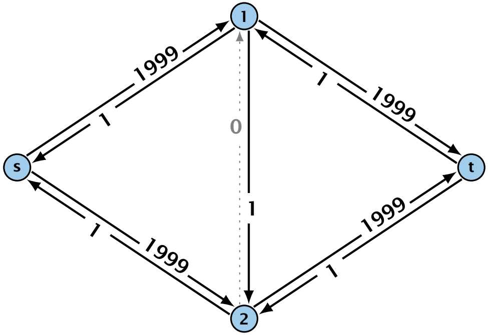

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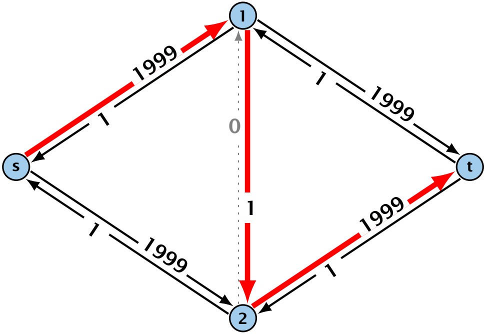

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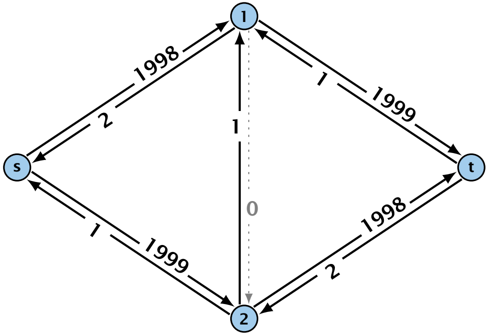

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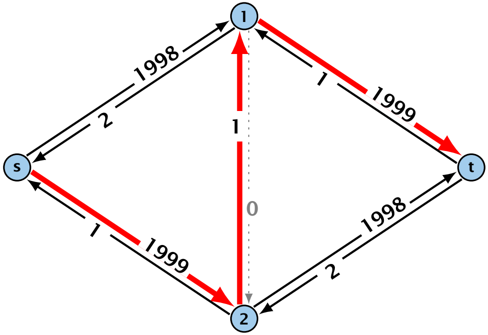

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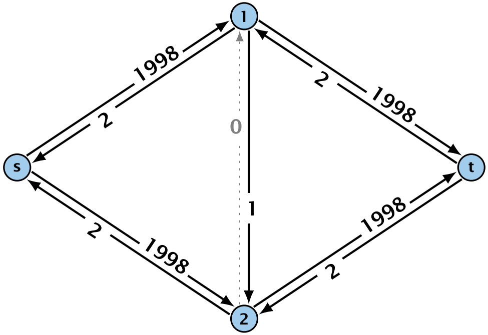

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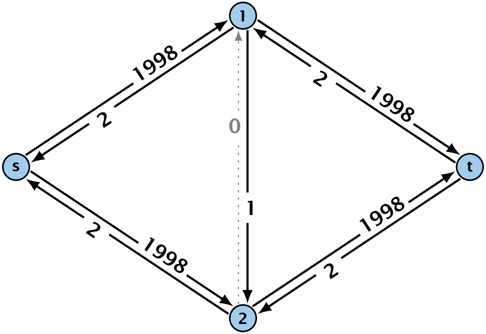

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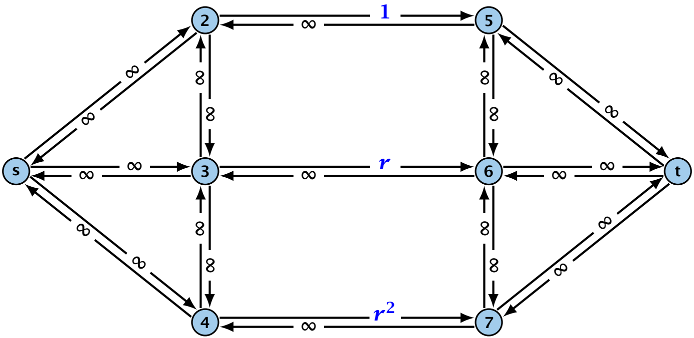
Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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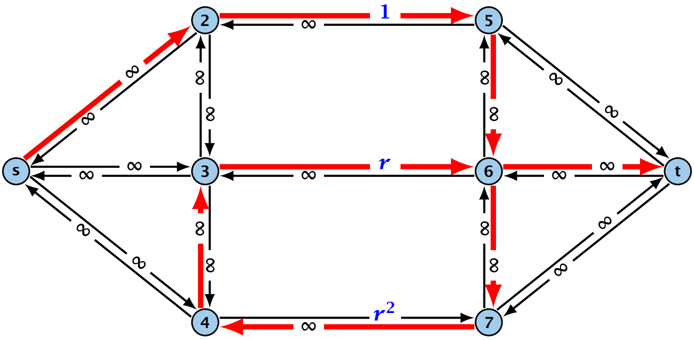

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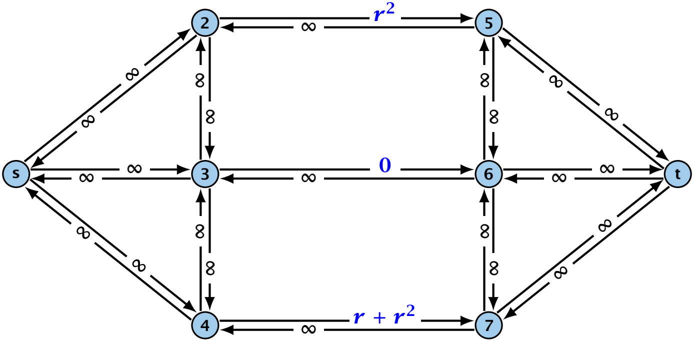

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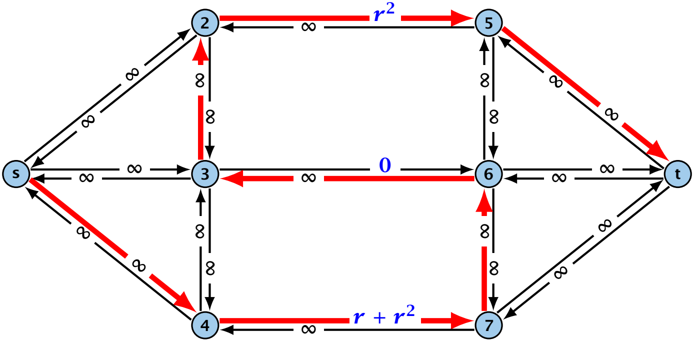

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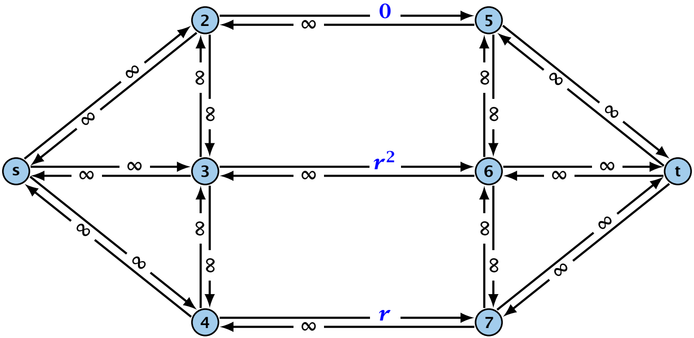

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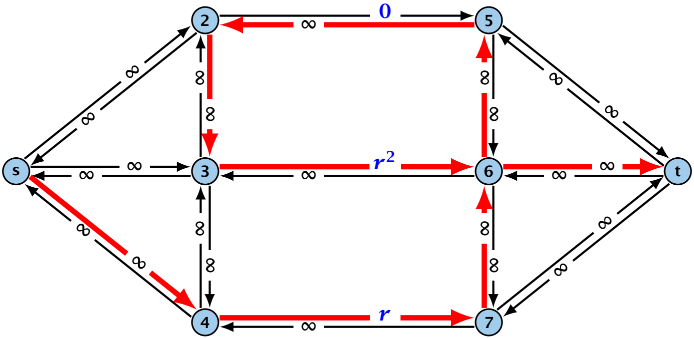

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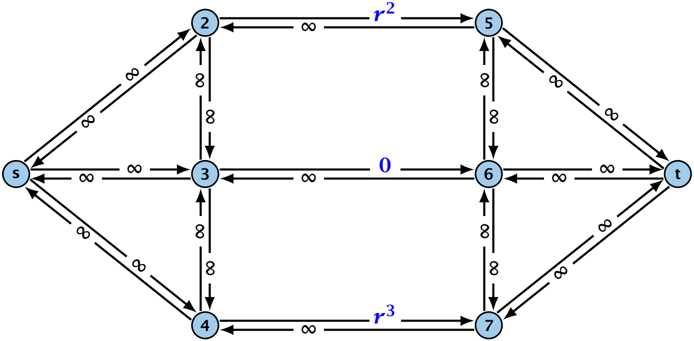

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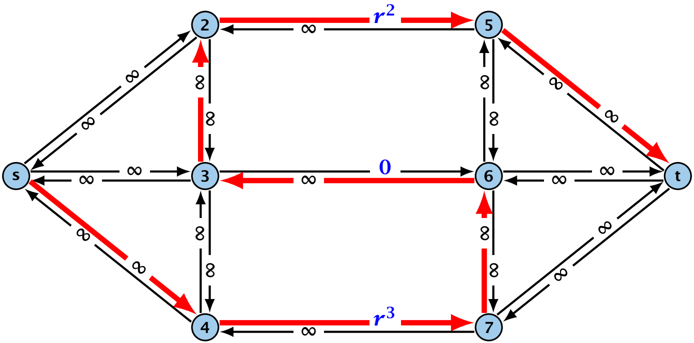

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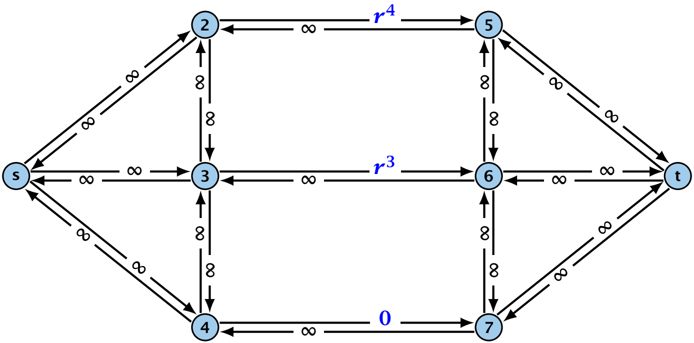

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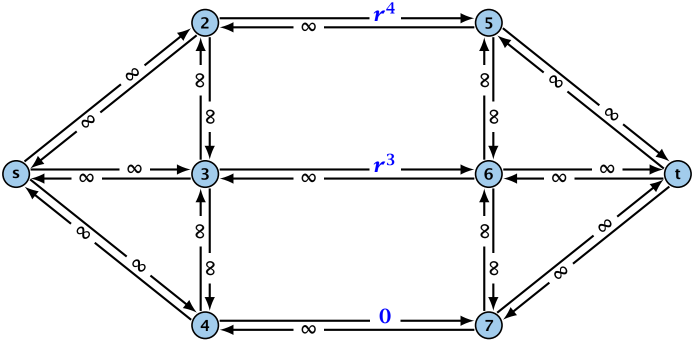

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

Lemma 56
The length of the shortest augmenting path never decreases.

Lemma 57
After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

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The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

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- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
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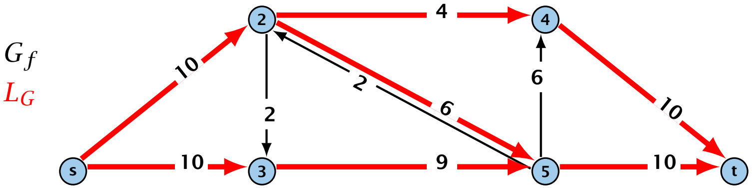

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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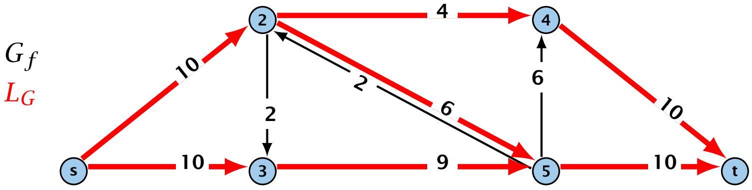

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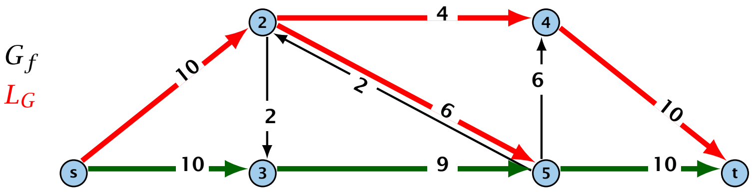

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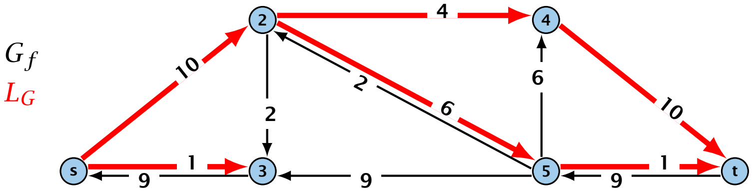

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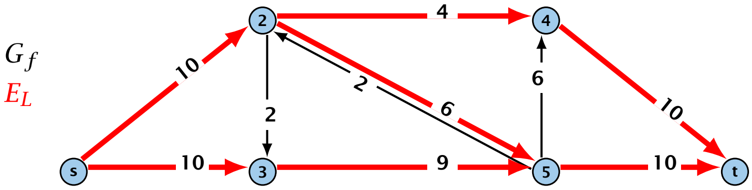

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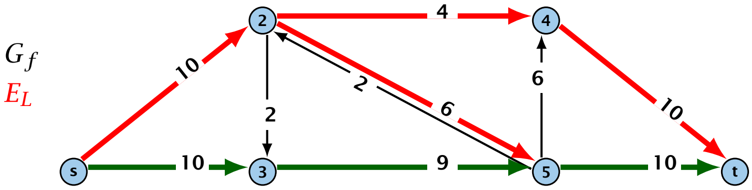

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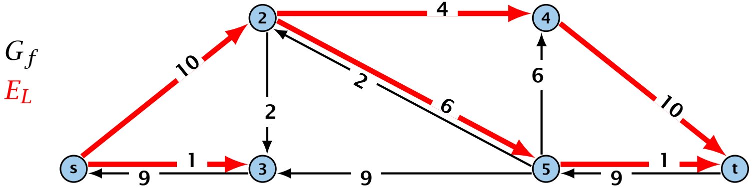

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The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 60 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

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We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s-t path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L.

When E_L does not contain an s-t path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

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Suppose that the initial distance between s and t in G_f is k.

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Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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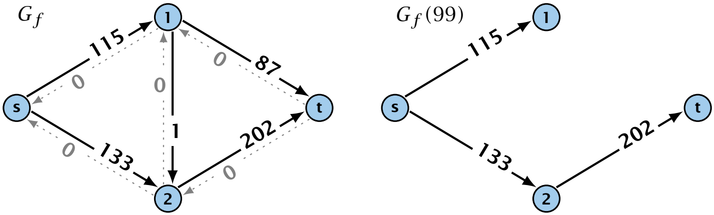
Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.



Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

```
foreach e ∈ E do f_e ← 0;
Δ ← 2^⌈log₂ C⌉
while Δ ≥ 1 do
    G_f(Δ) ← Δ-residual graph
    while there is augmenting path P in G_f(Δ) do
        f ← augment(f, c, P)
        update(G_f(Δ))
    Δ ← Δ/2
return f
```

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All capacities are integers between 1 and C.

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All flows and capacities are/remain integral throughout the algorithm.

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The algorithm computes a maxflow:

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- therefore after the last phase there are no augmenting paths anymore

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Correctness:
The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

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There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

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Proof: less obvious, but simple:

- There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.
- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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There are at most $2m$ *augmentations per scaling-phase.*

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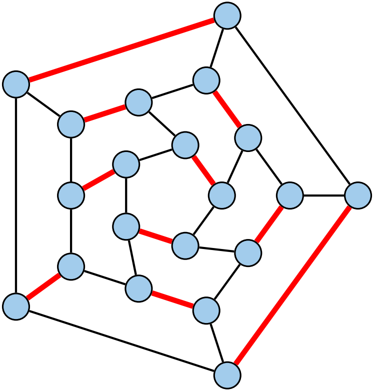
Theorem 64

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
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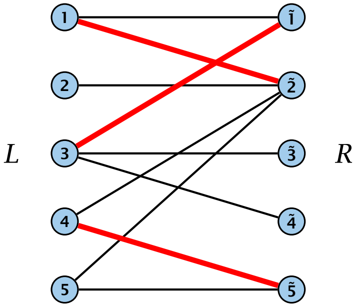

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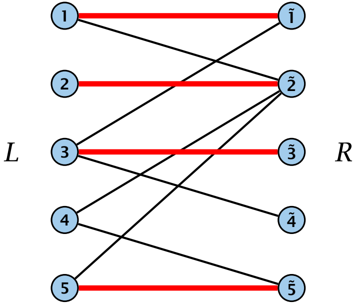
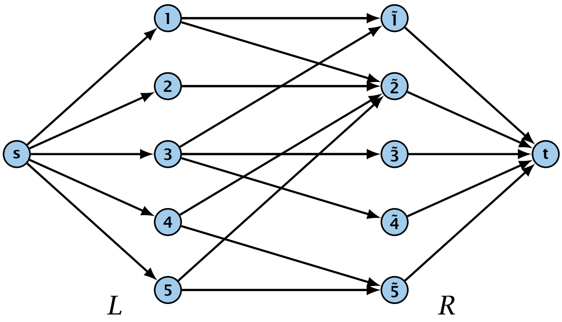


Maxflow Formulation

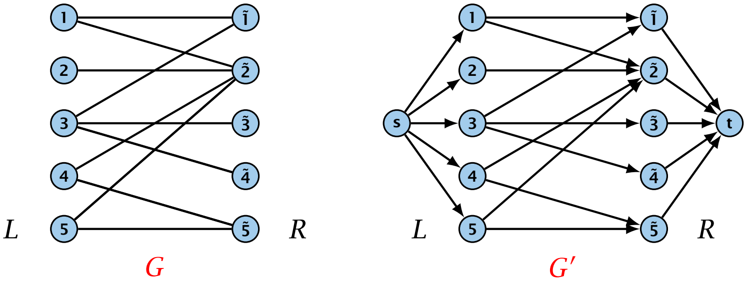
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s,t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.



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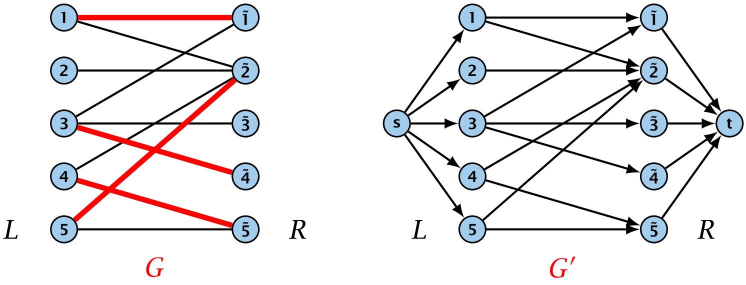

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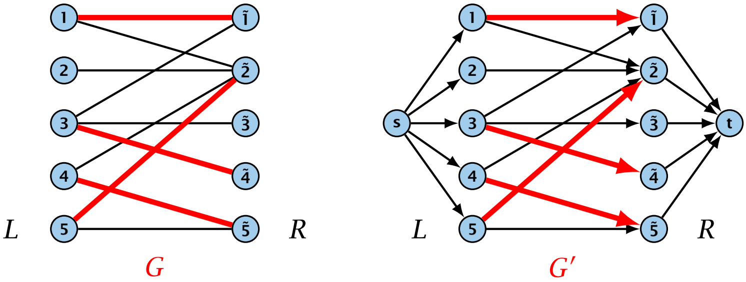

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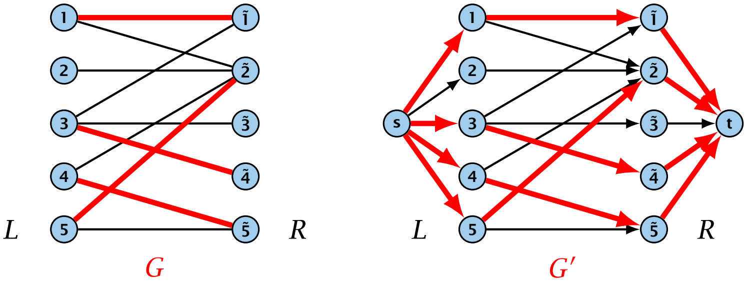

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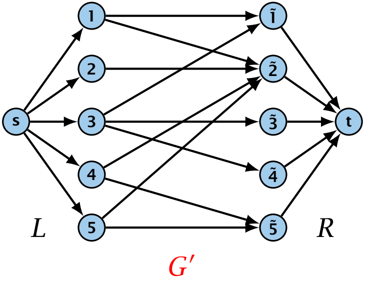
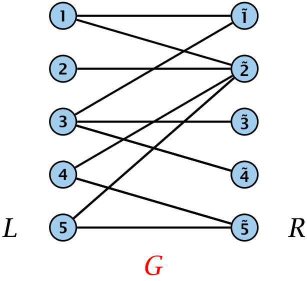
- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.





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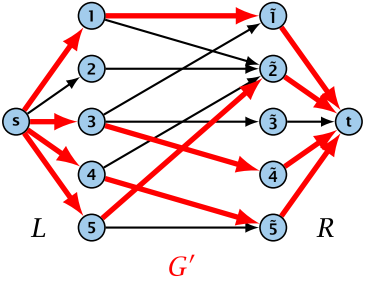

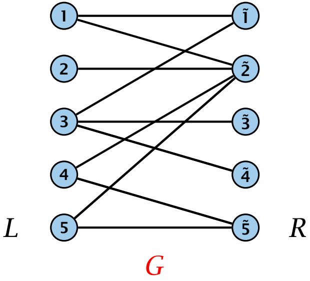

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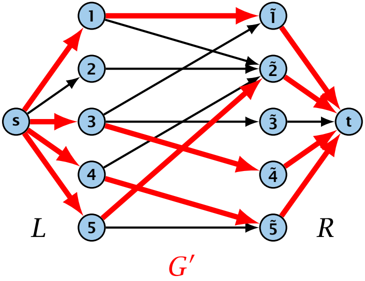
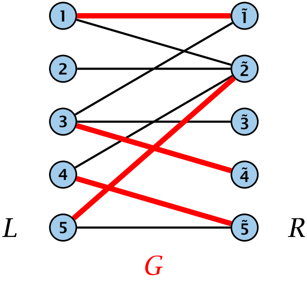




12.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

Baseball Elimination

team i	wins w_i	losses ℓ_i	remaining games Atl	Phi	NY	Mon
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

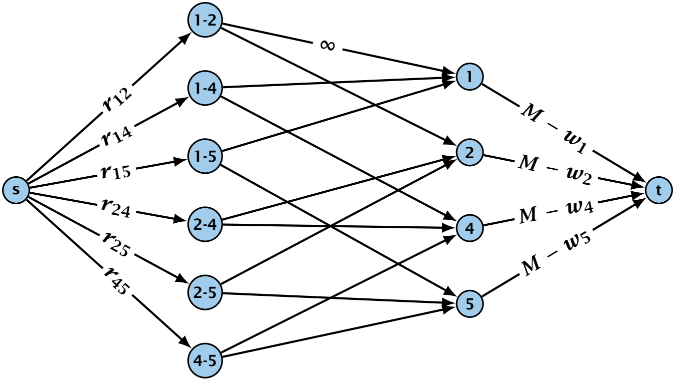
Baseball Elimination

Formal definition of the problem:

- Given a set S of teams, and one specific team $z \in S$.
- Team x has already won w_x games.
- Team x still has to play team y, r_{xy} times.
- Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.



Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i<j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 65

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i<j} r_{ij}$.

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$$r(S \setminus \{z\})$$

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$$\begin{aligned} r(S \setminus \{z\}) &> \mathrm{cap}(A, V \setminus A) \\ &\geq \sum_{i<j:\, i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \end{aligned}$$

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- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

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Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

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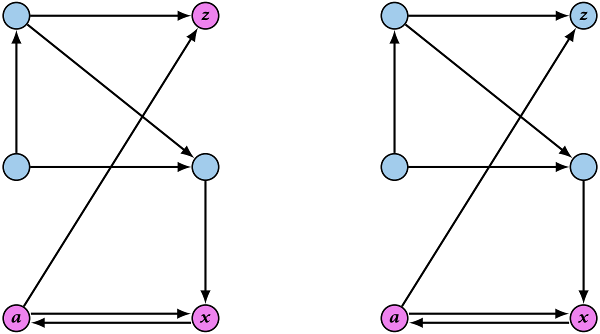
The prerequisite graph:

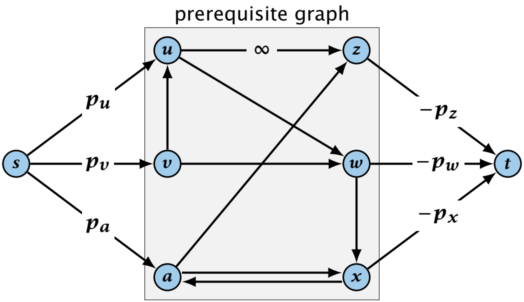
- $\{x, a, z\}$ is a feasible subset.
- $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- Edges in the prerequisite graph get infinite capacity.
- Add edge (s, v) with capacity p_v for nodes v with positive profit.
- Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



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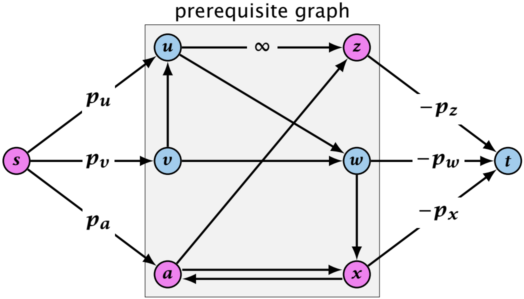

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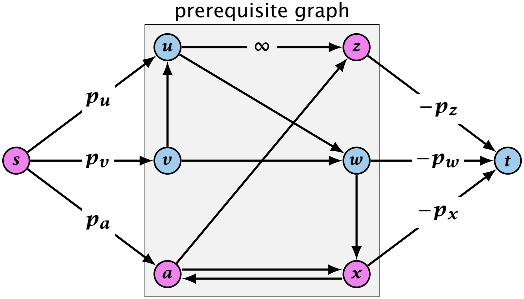

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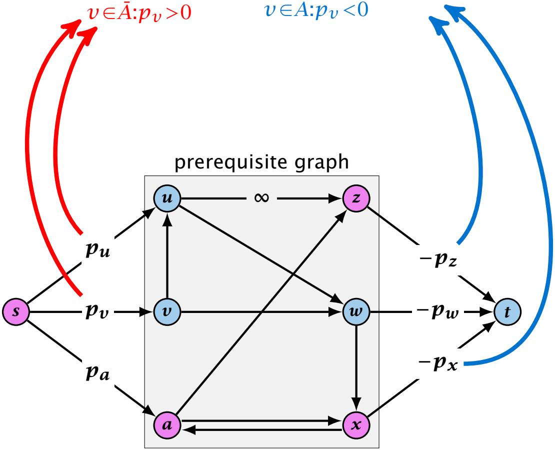
- $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$



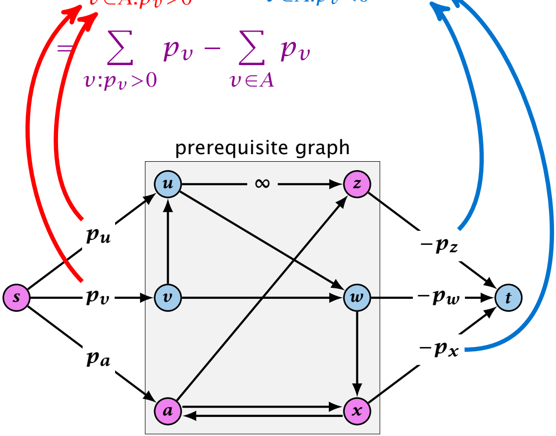
Theorem 66

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- A is feasible because of capacity infinity edges.
- $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$

$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$



Preflows

Definition 67
An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \leq f(e) \leq c(e) \; .$$
(capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) \; .$$

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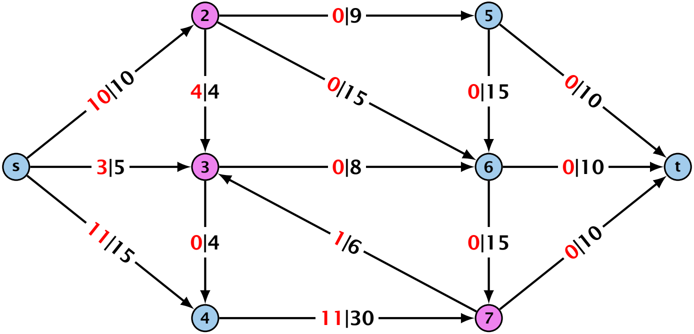
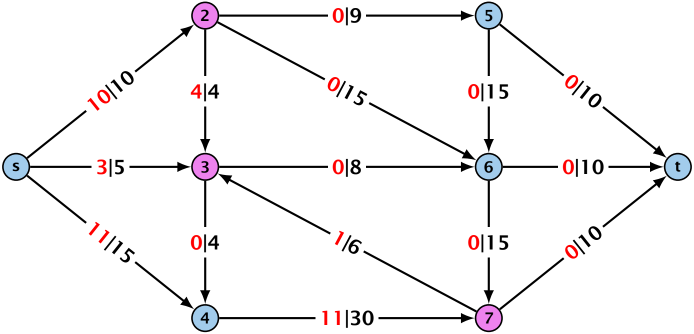
Preflows

Example 68



Preflows

Example 68



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

Preflows

Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \le \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)

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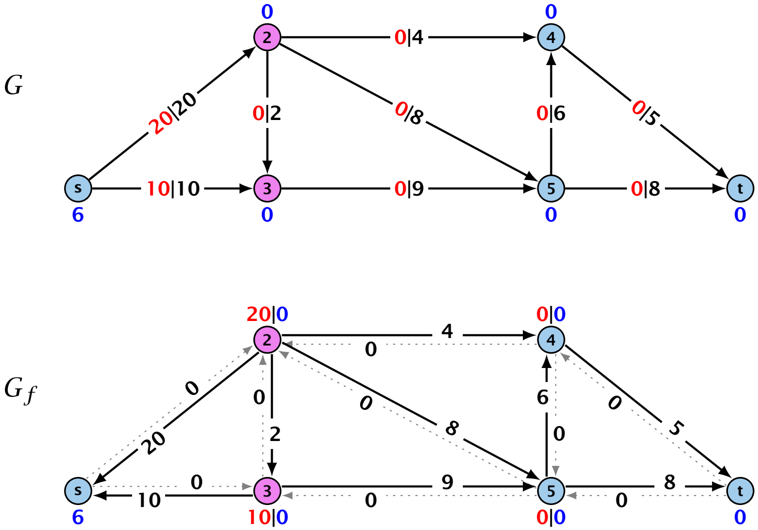
Intuition:
The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows


G
0
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G_f
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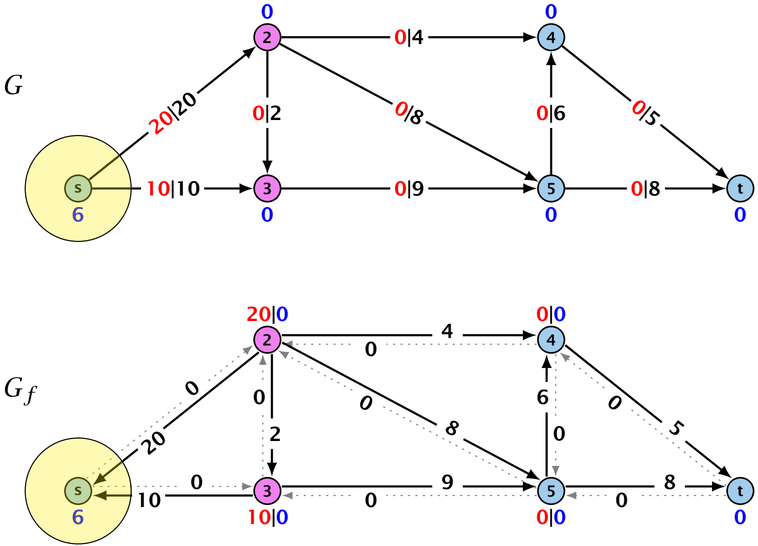
Preflows


G
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Preflows

Preflows

Lemma 69

A preflow that has a valid labelling saturates a cut.

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Proof:

- There are n nodes but $n + 1$ different labels from $0, \dots, n$.

Preflows

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- There are n nodes but $n+1$ different labels from $0,\ldots,n$.
- There must exist a label $d \in \{0,\ldots,n\}$ such that none of the nodes carries this label.

Preflows

Lemma 69

A preflow that has a valid labelling saturates a cut.

Proof:

- There are n nodes but $n+1$ different labels from $0,\ldots,n$.
- There must exist a label $d\in\{0,\ldots,n\}$ such that none of the nodes carries this label.
- Let $A=\{v\in V\mid \ell(v)>d\}$ and $B=\{v\in V\mid \ell(v)<d\}$.

Preflows

Lemma 69

A *preflow* *that has a valid labelling saturates a cut.*

Proof:

- There are n nodes but $n+1$ different labels from $0,\dots,n$.
- There must exist a label $d \in \{0,\dots,n\}$ such that none of the nodes carries this label.
- Let $A = \{v \in V \mid \ell(v) > d\}$ and $B = \{v \in V \mid \ell(v) < d\}$.
- We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f; this means that (A,B) is a saturated cut.

Preflows

Lemma 69

A preflow that has a valid labelling saturates a cut.

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- There are n nodes but $n+1$ different labels from $0, \dots, n$.
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- We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f; this means that (A, B) is a saturated cut.

Lemma 70

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

Push Relabel Algorithms

Idea:

- start with some preflow and some valid labelling

Push Relabel Algorithms

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling

Push Relabel Algorithms

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

- saturating push: $\min\{f(u), c_f(e)\} = c_f(e)$
  the arc e is deleted from the residual graph
- non-saturating push: $\min\{f(u), c_f(e)\} = f(u)$
  the node u becomes inactive

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Push Relabel Algorithms

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The relabel operation

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- Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.

Push Relabel Algorithms

The relabel operation

Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:
We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u,v) in the residual graph $\ell(u) \leq \ell(v) + 1$.
- An arc (u,v) in residual graph is admissible if $\ell(u) = \ell(v) + 1$.
- A saturating push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A non-saturating push along $e = (u,v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 3 maxflow(G, s, t, c)

```
find initial preflow f
while there is active node u do
    if there is admiss. arc e out of u then
        push(G, e, f, c)
    else
        relabel(u)
return f
```

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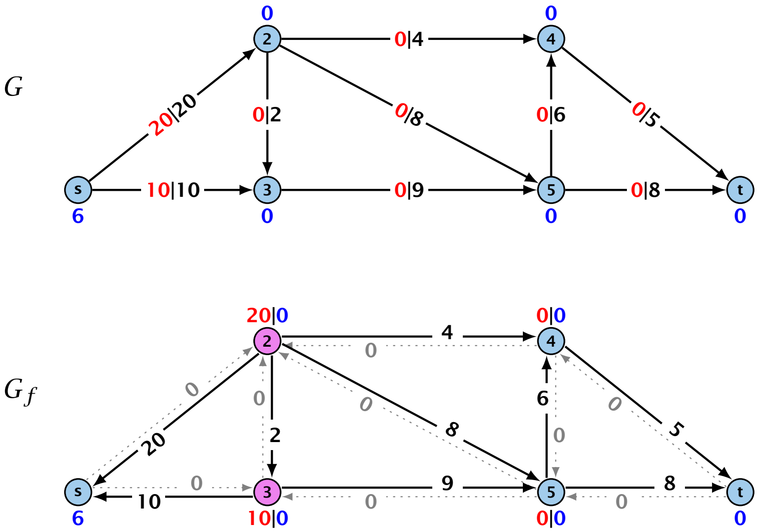
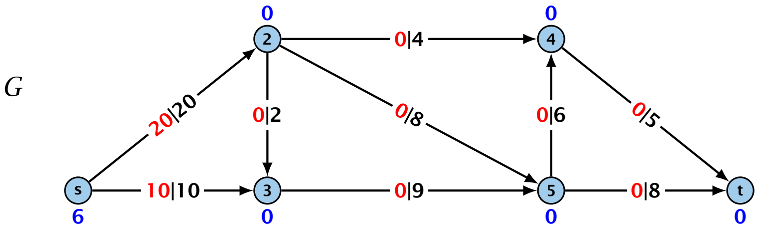
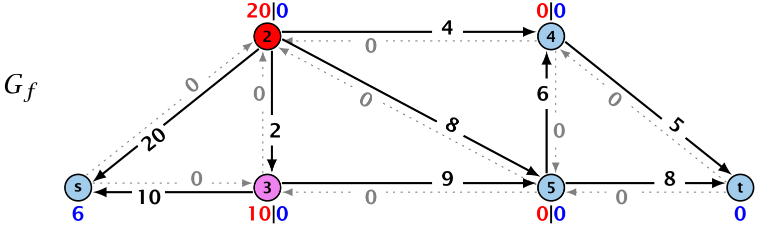
In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push Algorithm



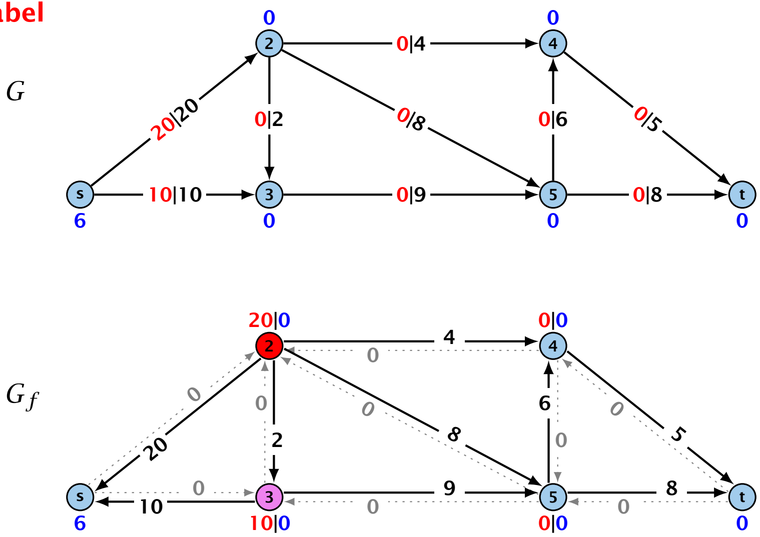
Preflow Push Algorithm



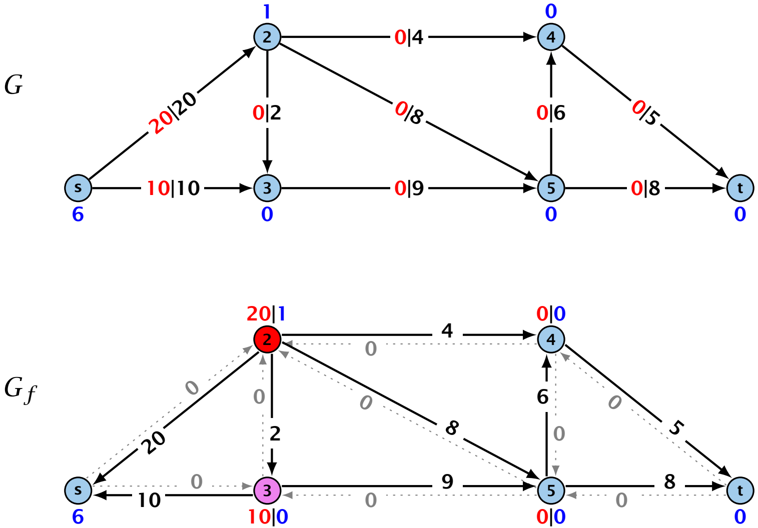


Preflow Push Algorithm

relabel

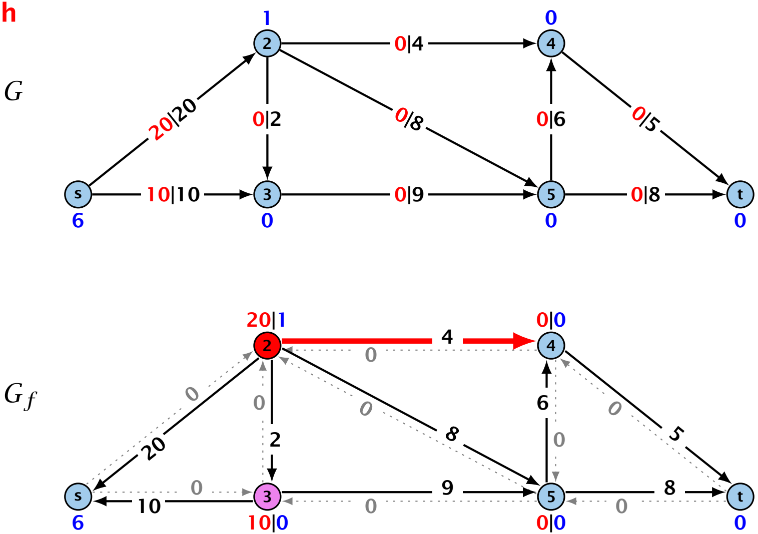


Preflow Push Algorithm

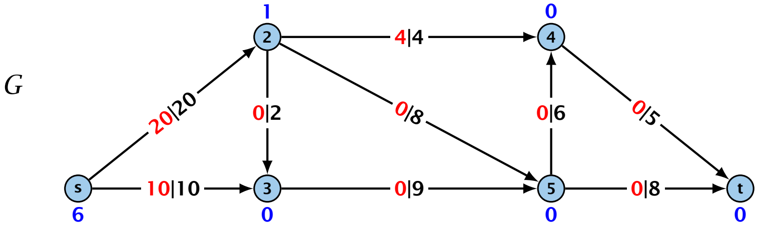


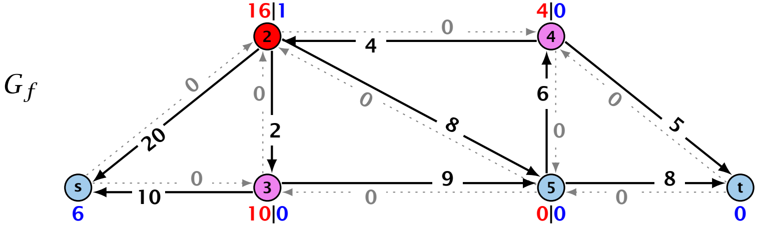
Preflow Push Algorithm

push



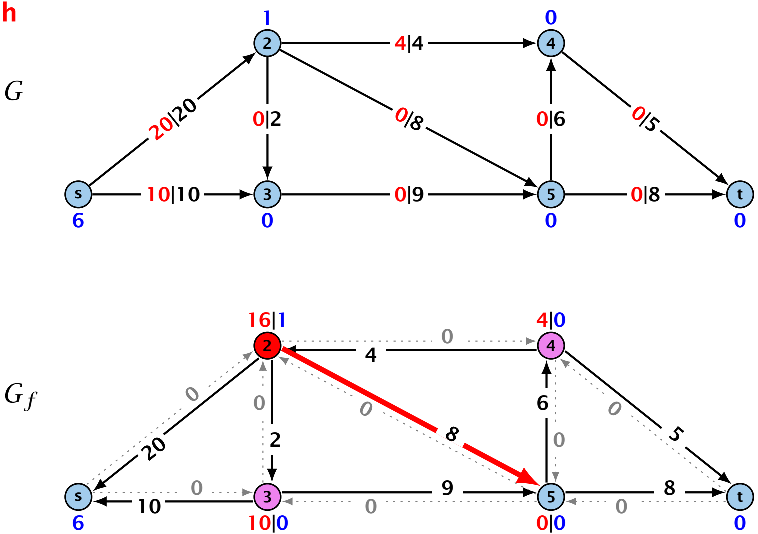
Preflow Push Algorithm



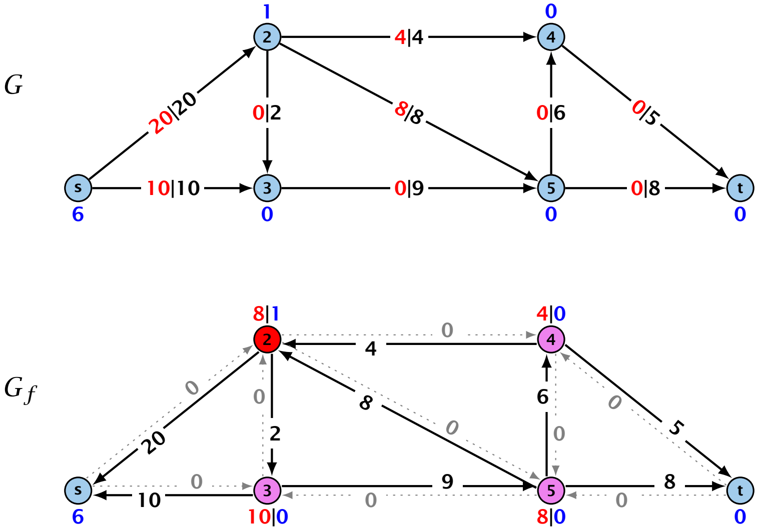


Preflow Push Algorithm

push

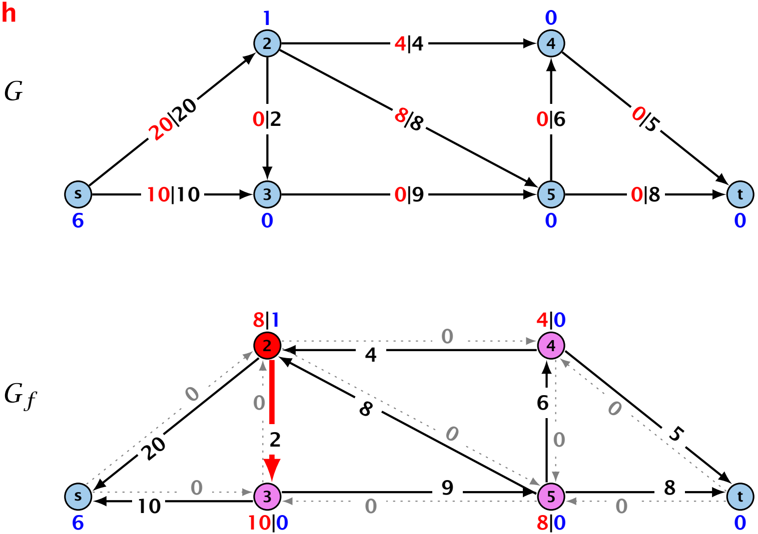


Preflow Push Algorithm

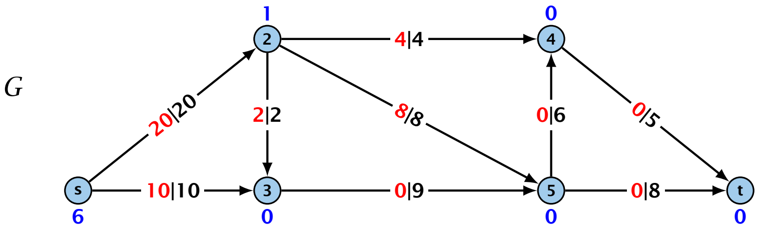


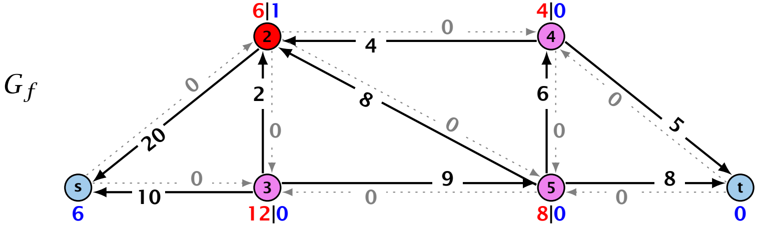
Preflow Push Algorithm

push



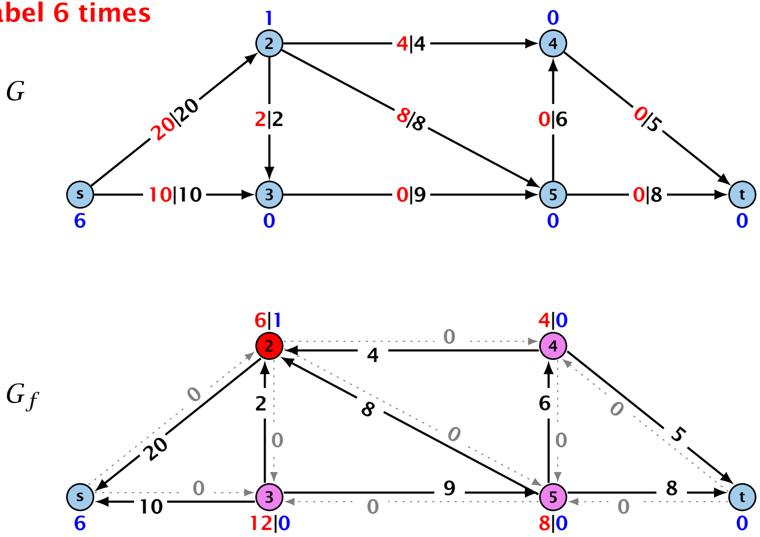
Preflow Push Algorithm



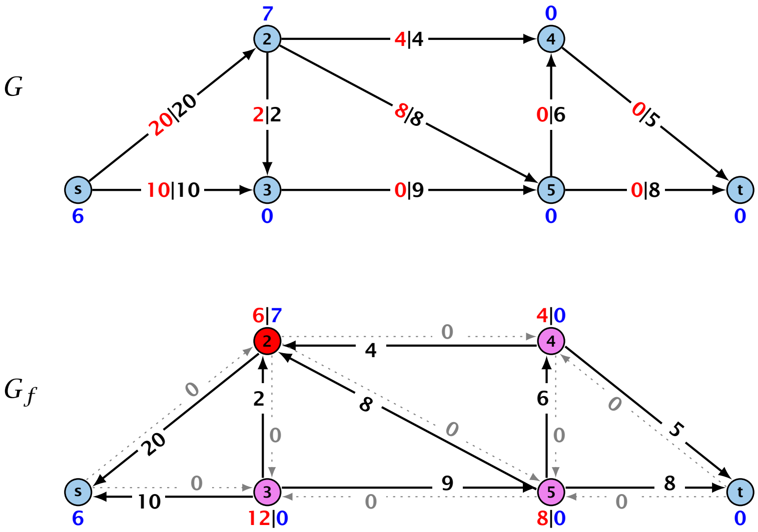


Preflow Push Algorithm

relabel 6 times

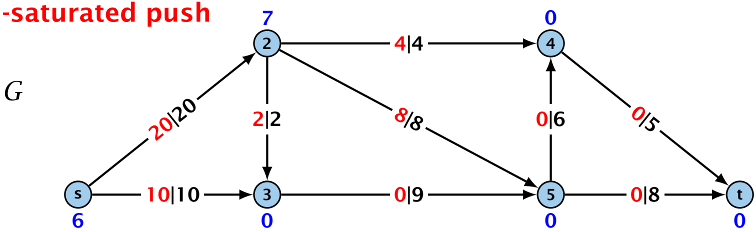


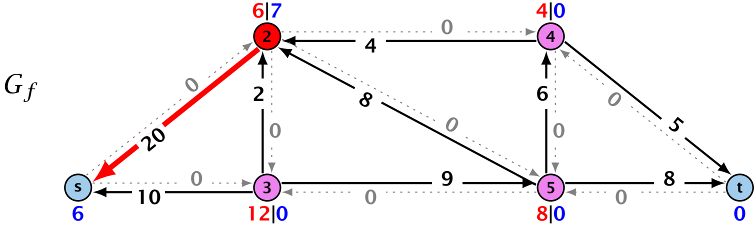
Preflow Push Algorithm



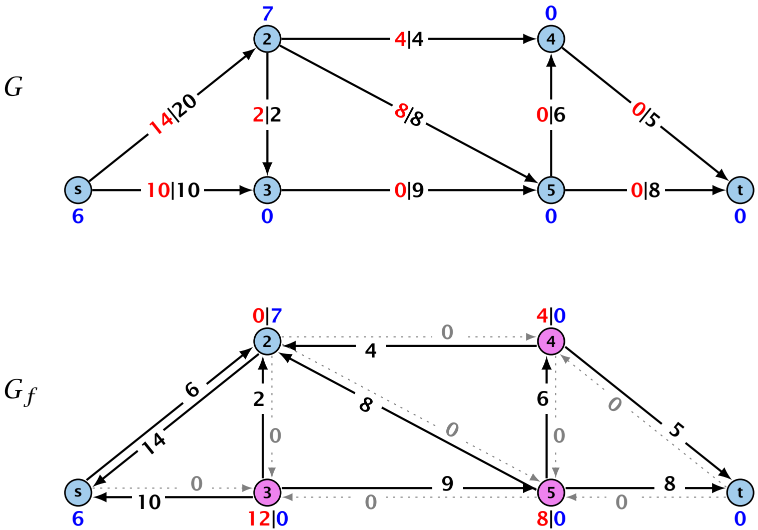
Preflow Push Algorithm

non-saturated push



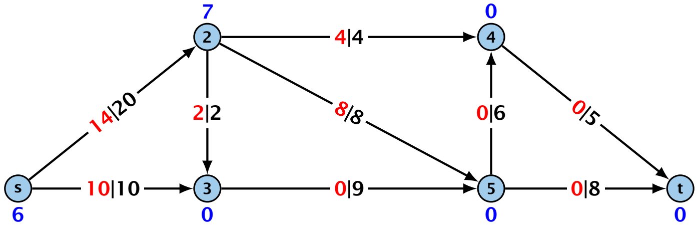


Preflow Push Algorithm

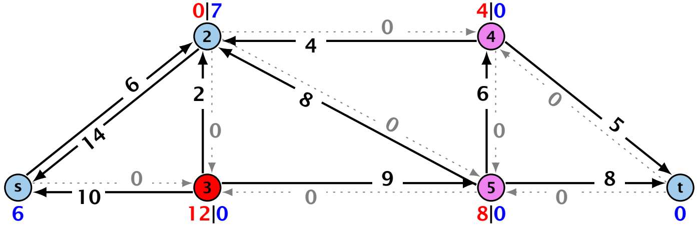


Preflow Push Algorithm

G



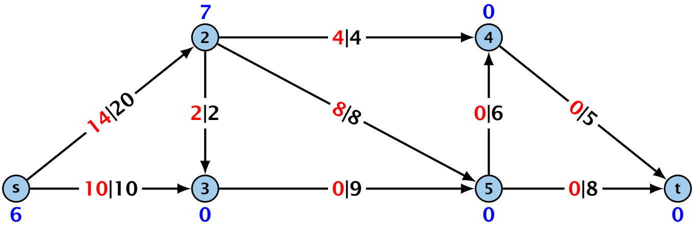
G_f



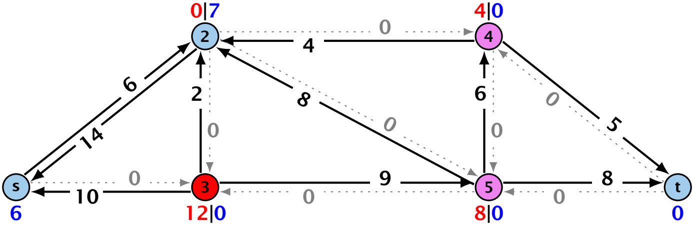
Preflow Push Algorithm

relabel

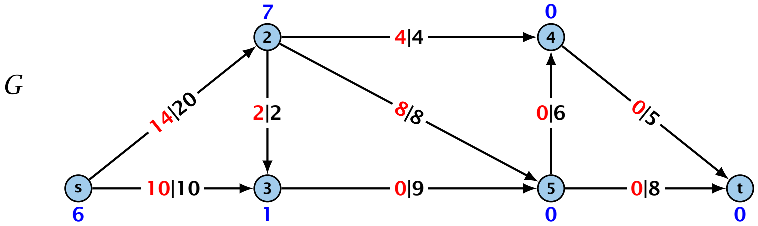
G

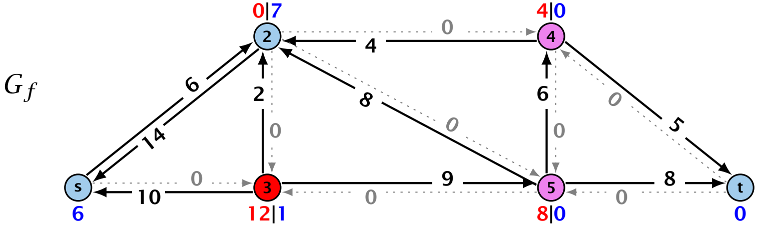


G_f



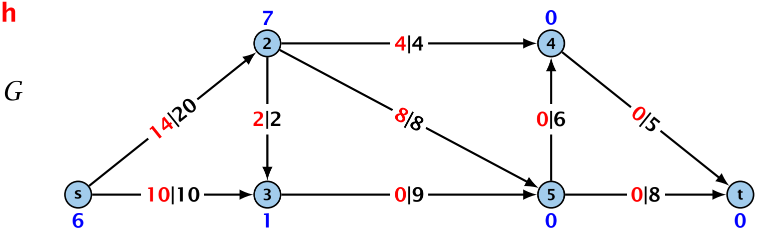
Preflow Push Algorithm

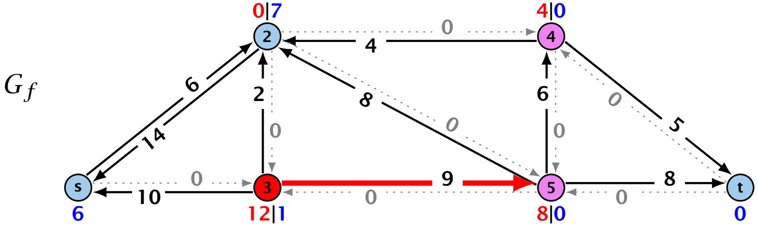




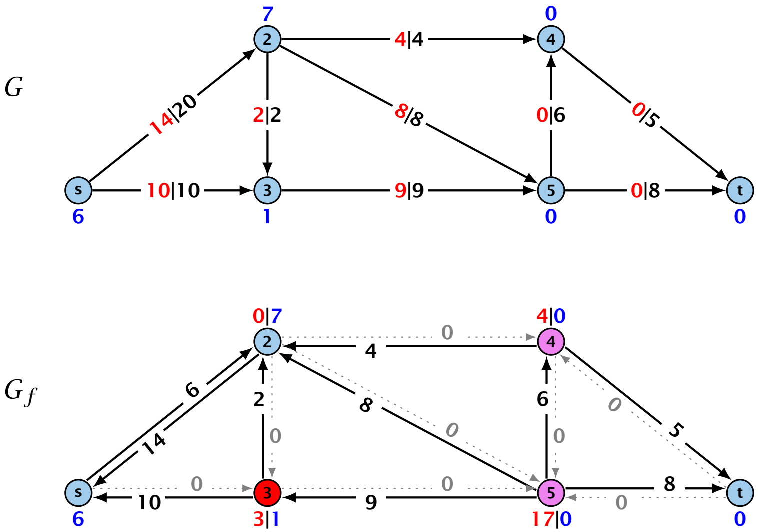
Preflow Push Algorithm

push



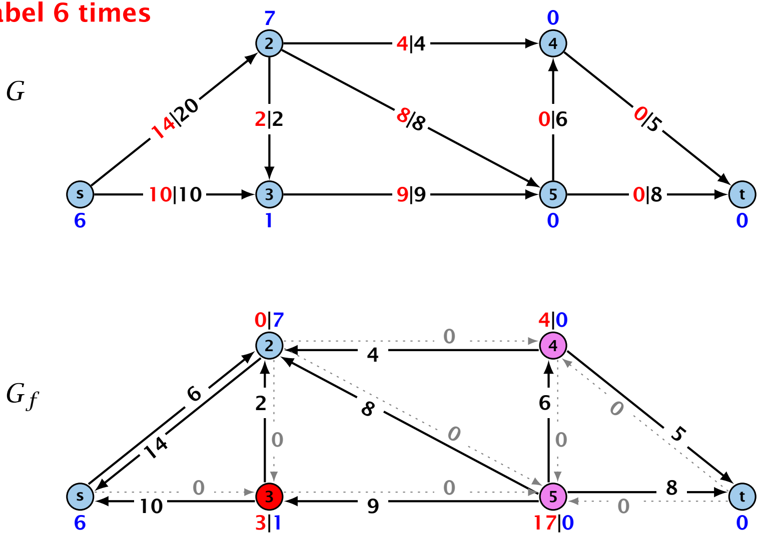


Preflow Push Algorithm

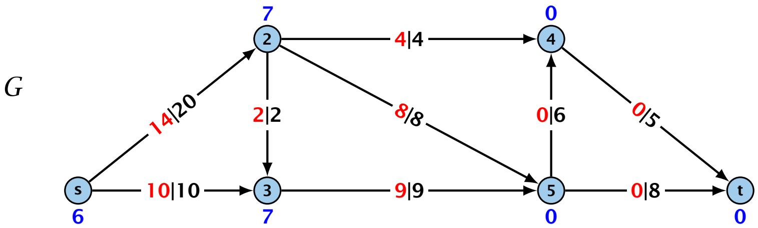


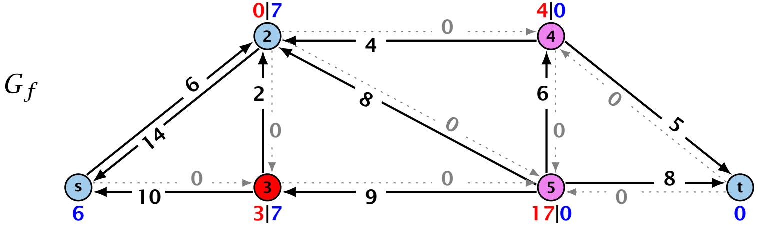
Preflow Push Algorithm

relabel 6 times



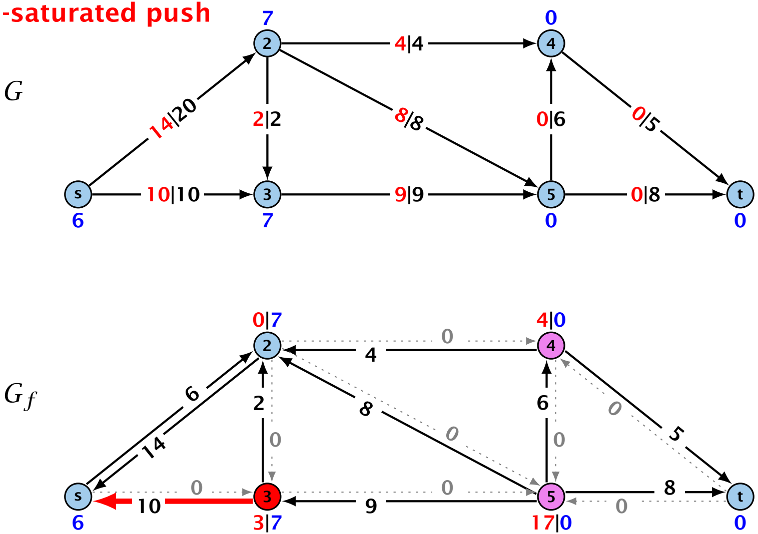
Preflow Push Algorithm





Preflow Push Algorithm

non-saturated push



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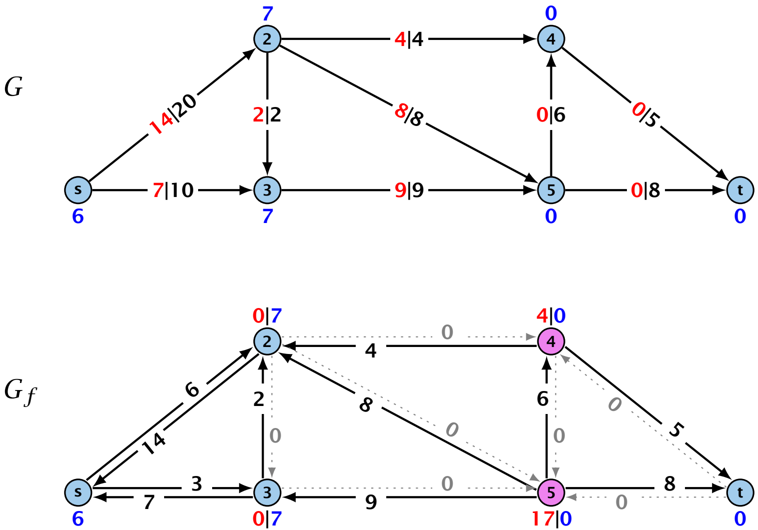

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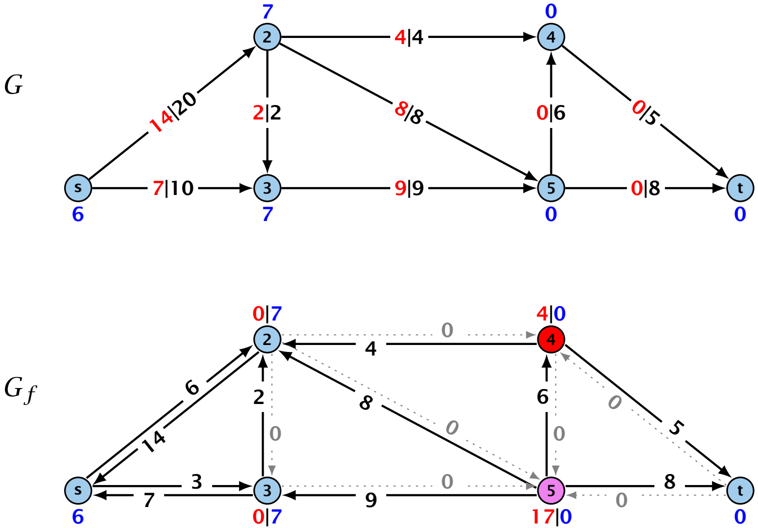
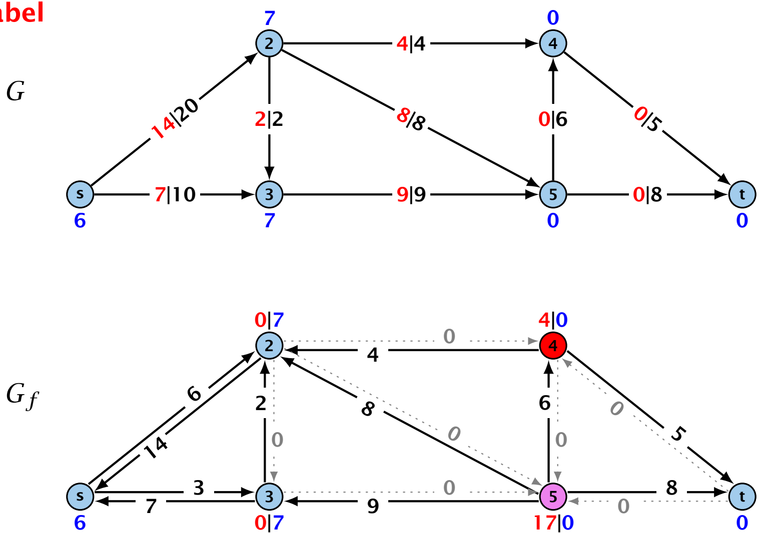


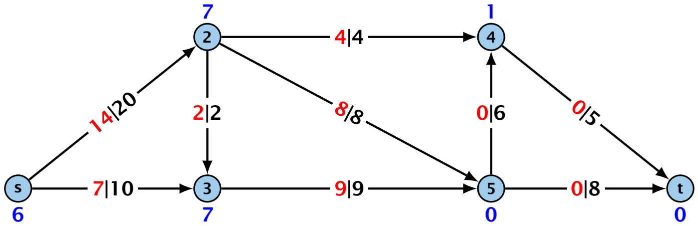
Preflow Push Algorithm

relabel

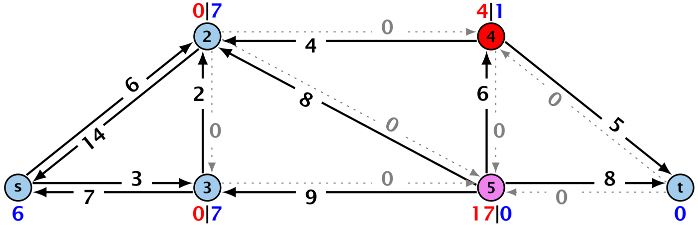


Preflow Push Algorithm

G

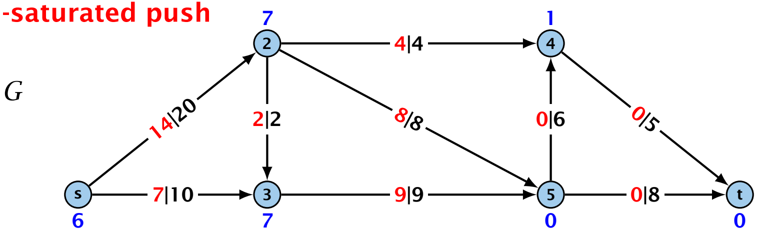


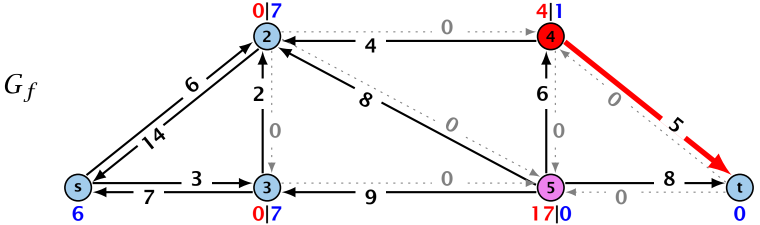
G_f



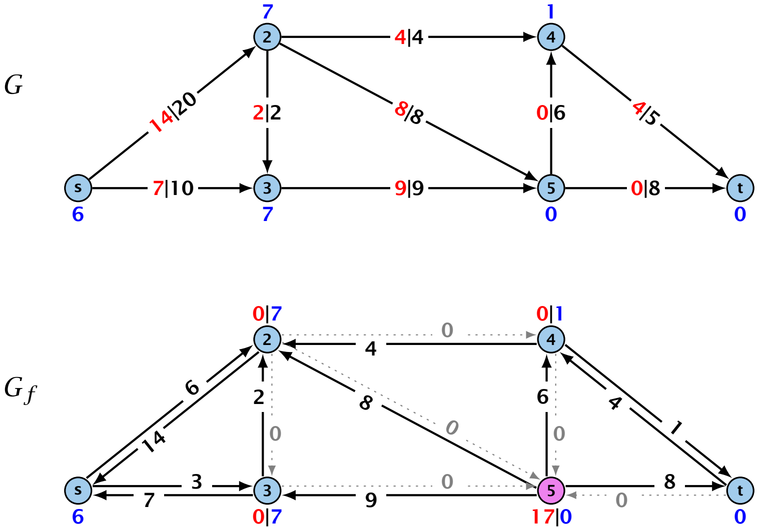
Preflow Push Algorithm

non-saturated push

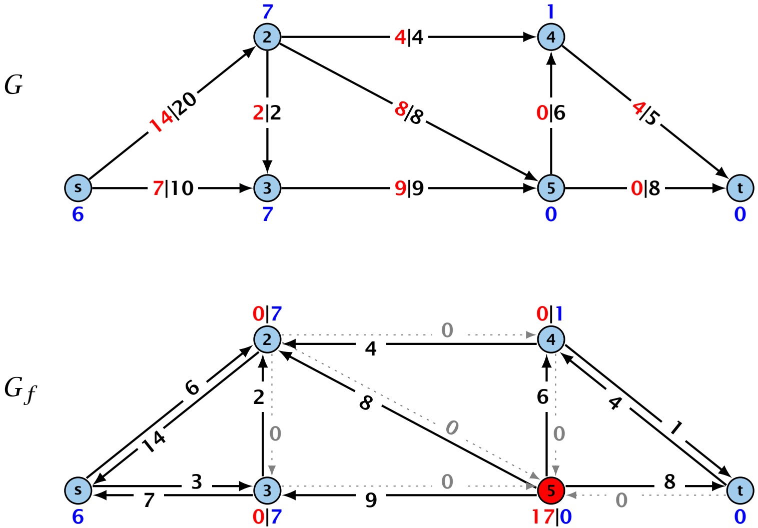




Preflow Push Algorithm

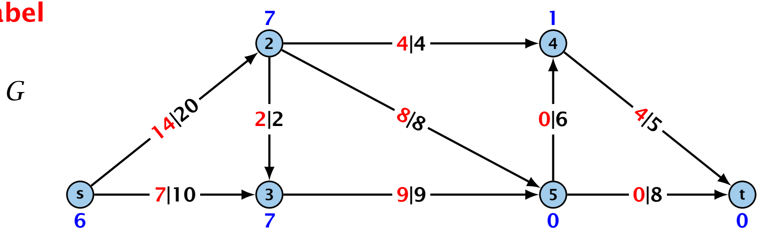


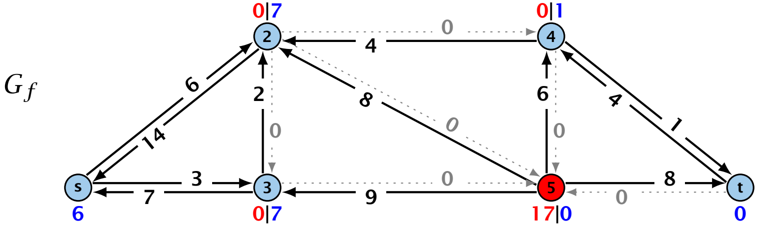
Preflow Push Algorithm



Preflow Push Algorithm

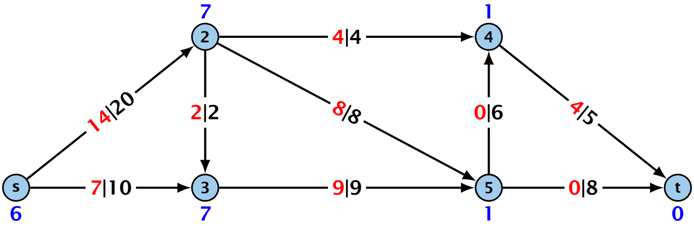
relabel



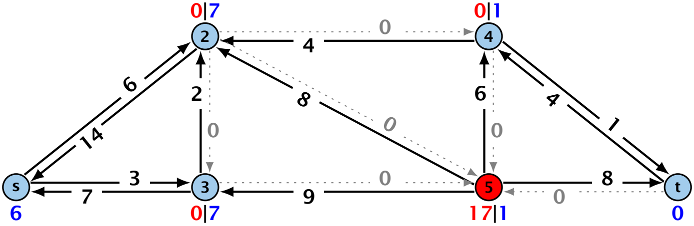


Preflow Push Algorithm

G

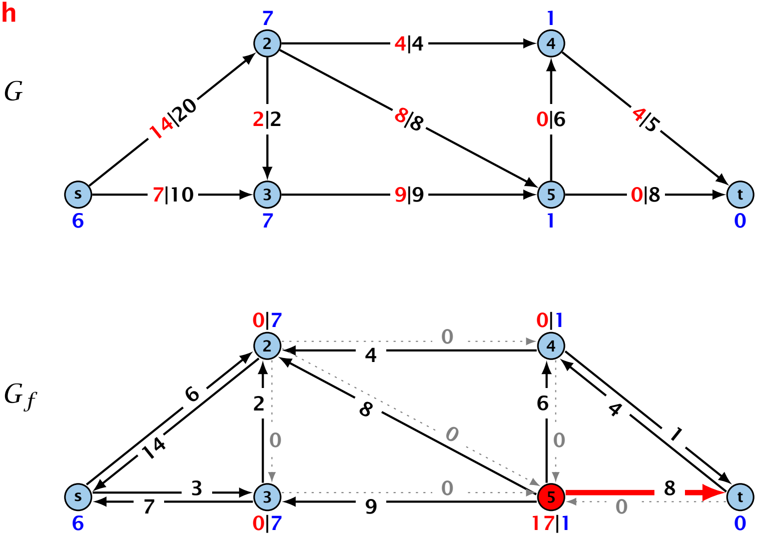


G_f

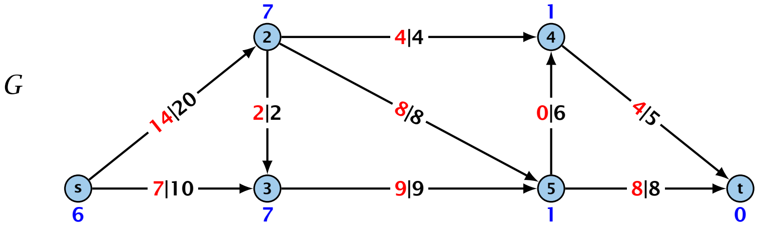


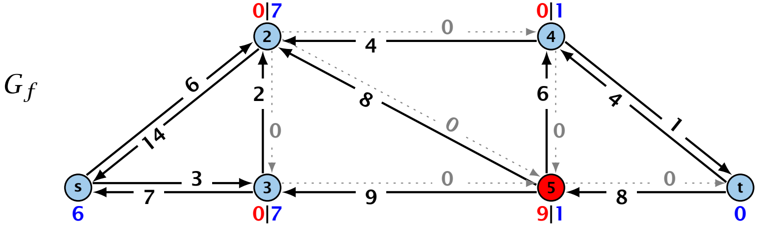
Preflow Push Algorithm

push



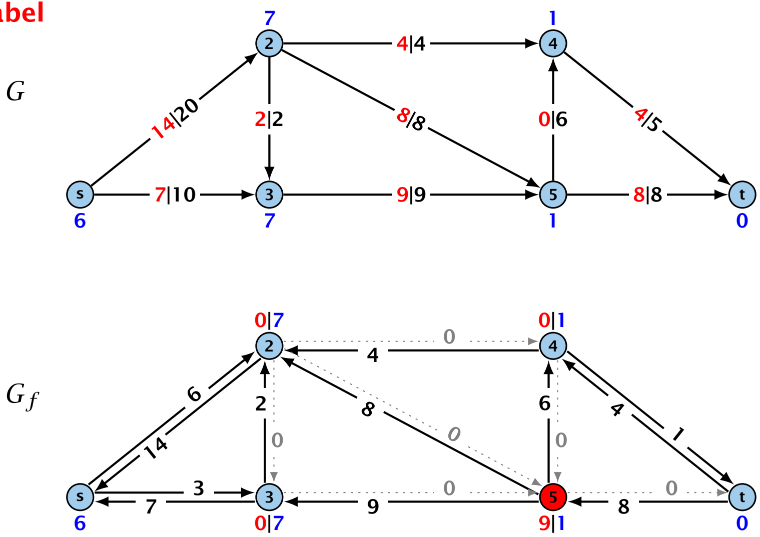
Preflow Push Algorithm



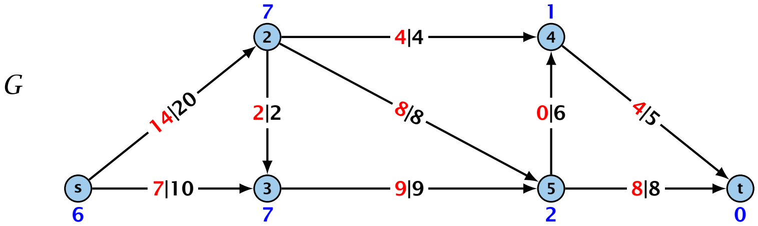


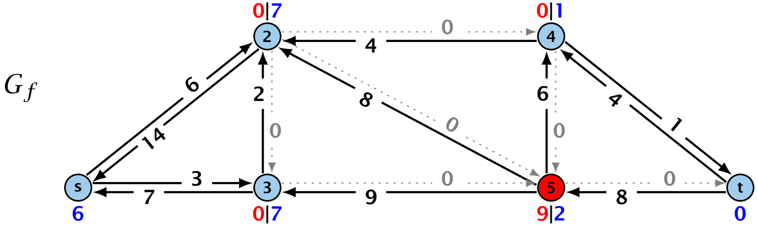
Preflow Push Algorithm

relabel



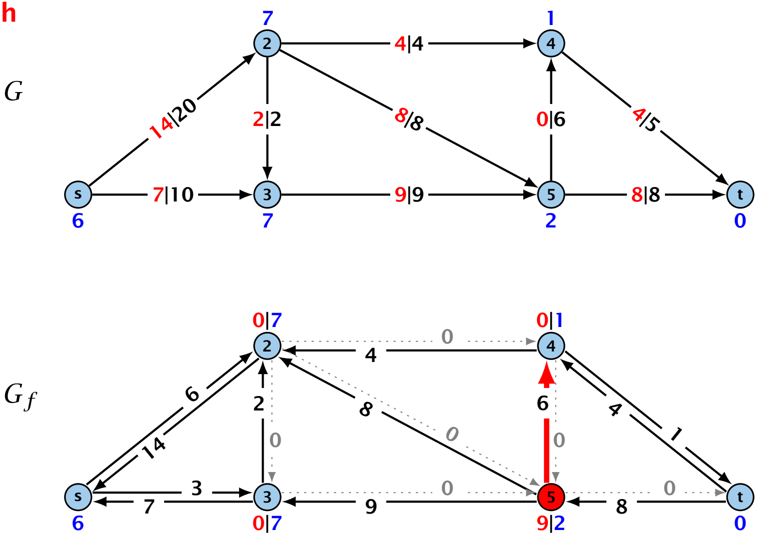
Preflow Push Algorithm



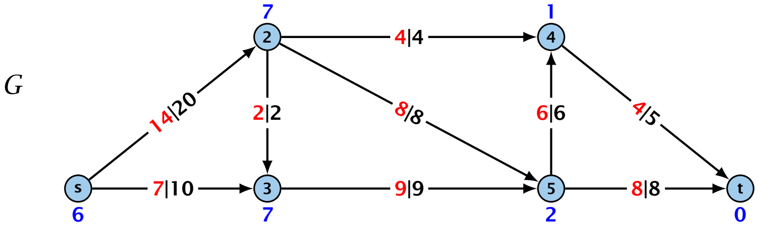


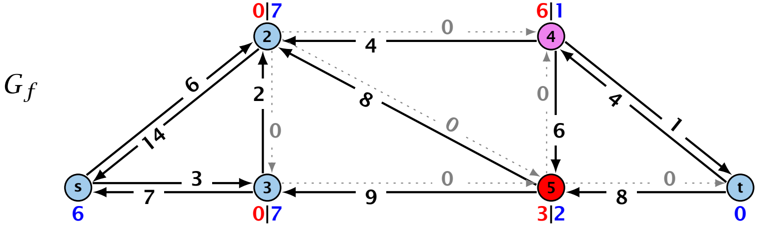
Preflow Push Algorithm

push



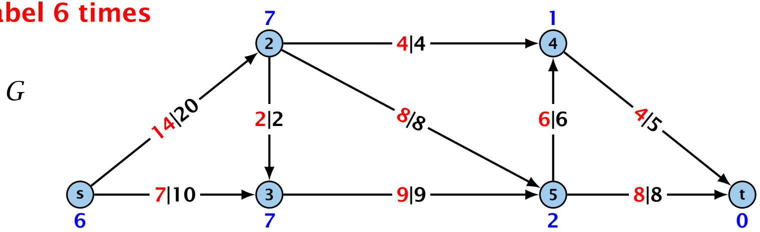
Preflow Push Algorithm

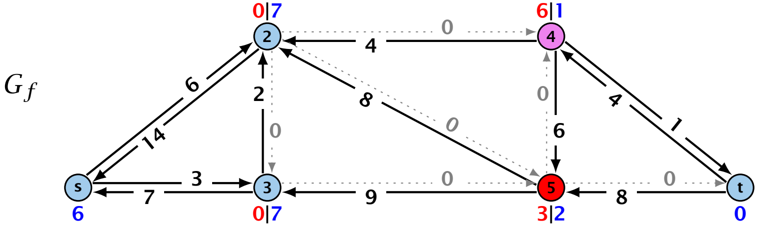




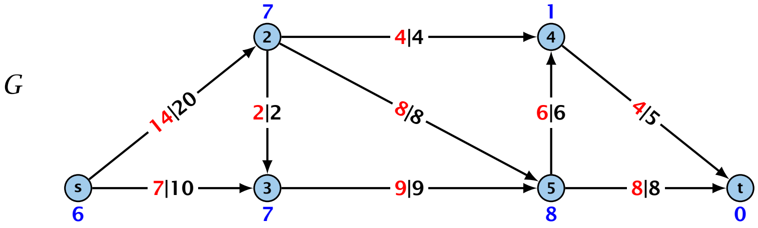
Preflow Push Algorithm

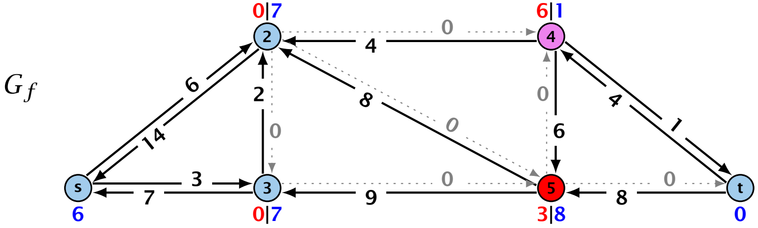
relabel 6 times





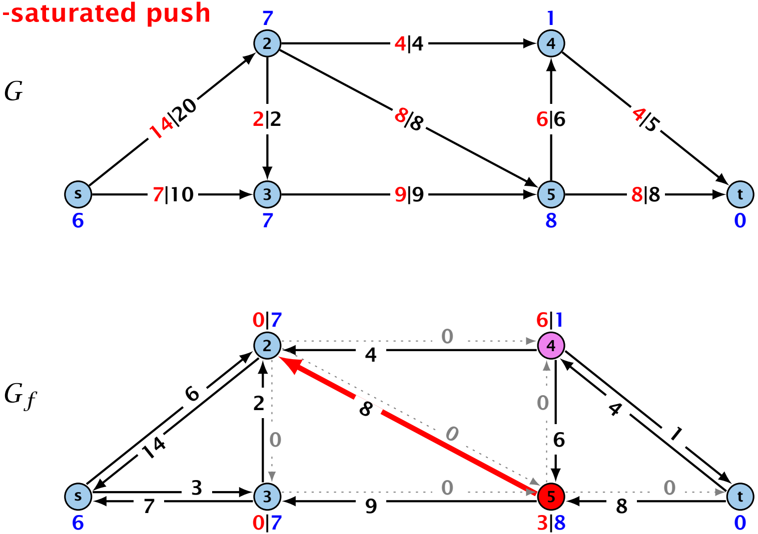
Preflow Push Algorithm



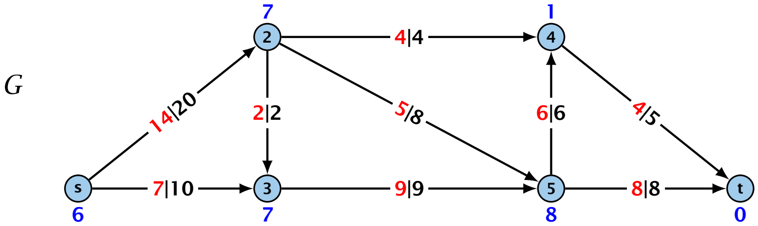


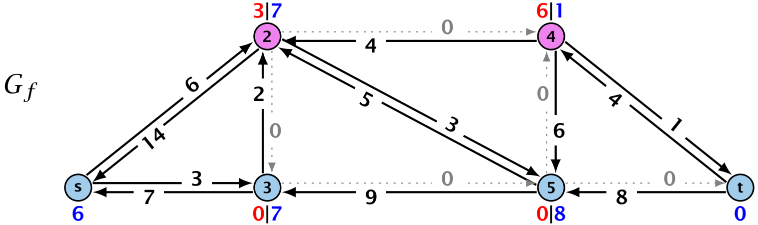
Preflow Push Algorithm

non-saturated push



Preflow Push Algorithm





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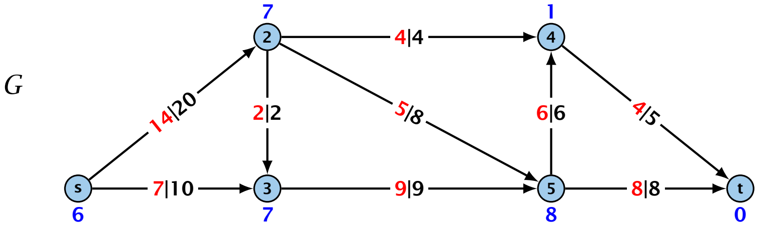
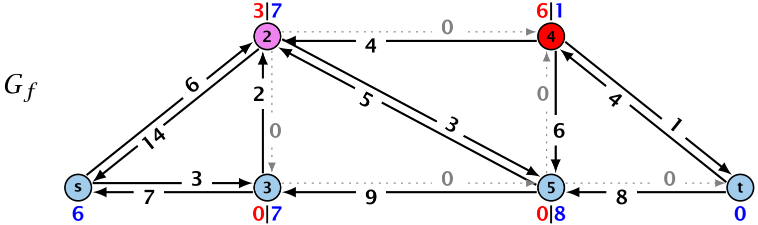
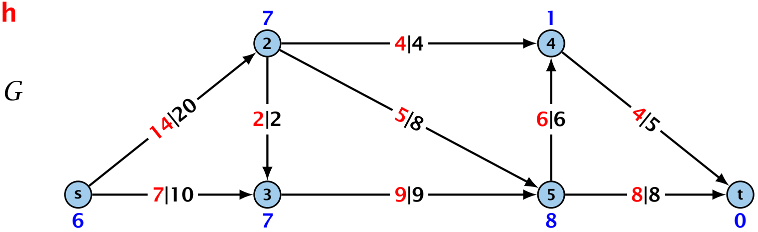
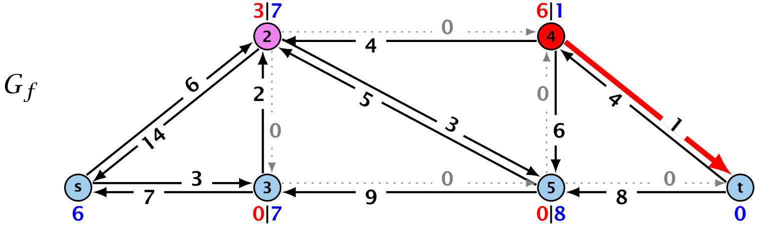




Preflow Push Algorithm

push





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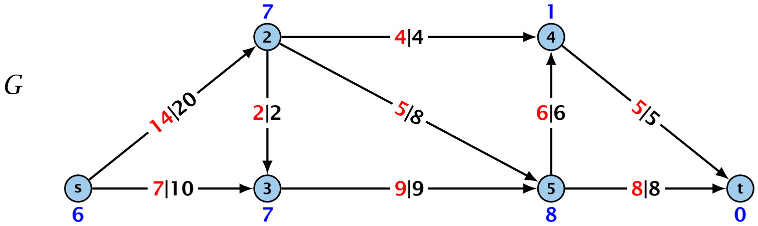
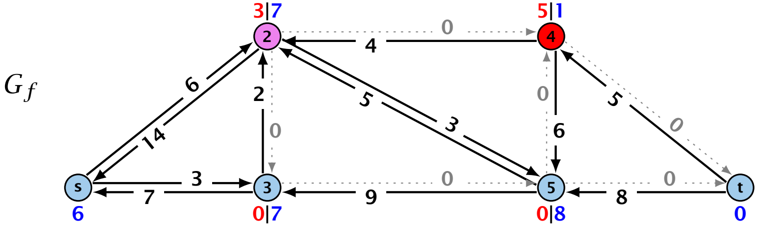
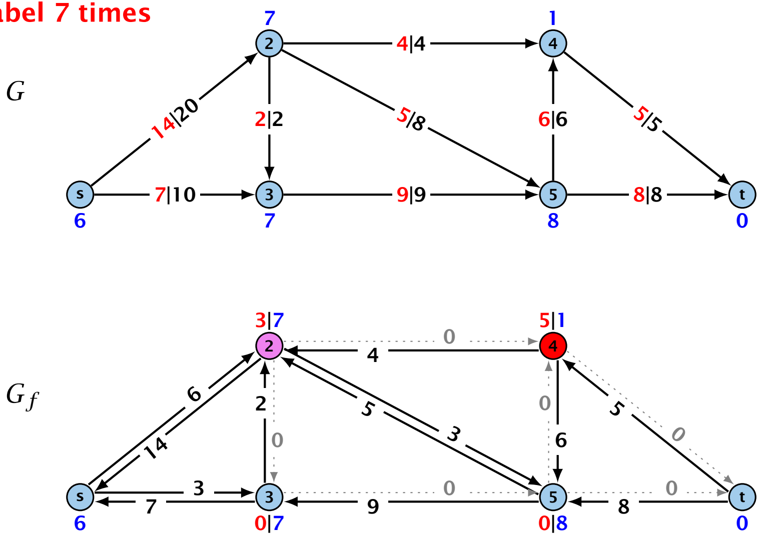


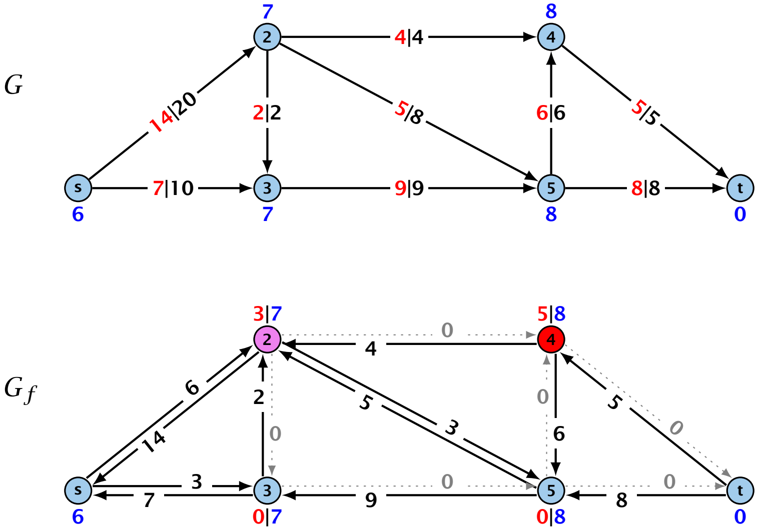


Preflow Push Algorithm

relabel 7 times



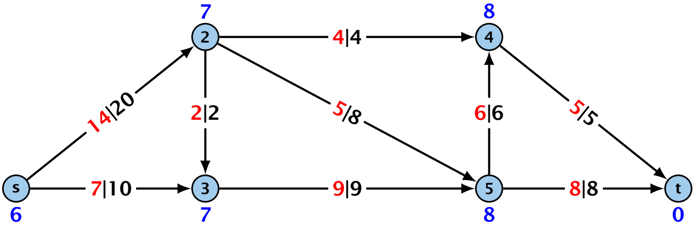
Preflow Push Algorithm



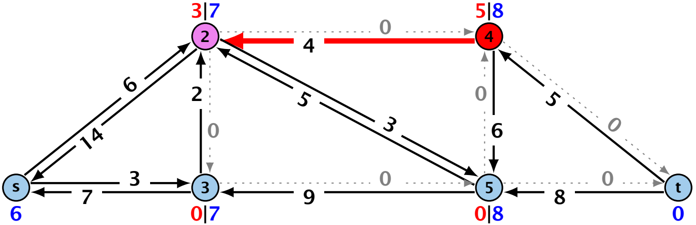
Preflow Push Algorithm

push

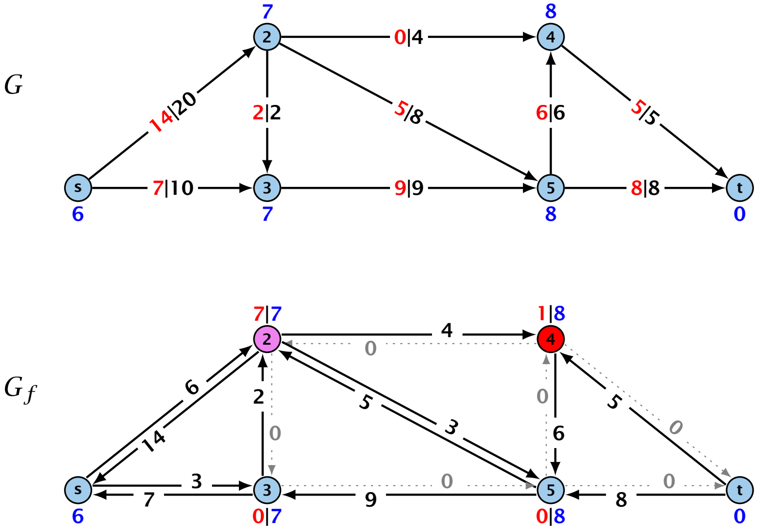
G



G_f

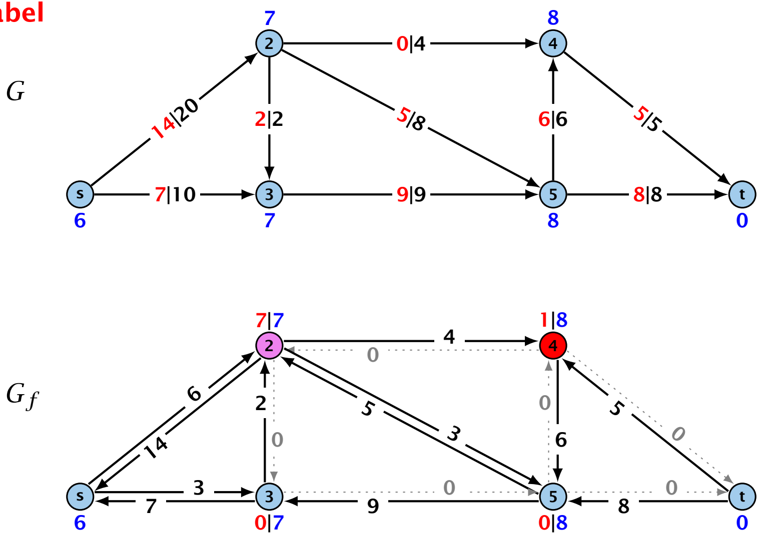


Preflow Push Algorithm



Preflow Push Algorithm

relabel



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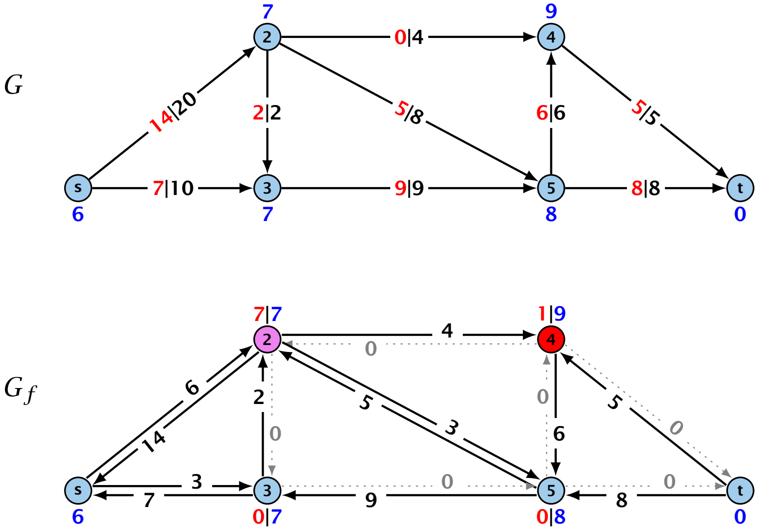
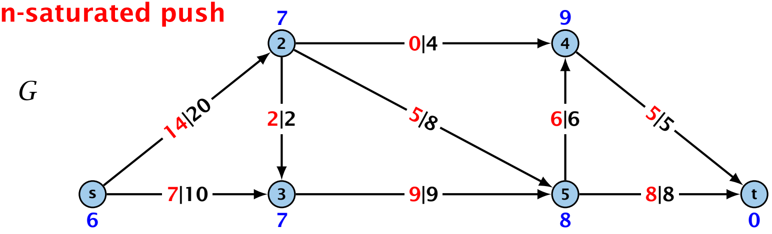
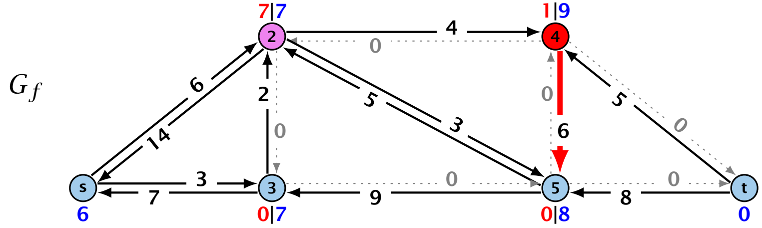


Preflow Push Algorithm

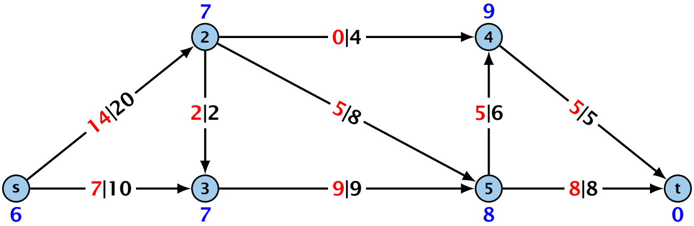
non-saturated push



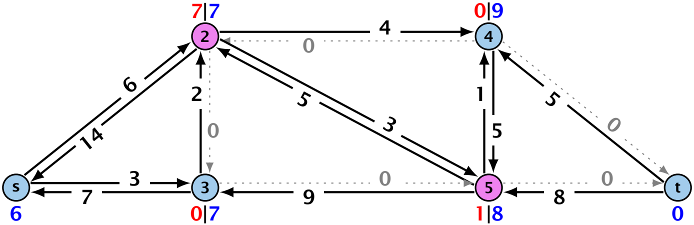


Preflow Push Algorithm

G



G_f



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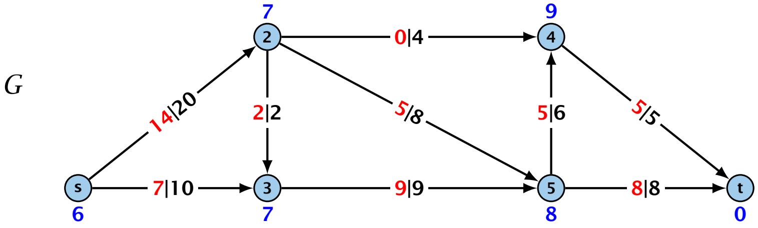
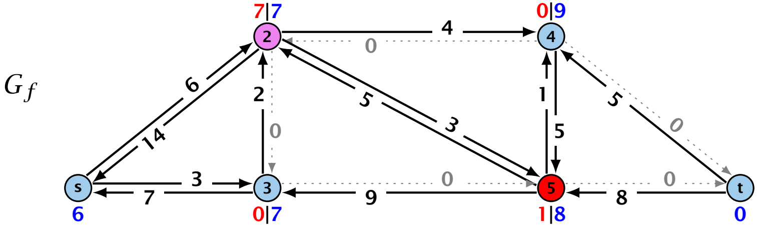
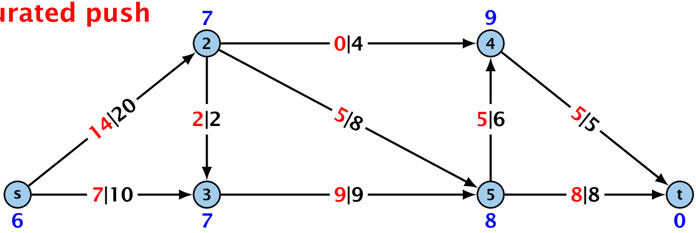




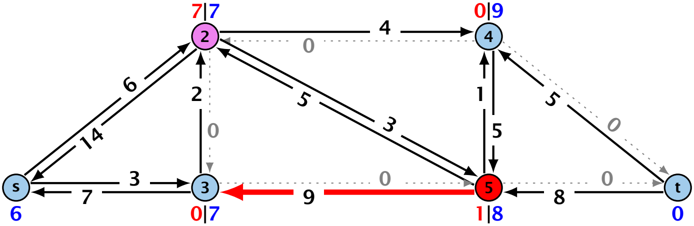
Preflow Push Algorithm

non-saturated push

G



G_f



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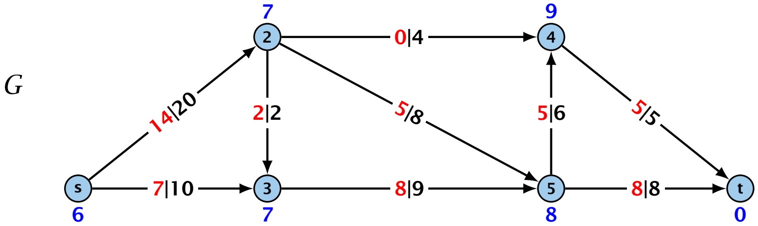

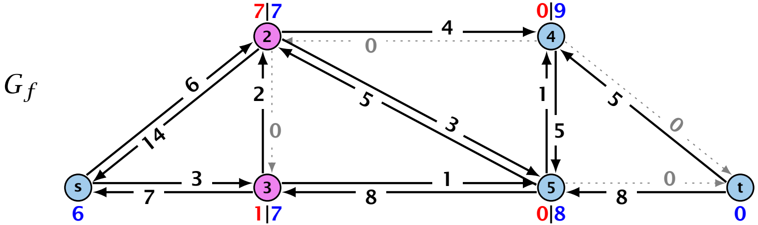

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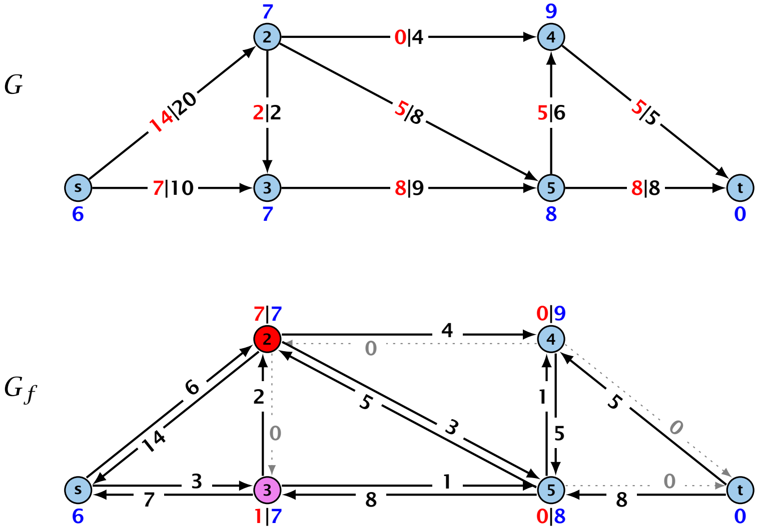

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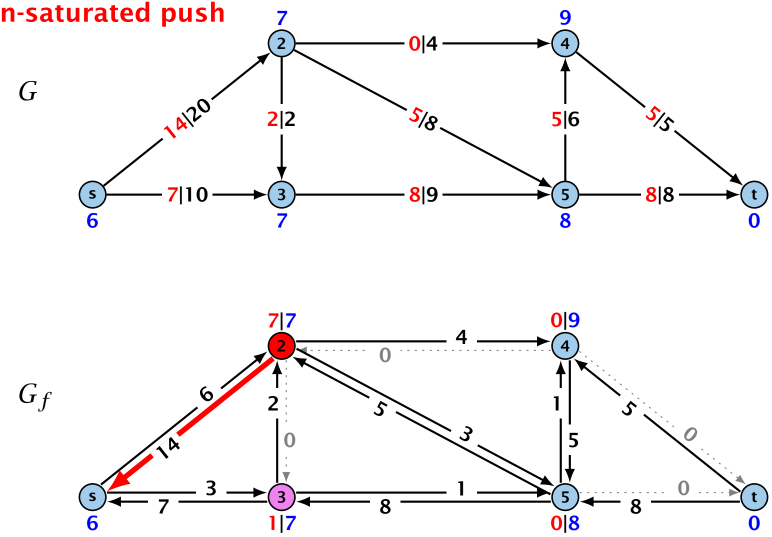
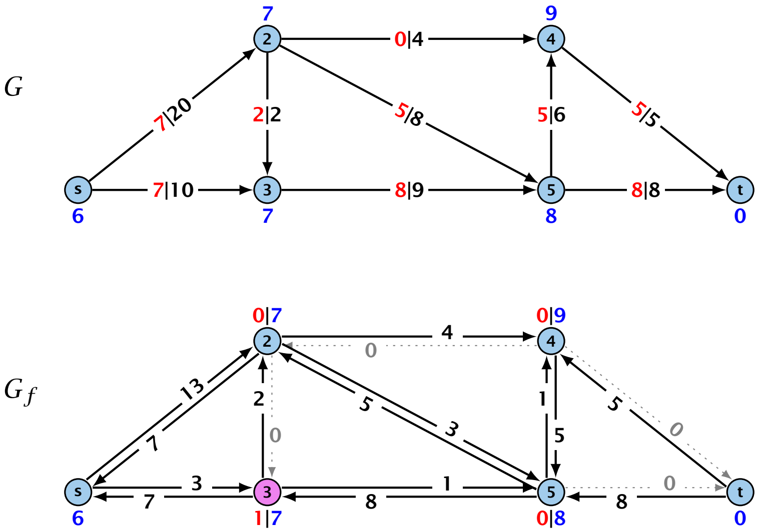


Preflow Push Algorithm



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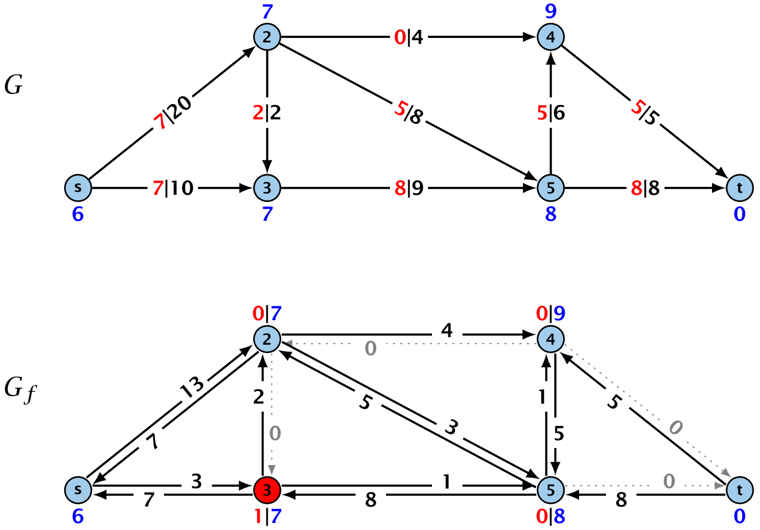
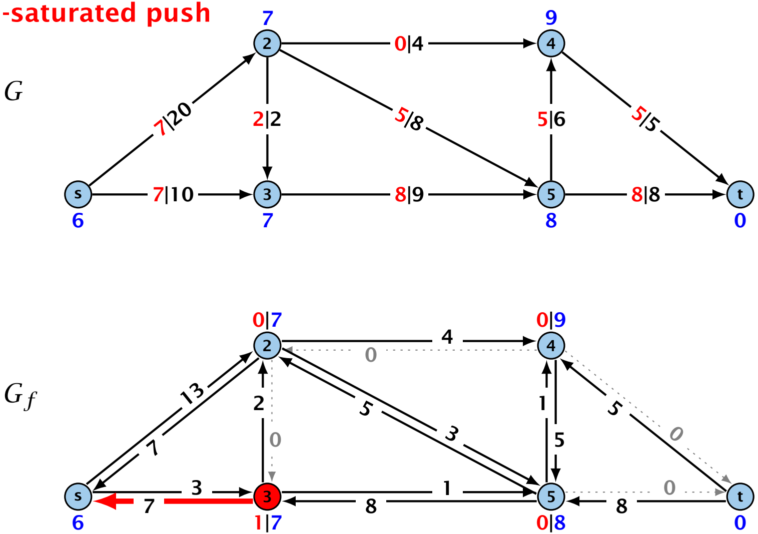


Preflow Push Algorithm

non-saturated push



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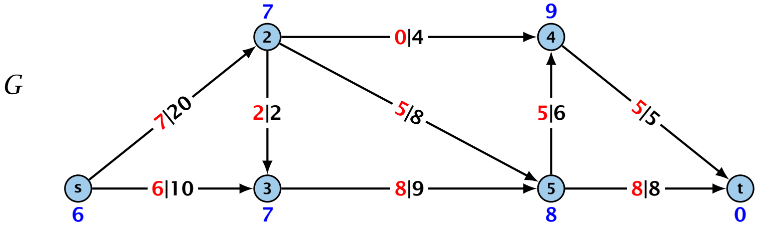

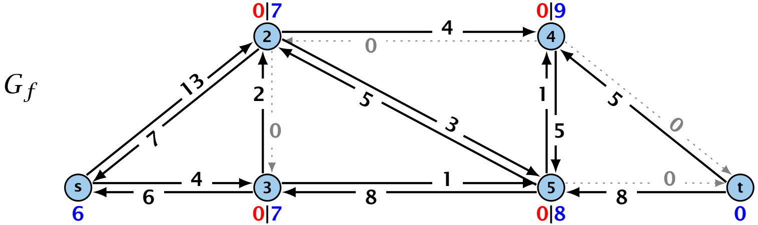

Analysis

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An active node has a path to s in the residual graph.

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Proof.

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- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

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$$\begin{aligned} f(B) &= \sum_{b\in B} f(b) \\ &= \sum_{b\in B}\left(\sum_{v\in V} f(v,b) - \sum_{v\in V} f(b,v)\right) \\ &= \sum_{b\in B}\left(\sum_{v\in A} f(v,b) + \sum_{v\in B} f(v,b) - \sum_{v\in A} f(b,v) - \sum_{v\in B} f(b,v)\right) \\ &= \sum_{b\in B}\sum_{v\in A} f(v,b) - \sum_{b\in B}\sum_{v\in A} f(b,v) + \sum_{b\in B}\sum_{v\in B} f(v,b) - \sum_{b\in B}\sum_{v\in B} f(b,v) \end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

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&= \qquad\qquad - \sum_{b\in B} \sum_{v\in A} f(b,v) \\
&\le 0
\end{aligned}$$

Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 72

The label of a node cannot become larger than $2n - 1$.

Analysis

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Proof.

- When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

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Lemma 73

There are only $\mathcal{O}(n^2)$ relabel operations.

Analysis

Lemma 74

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

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Analysis

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- Suppose that we just made a saturating push along (u, v).
- Hence, the edge (u, v) is deleted from the residual graph.
- For the edge to appear again, a push from v to u is required.

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- Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.

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- For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.
- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

Lemma 75
The number of non-saturating pushes performed is at most $\mathcal{O}(n^2m)$.

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- A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- Hence,

$$\begin{aligned}\text{\#non-saturating\_pushes} &\leq \text{\#relabels} + 2n \cdot \text{\#saturating\_pushes} \\ &\leq \mathcal{O}(n^2 m)\ .\end{aligned}$$

Analysis

Theorem 76

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2 m)$.

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u,v) can be performed in constant time

- check whether edge (v,u) needs to be added to G_f
- check whether (u,v) needs to be deleted (saturating push)
- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible
- check for all incoming edges if they become non-admissible

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Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 4 discharge(u)

```
while u is active do
    v ← u.current-neighbour
    if v = null then
        relabel(u)
        u.current-neighbour ← u.neighbour-list-head
    else
        if (u, v) admissible then push(u, v)
        else u.current-neighbour ← v.next-in-list
```

Note that u.*current-neighbour* is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 77

If $v = \mathrm{null}$ in Line 3, then there is no outgoing admissible edge from u.

Proof.

- While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissible.
- The only thing that could make the edge admissible again would be a relabel at u.
- If we reach the end of the list ($v = \mathrm{null}$) all edges are not admissible. □

This shows that discharge(u) is correct, and that we can perform a relabel in Line 4.

13.2 Relabel to Front

Algorithm 21 relabel-to-front(G, s, t)

initialize preflow
initialize node list L containing $V \setminus \{s, t\}$ in any order
foreach $u \in V \setminus \{s, t\}$ **do**
    $u.\textit{current-neighbour} \leftarrow u.\textit{neighbour-list-head}$
$u \leftarrow L.\textit{head}$
while $u \neq \text{null}$ **do**
    $\textit{old-height} \leftarrow \ell(u)$
    discharge(u)
    if $\ell(u) > \textit{old-height}$ **then** // relabel happened
        move u to the front of L
    $u \leftarrow u.\textit{next}$

13.2 Relabel to Front

Lemma 78 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1. - Pushes do no create any new admissible edges. Therefore, if discharge$()$ does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to Front

Proof:

- Maintenance:
  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to Front

Lemma 79
There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to Front

Lemma 80
The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.current\text{-}neighbour$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to Front

Note that by definition a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can at the same time be a non-saturating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 81

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

13.2 Relabel to Front

Lemma 82
The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 83
The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Algorithm 6 highest-label(G, s, t)

```
initialize preflow
foreach u ∈ V \ {s, t} do
    u.current-neighbour ← u.neighbour-list-head
while ∃ active node u do
    select active node u with highest label
    discharge(u)
```

13.3 Highest Label

Lemma 84
When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only “activate” nodes on levels strictly less than ℓ.

This means, after a non-saturating push from u a relabel is required to make u active again.

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest Label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:

How do we find the next node for a discharge operation?

13.3 Highest Label

Maintain lists L_i, $i \in \{0, \dots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \dots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#non\text{-}saturating\text{-}pushes\text{-}to\text{-}s\text{-}or\text{-}t)$$

Lemma 85

The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 86

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Mincost Flow

Problem Definition:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; 0 \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

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- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
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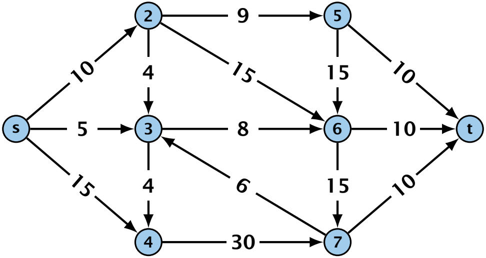
Mincost Flow

Problem Definition:

$$\begin{aligned} \text{min} \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad 0 \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow



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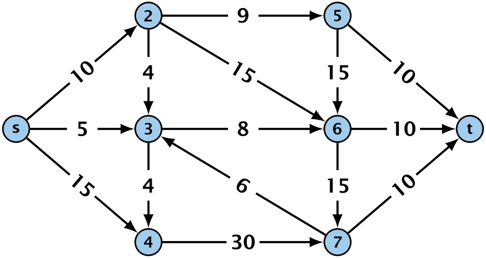

- Given a flow network for a standard maxflow problem.

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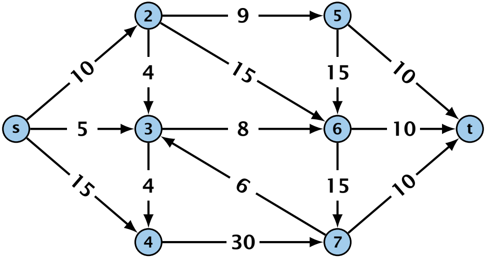

- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

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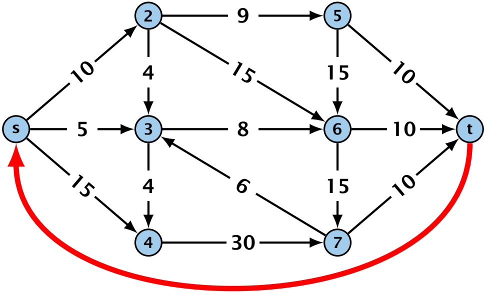

- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.

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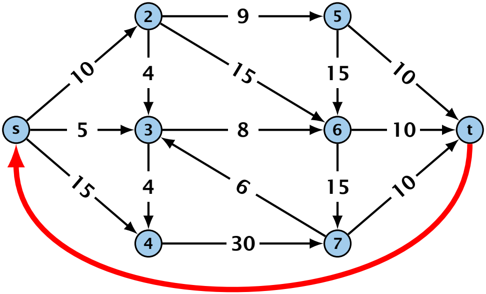

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- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\text{val}(f^*) = -\text{cost}(f_{\text{min}})$, where f^* is a maxflow, and f_{min} is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value at least k if and only if the mincost-flow problem is feasible.

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Generalization

Our model:

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; 0 \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

A more general model?

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{array}$$

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Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

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Add edge (r, v) for all $v \in V$.

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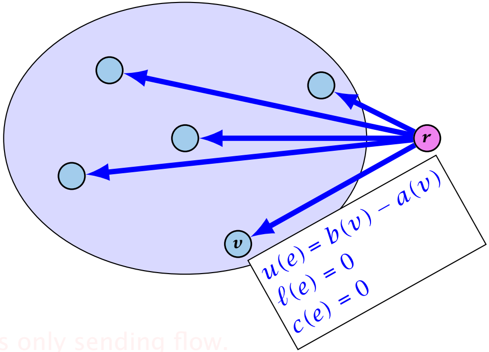
$-\sum_v b(v)$ is negative; hence r is only sending flow.



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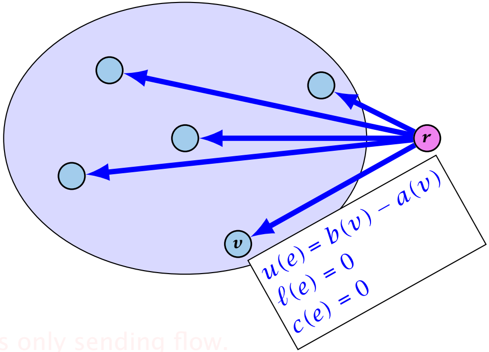

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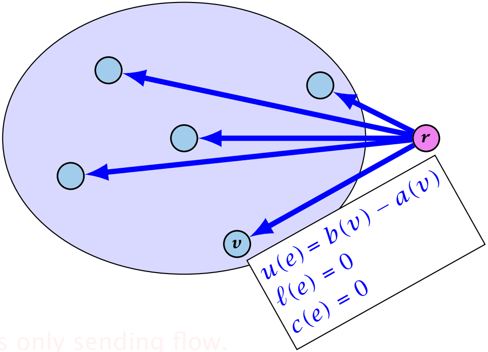

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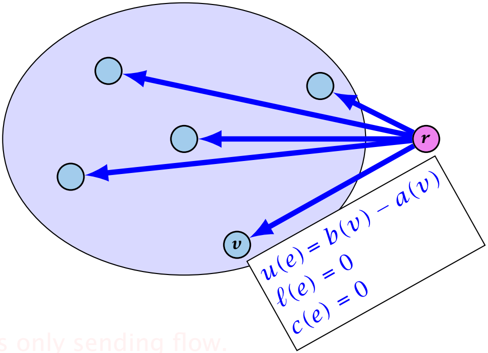

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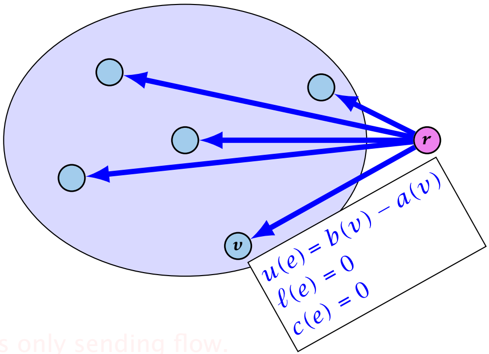

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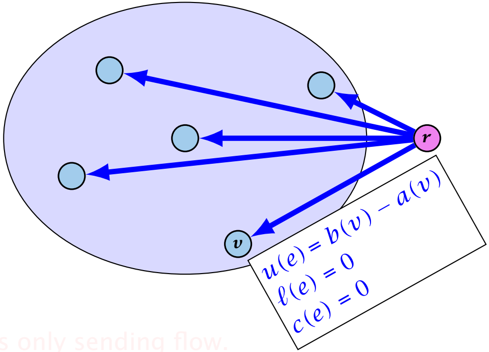

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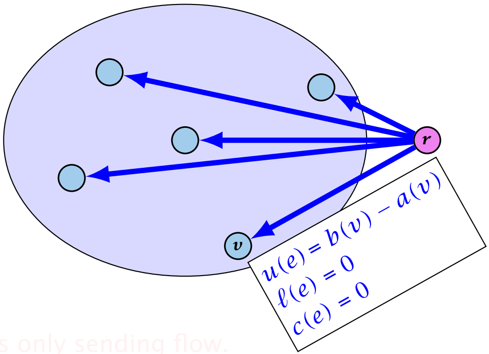

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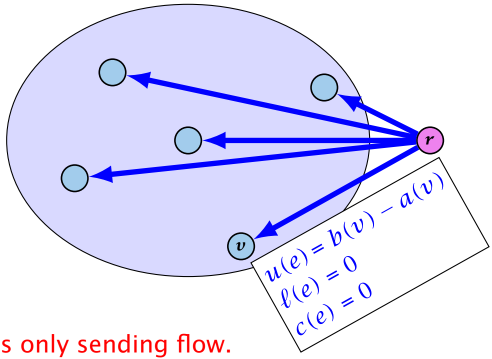

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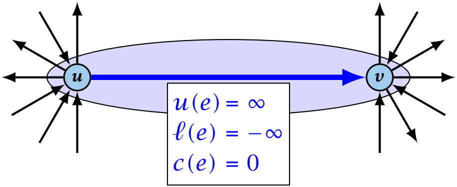
We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:



If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

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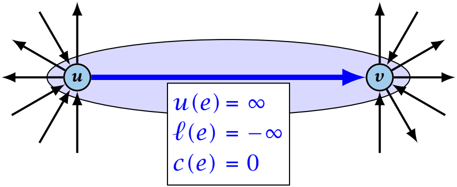

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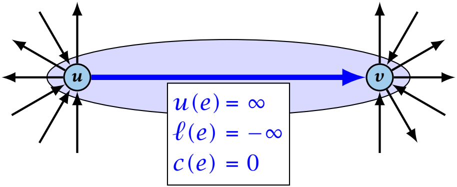

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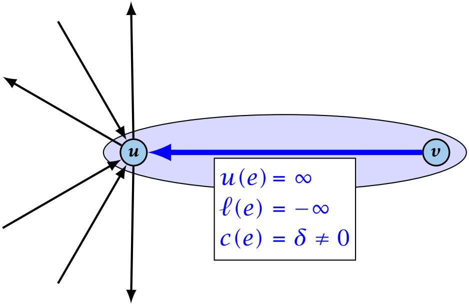
Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:



Additionally we set $b(u) = 0$.

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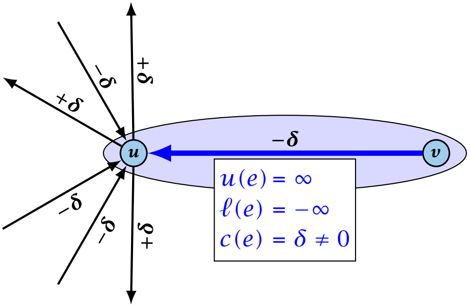

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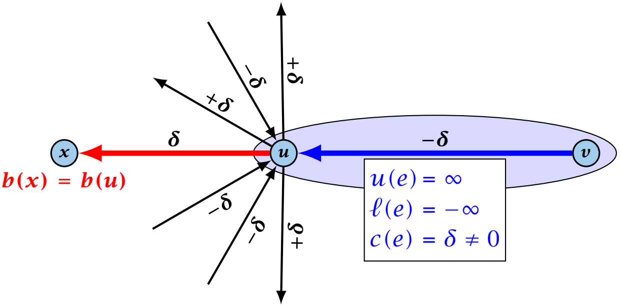

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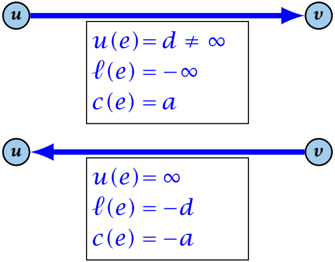

Replace the edge by an edge in opposite direction.

Reduction IV

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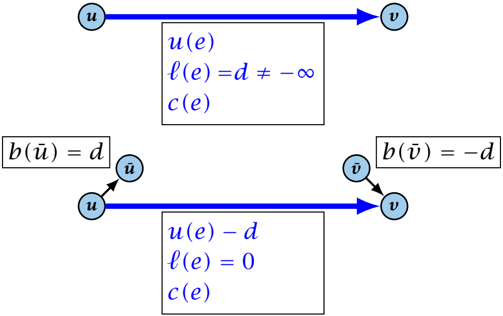

The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
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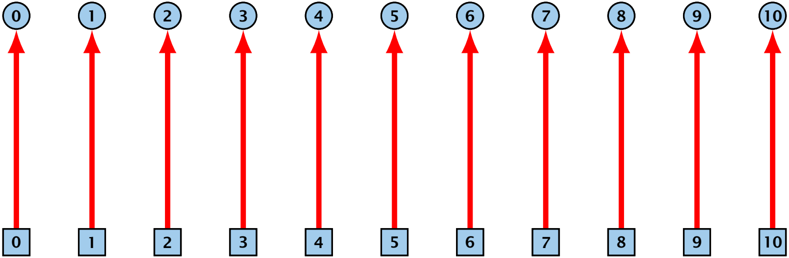
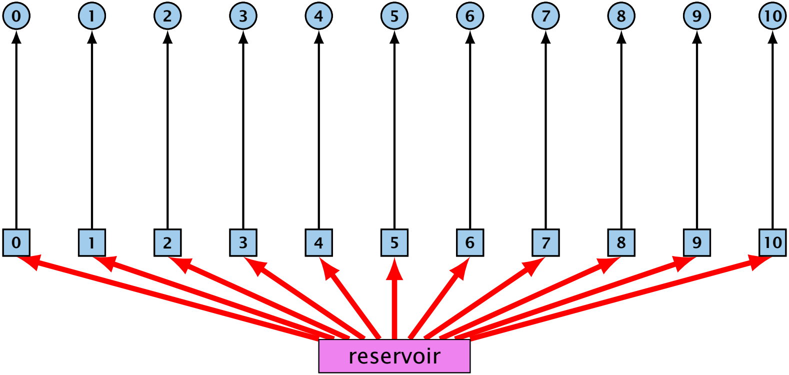


day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



buy edges:

upper bound: $u(e_i) = \infty$;
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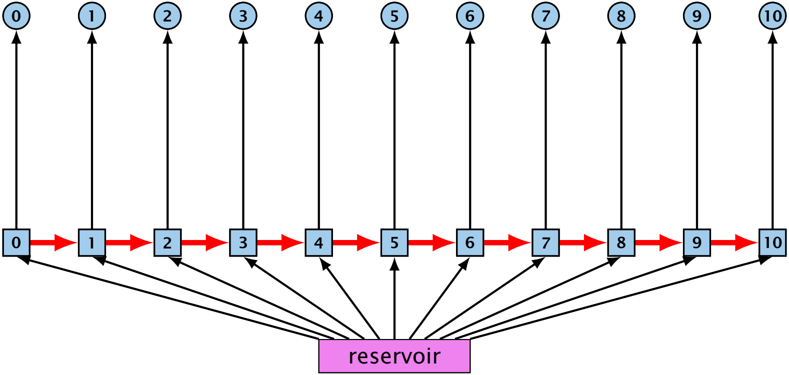
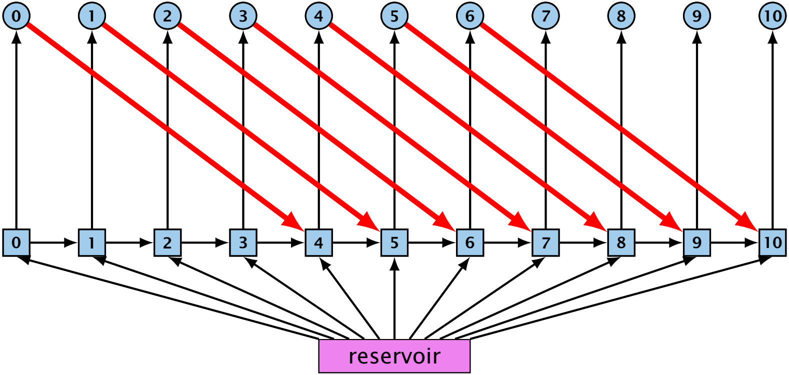
cost: $c(e) = p$



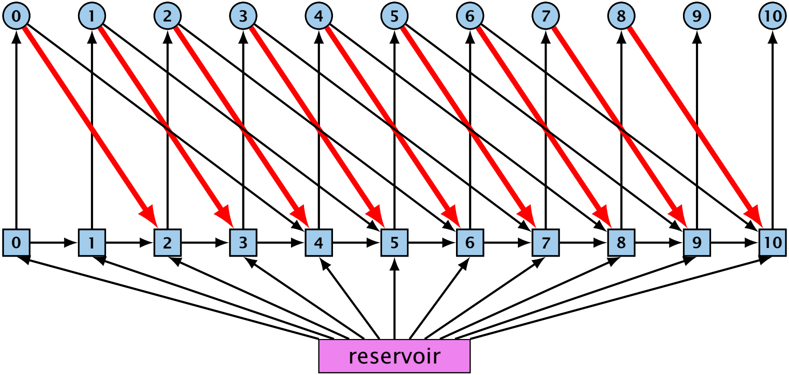
forward edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$



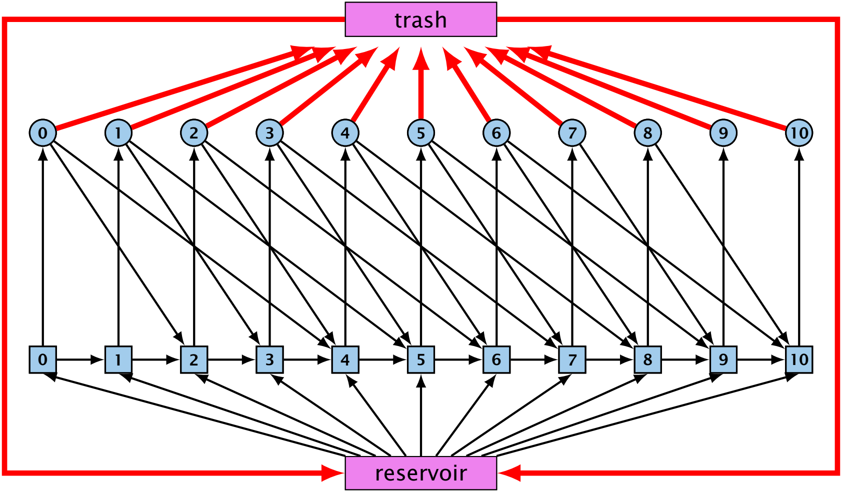
slow edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = s$



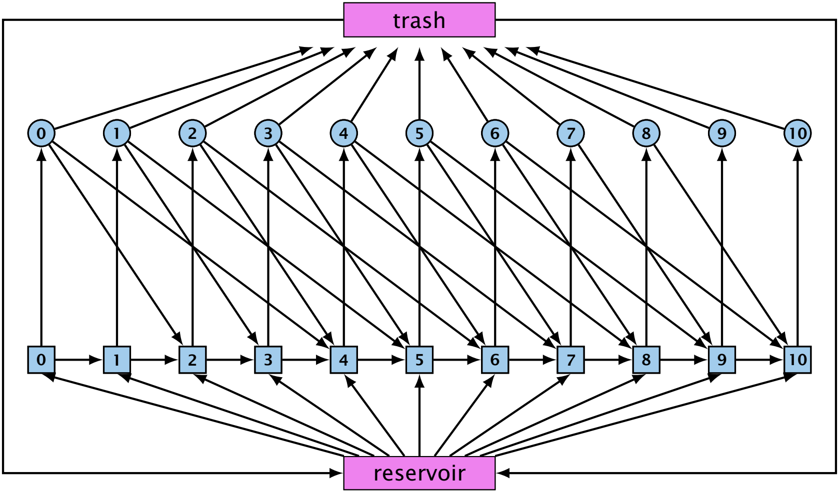
fast edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$


trash
0
1
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3
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5
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7
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9
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0
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7
8
9
10
reservoir

Residual Graph

Version A:

The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

Version B:

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v,u) has capacity z and a cost of $-c((u,v))$.

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14 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

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Lemma 87

A given flow is a mincost-flow if and only if the corresponding residual graph G_f *does not have a feasible circulation of negative cost.*

- Suppose that g is a feasible circulation of negative cost in the residual graph.

- Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph G_f has a feasible circulation with negative cost.

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- ⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

  Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

- ⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

  Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

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A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed cycle only using edges that have non-zero flow.
- If this cycle has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
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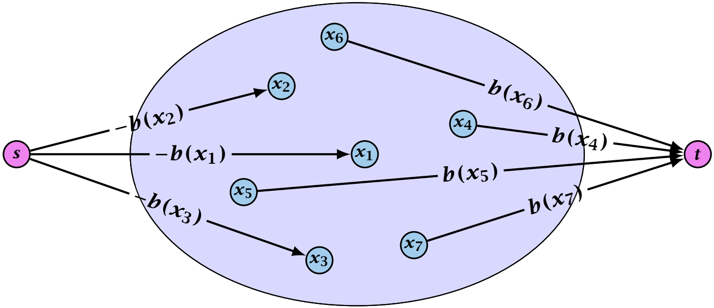
14 Mincost Flow

Algorithm 23 CycleCanceling$(G = (V, E), c, u, b)$

establish a feasible flow f in G
while G_f contains negative cycle **do**
    use Bellman-Ford to find a negative circuit Z
    $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
    augment δ units along Z and update G_f

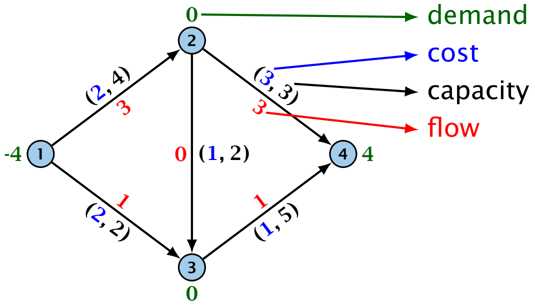
How do we find the initial feasible flow?



- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \;.$$

14 Mincost Flow



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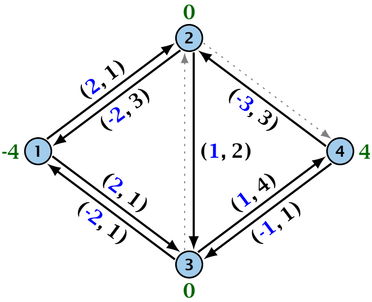
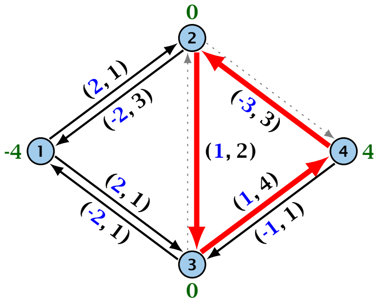
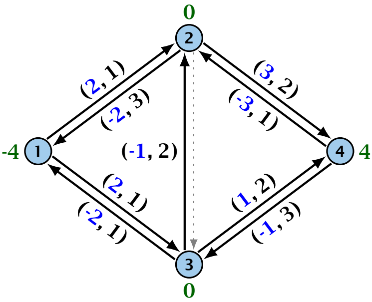



0
2
(2, 1)
(-2, 3)
-4
1
(1, 2)
(-3, 3)
4
4
(2, 1)
(-2, 1)
(1, 4)
(-1, 1)
3
0

14 Mincost Flow



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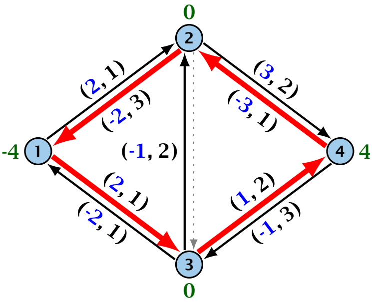

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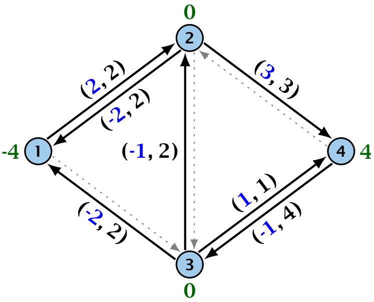


14 Mincost Flow

Lemma 89

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \le C$ and $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \dots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

14 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
$c : E \to \mathbb{R}$;

Lemma 90 (without proof)

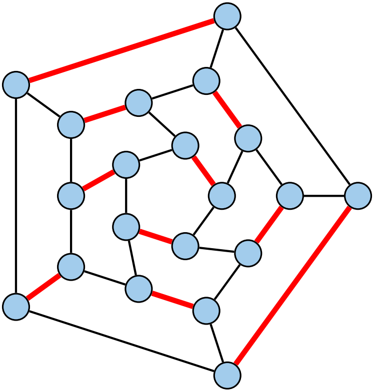
A general mincost flow problem can be solved in polynomial time.

Part V

Matchings

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



16 Bipartite Matching via Flows

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

17 Augmenting Paths for Matchings

Definitions.

- Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.
- For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.
- An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

Theorem 91

A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

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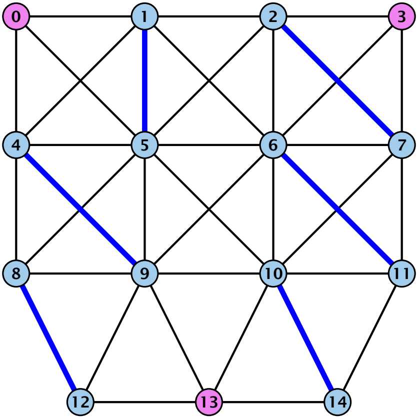

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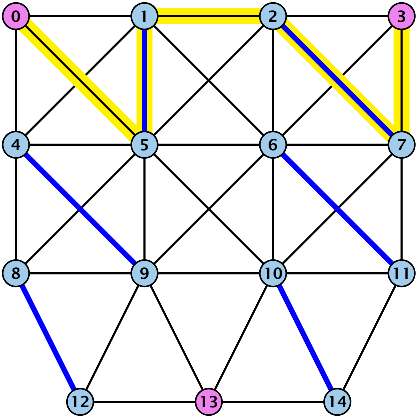

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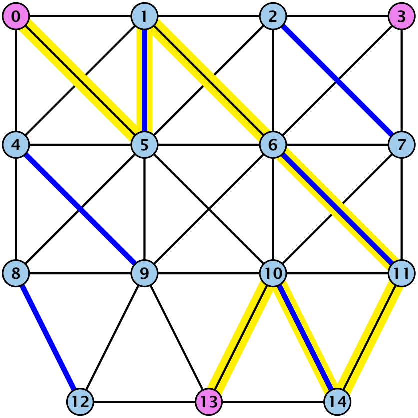

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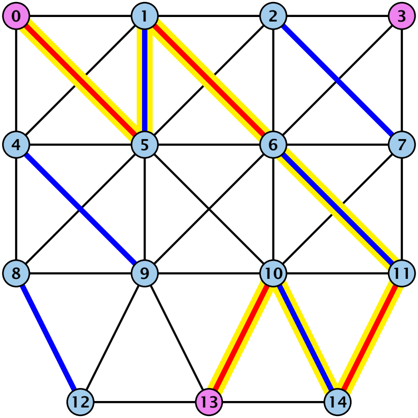

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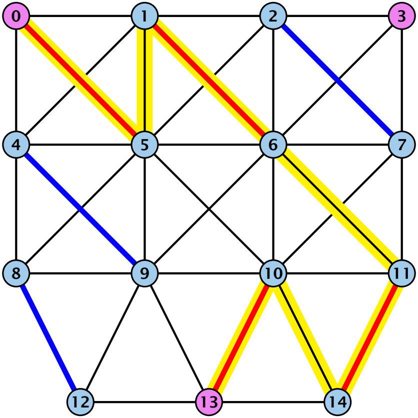

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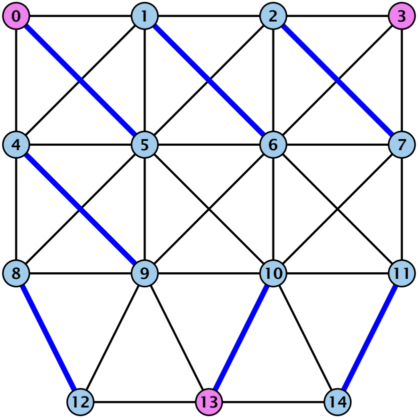

17 Augmenting Paths for Matchings

Proof.

- ⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.
- ⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

  Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

  As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

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17 Augmenting Paths for Matchings

Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 92

Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

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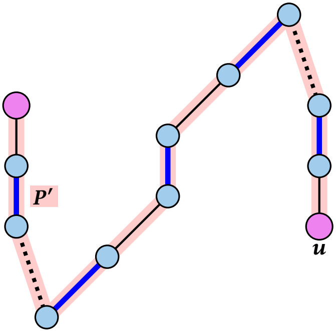

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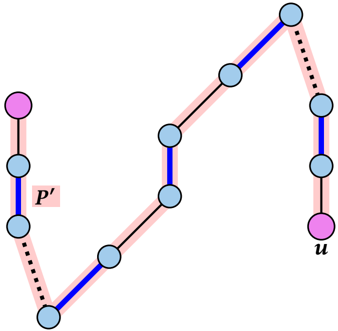

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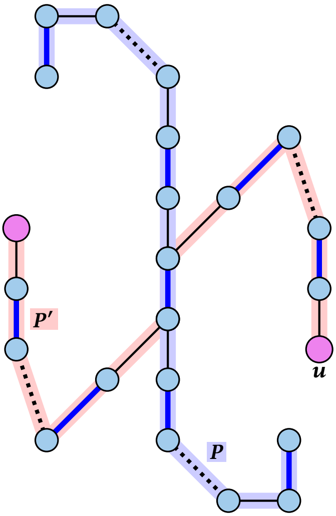

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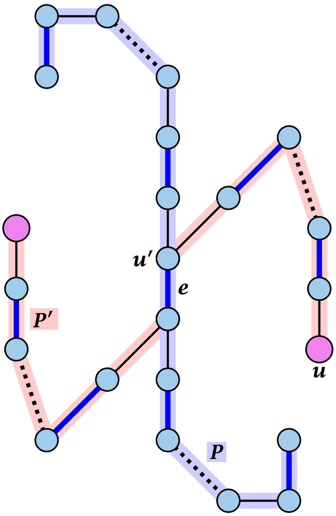

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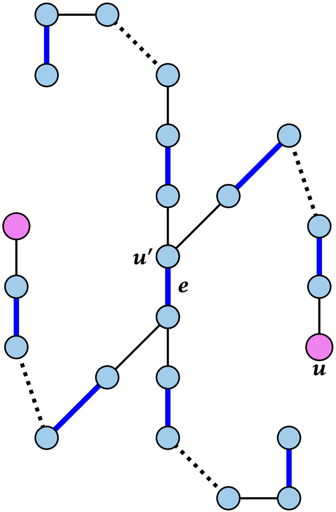

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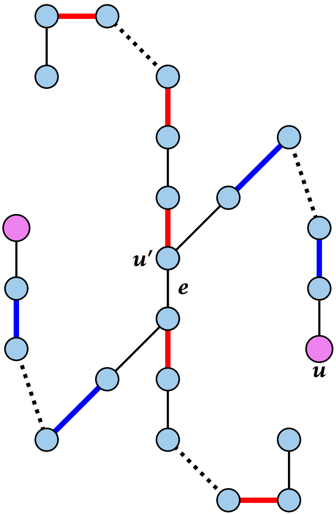

17 Augmenting Paths for Matchings

Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
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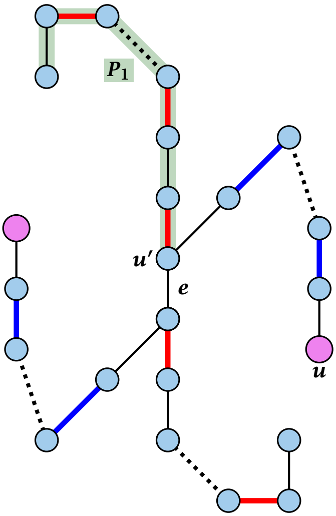

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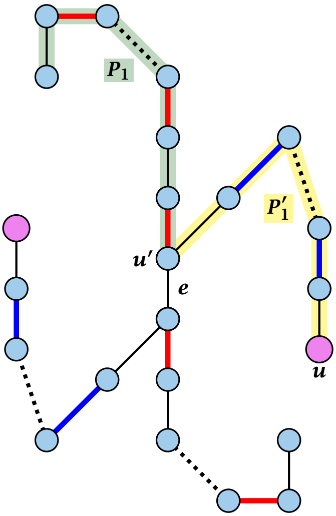

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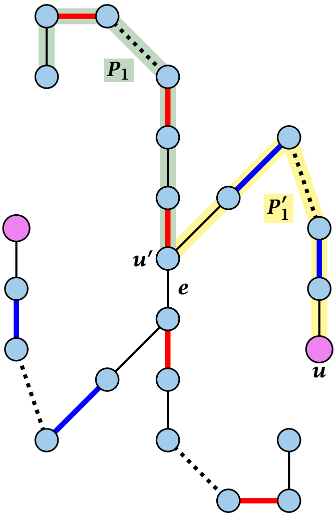
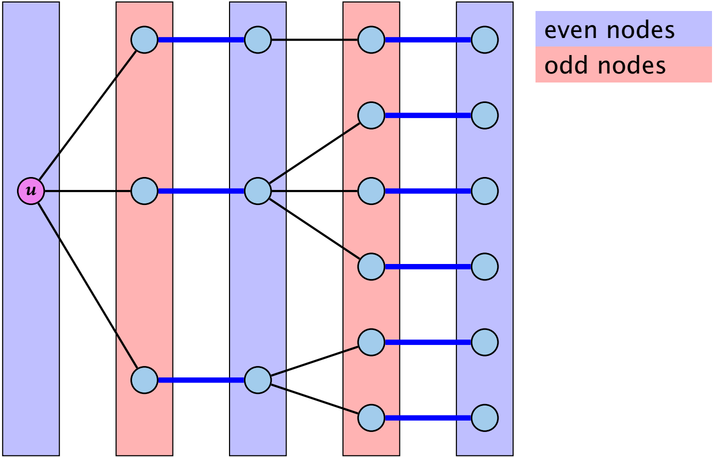
- u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P'_1.
- $P_1 \circ P'_1$ is augmenting path in M (↯).



How to find an augmenting path?

Construct an alternating tree.



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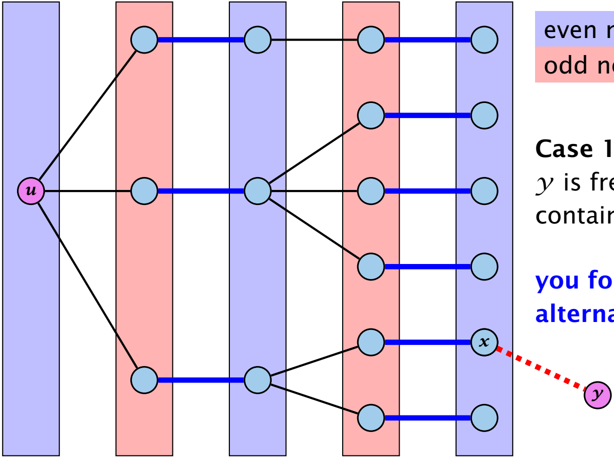


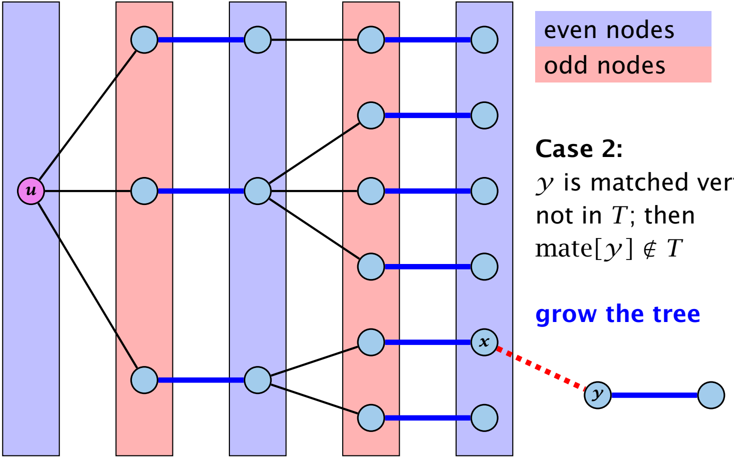
even nodes
odd nodes

Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.

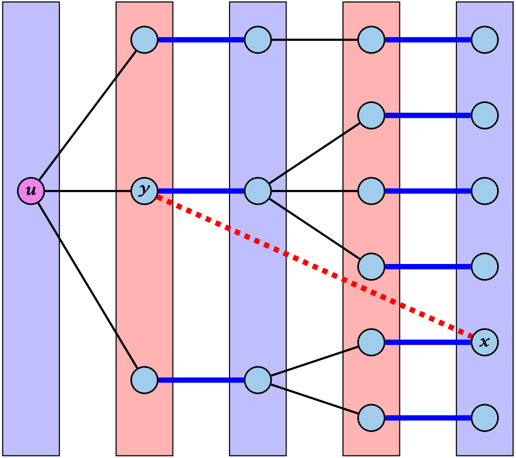


Case 2:
y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.



even nodes

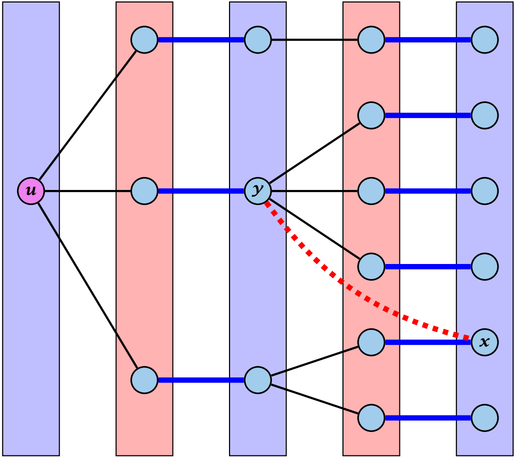
odd nodes

Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

Case 4:

y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 24 BiMatch(G, *match*)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

graph $G = (S \cup S', E)$

$$S = \{1, \ldots, n\}$$

$$S' = \{1', \ldots, n'\}$$

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                    free ← free − 1;
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```

start with an empty matching

Algorithm 24 BiMatch(G, $match$)

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                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

free: number of unmatched nodes in S

r: root of current tree

Algorithm 24 BiMatch(G, $match$)

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                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

Algorithm 24 BiMatch(G, *match*)

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```

r is the new node that we grow from.

Algorithm 24 BiMatch(G, $match$)

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```

If r is free start tree construction

Algorithm 24 BiMatch$(G, match)$

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for x ∈ V do mate[x] ← 0;
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                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0  then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

Initialize an empty tree. Note that only nodes i' have parent pointers.

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                else
                    if parent[y] = 0 then
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```

Q is a queue (BFS!!!).

aug is a Boolean that stores whether we already found an augmenting path.

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```

as long as we did not augment and there are still unexamined leaves continue...

Algorithm 24 BiMatch(G, $match$)

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                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

take next unexamined leaf

Algorithm 24 BiMatch(G, $match$)

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while free ≥ 1 and r < n do
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if x has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

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```

do an augmentation...

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```

setting aug = true ensures that the tree construction will not continue

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reduce number of free nodes

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```

if y is not in the tree yet

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```

...put it into the tree

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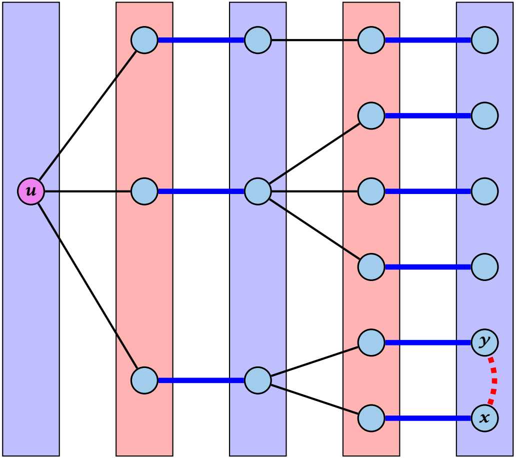
```

add its buddy to the set of unexamined leaves

How to find an augmenting path?

Construct an alternating tree.



even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

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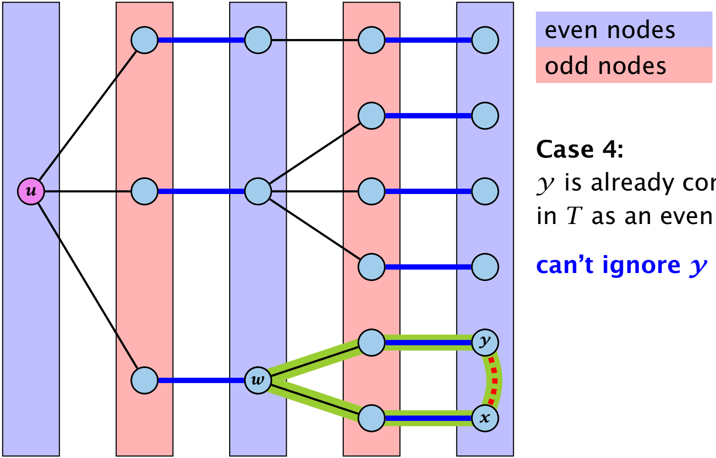
Construct an alternating tree.



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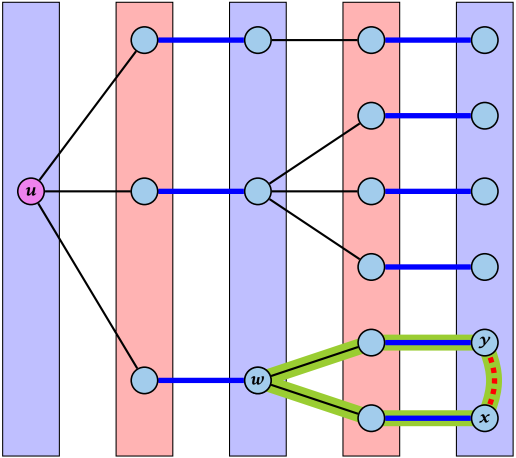

even nodes
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y is already contained in T as an even vertex

can't ignore y

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom. w is called the base of the blossom (even node!!!). The path u-w is called the stem of the blossom.

Flowers and Blossoms

Definition 93
A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

- A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

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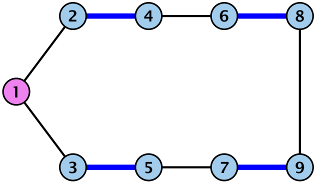
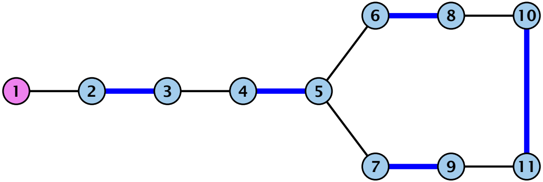
Flowers and Blossoms





Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.
2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
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Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

5. The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms

Properties:

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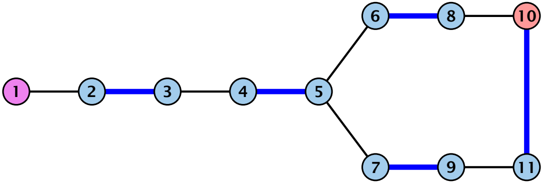
Flowers and Blossoms



Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- Delete all vertices in B (and its incident edges) from G.
- Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

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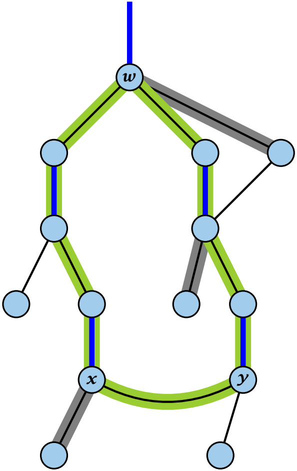
Shrinking Blossoms

- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.
- Nodes that are connected in G to at least one node in B become connected to b in G'.



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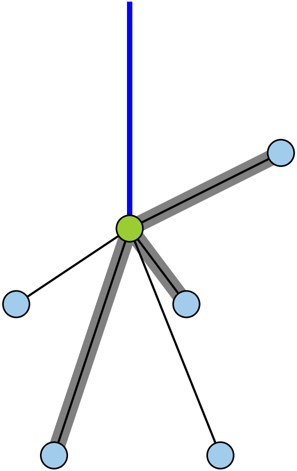
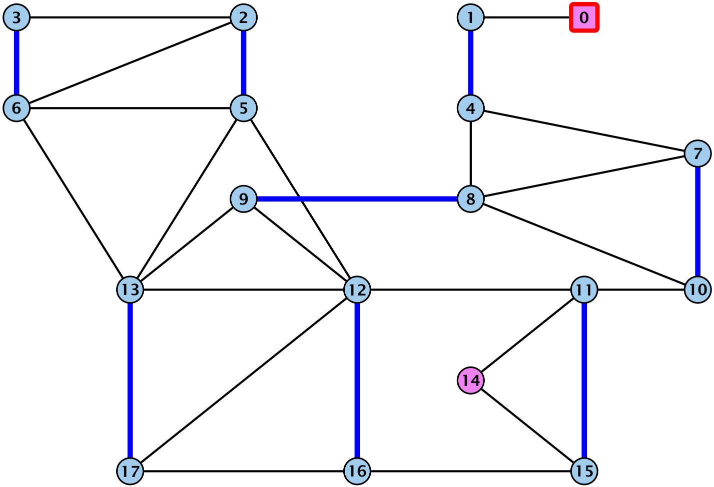
- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.
- Nodes that are connected in G to at least one node in B become connected to b in G'.



Example: Blossom Algorithm



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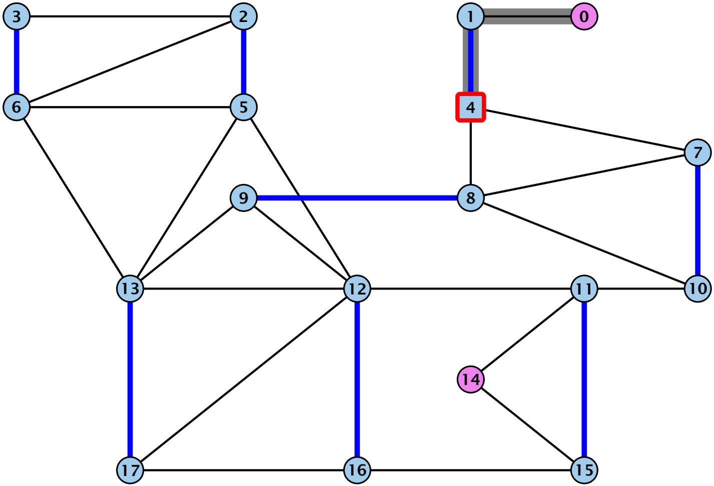

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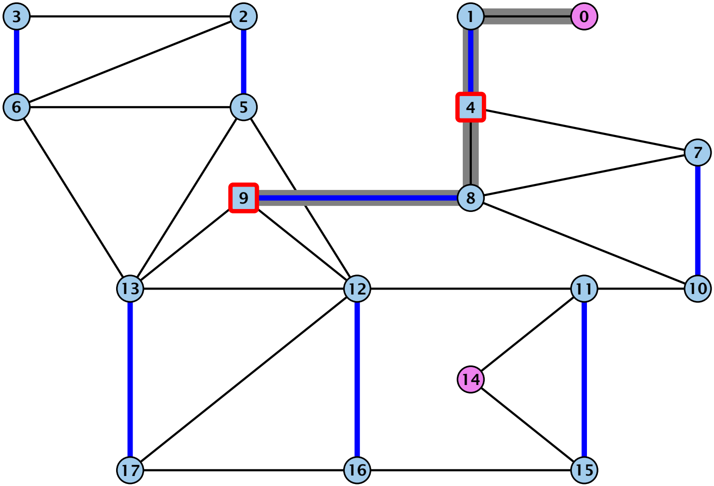

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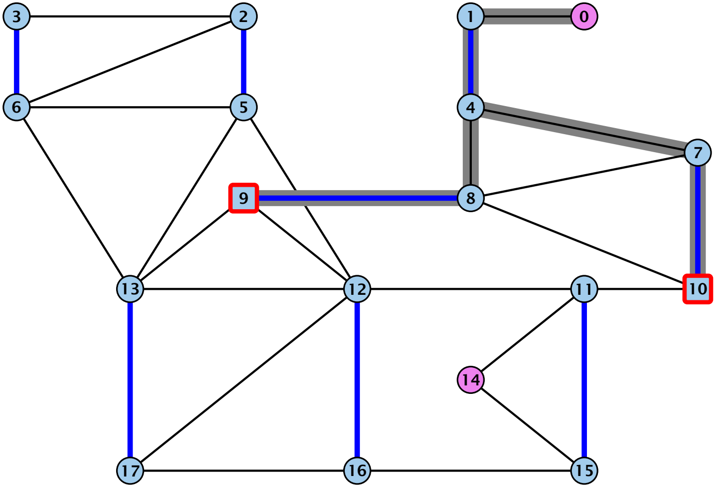

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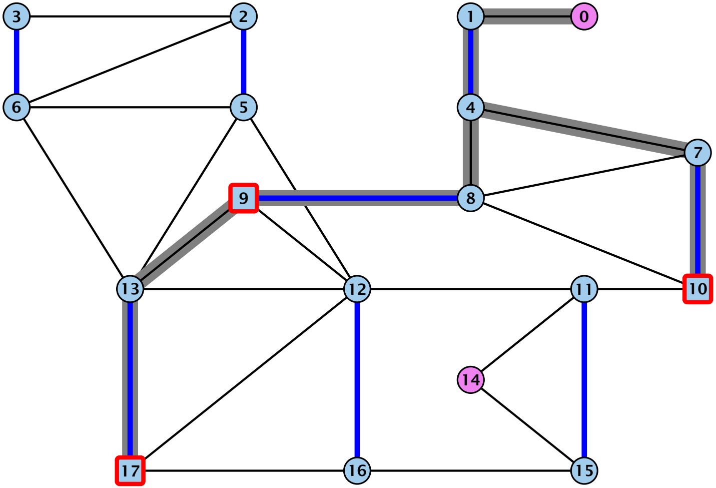

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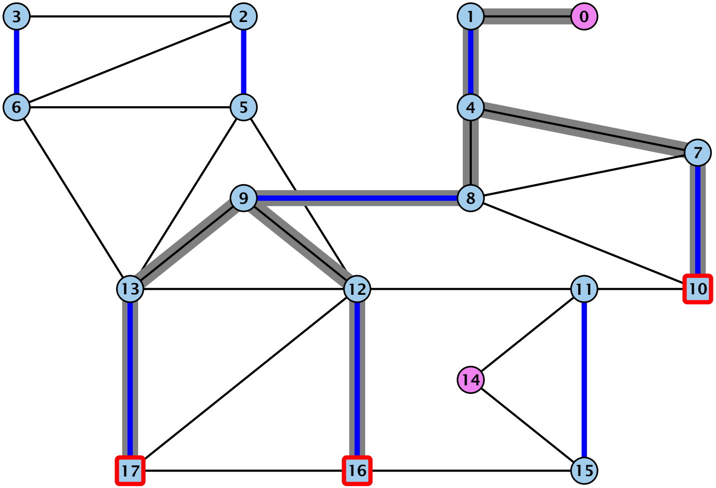

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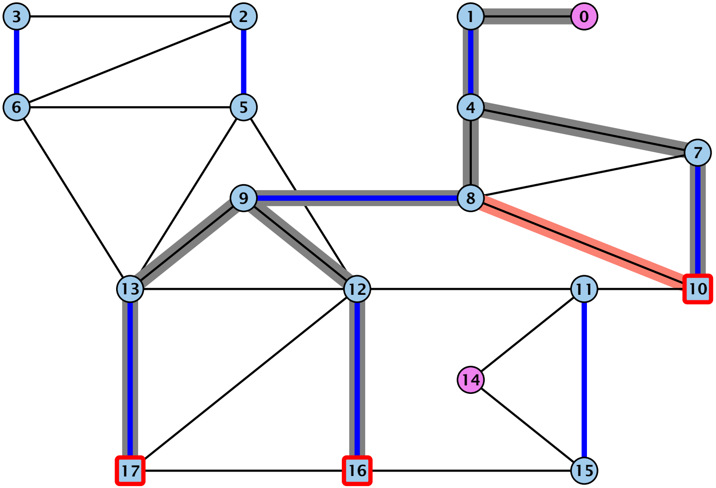

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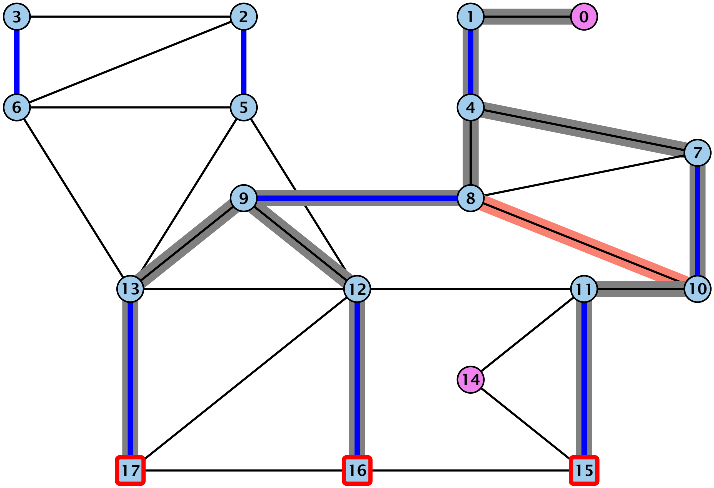

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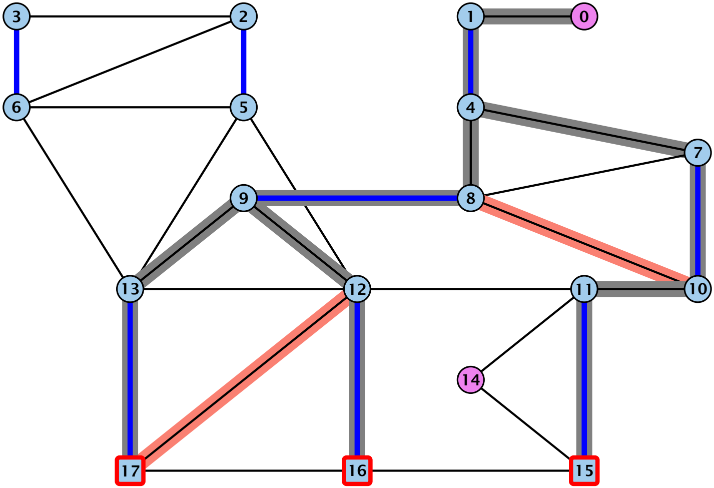

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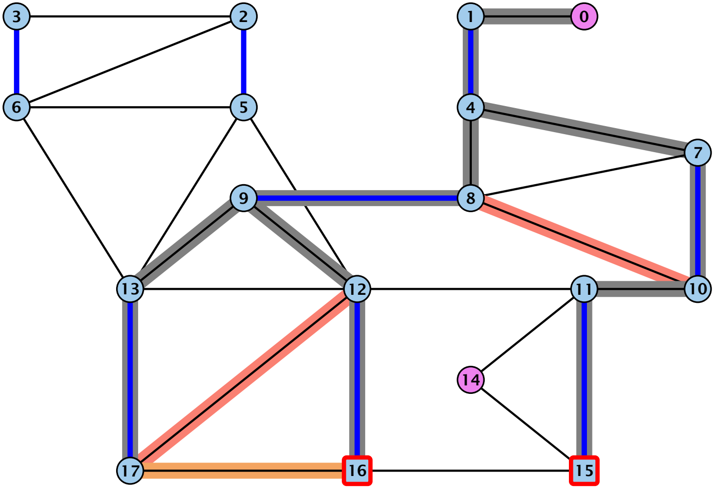

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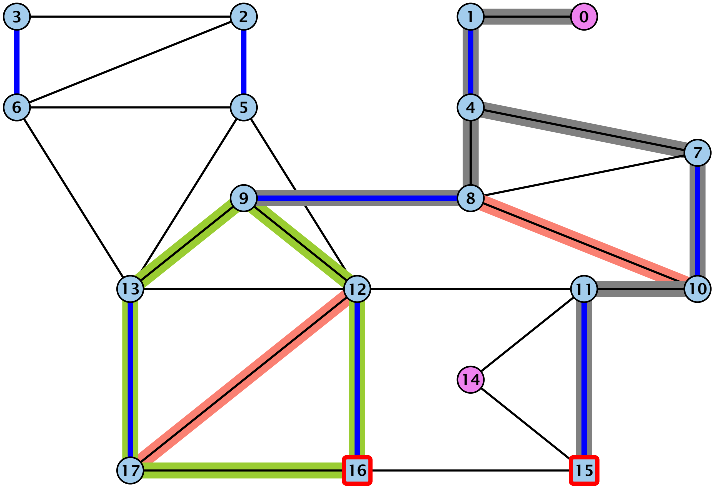

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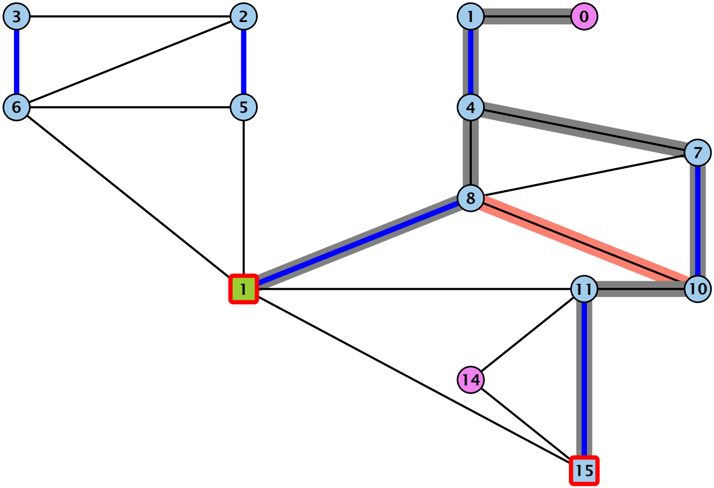

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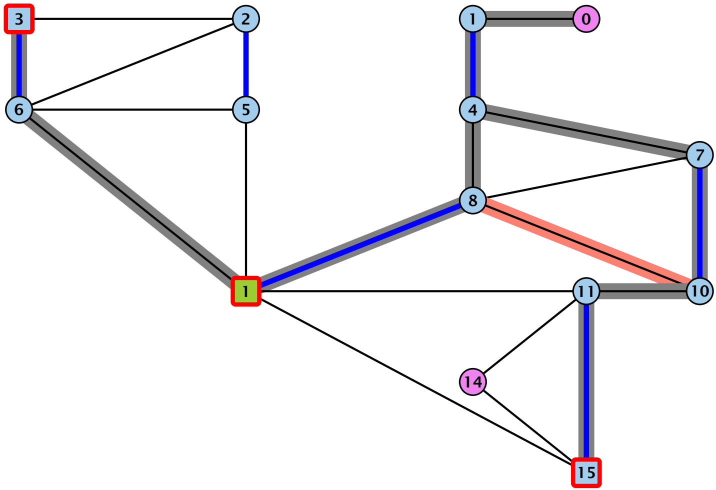

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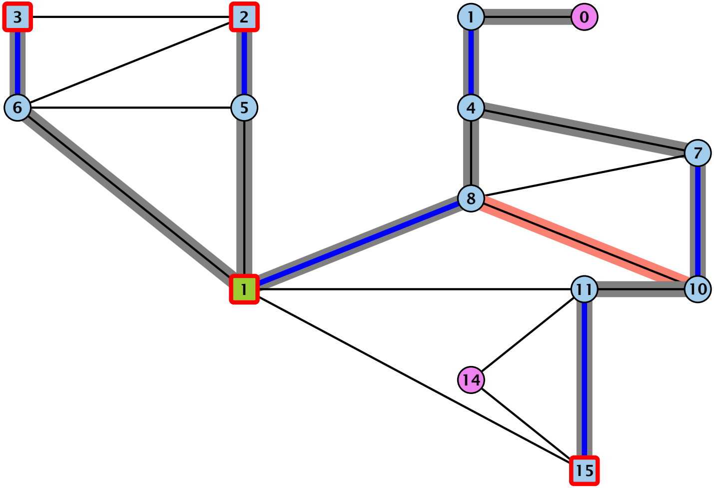

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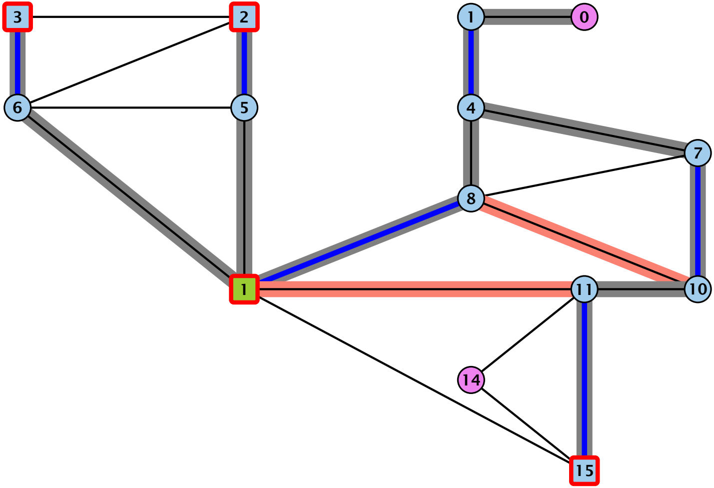

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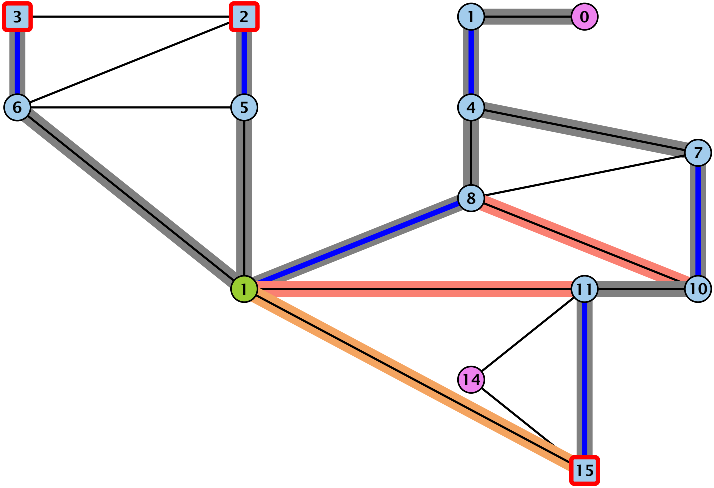

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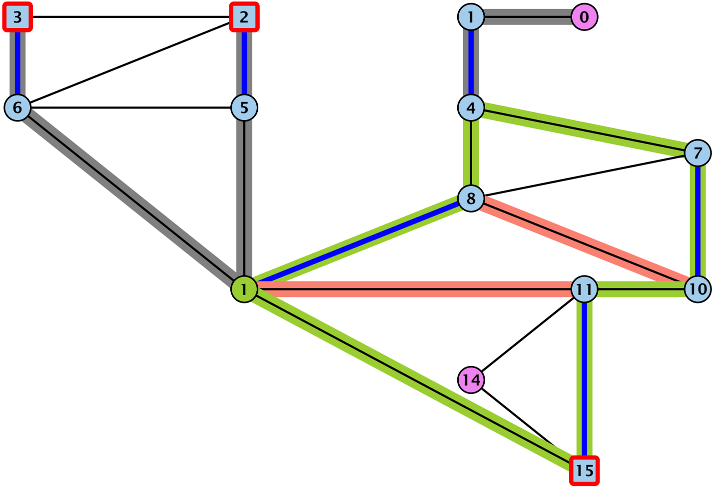

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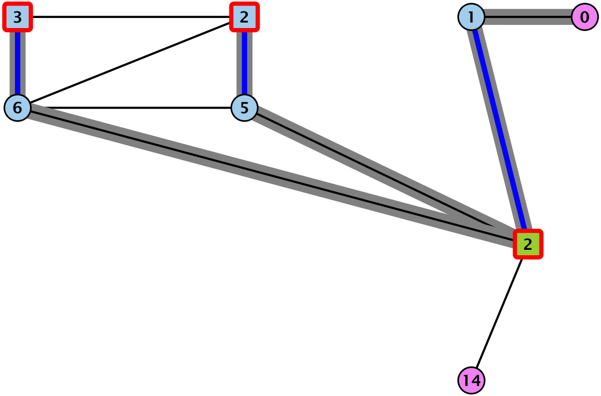

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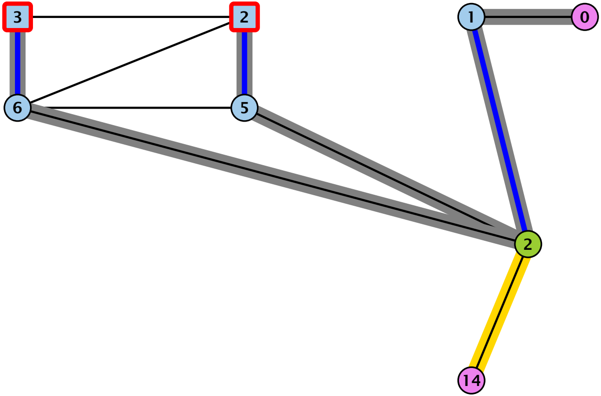

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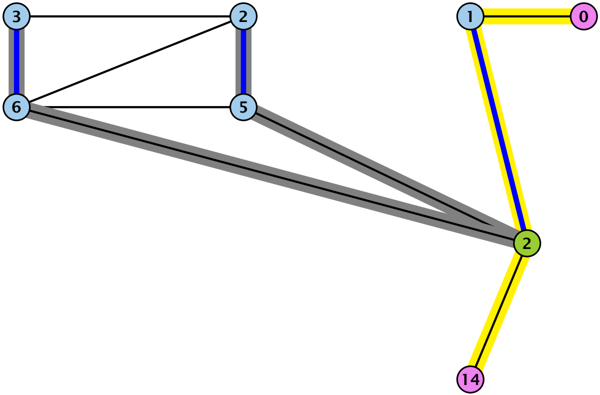

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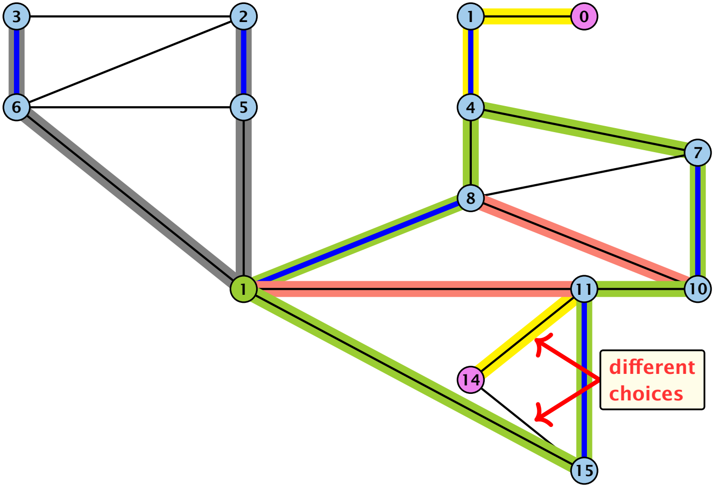

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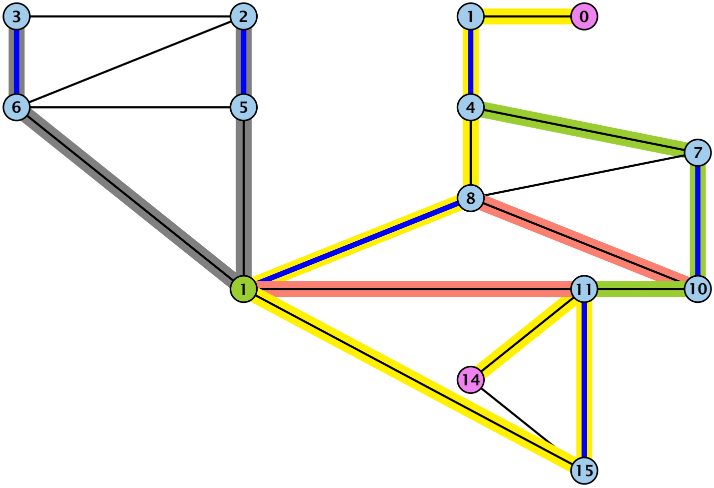

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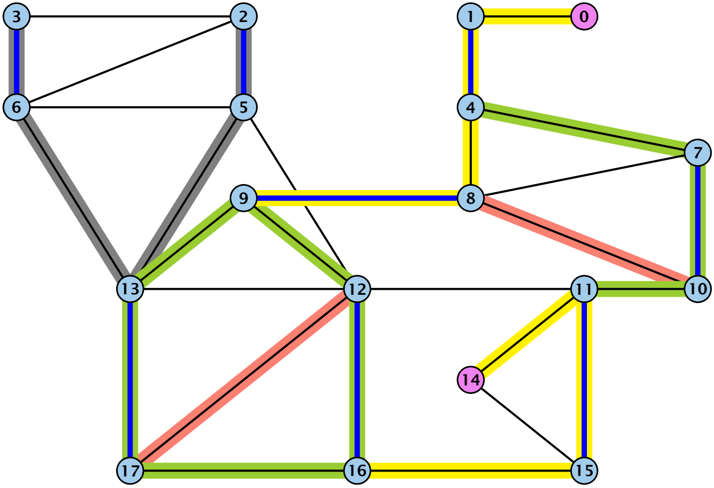
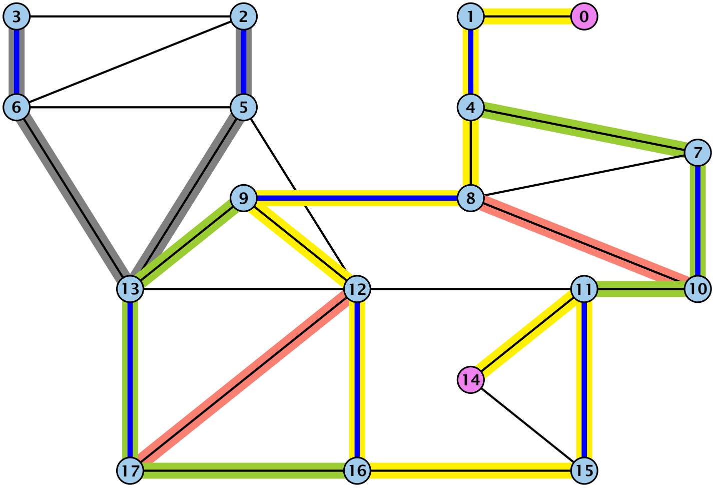


Example: Blossom Algorithm



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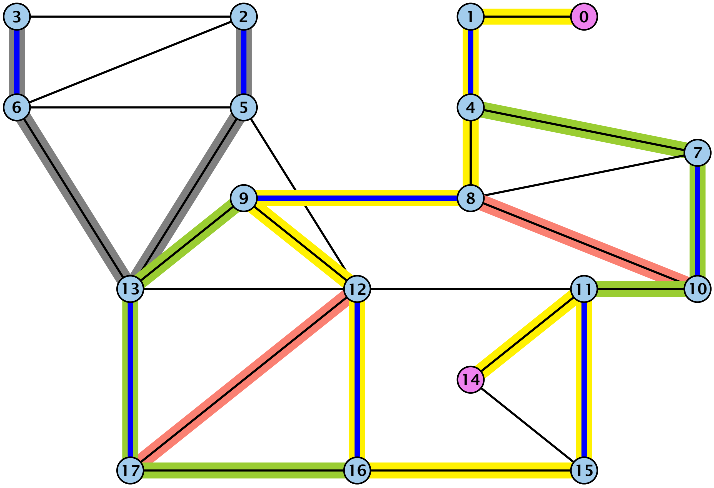

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 94

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

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Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

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Case 1: non-empty stem

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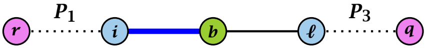

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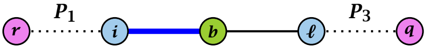

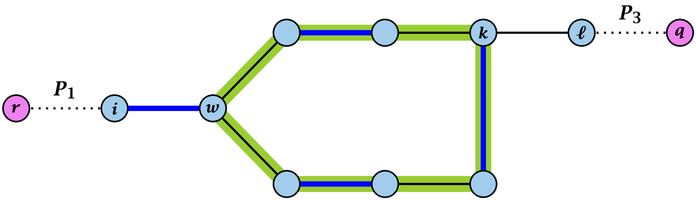

Correctness

- After the expansion ℓ must be incident to some node in the blossom. Let this node be k.
- If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.
- $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

Correctness

Proof.

Case 2: empty stem

- If the stem is empty then after expanding the blossom, $w = r$.

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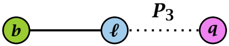

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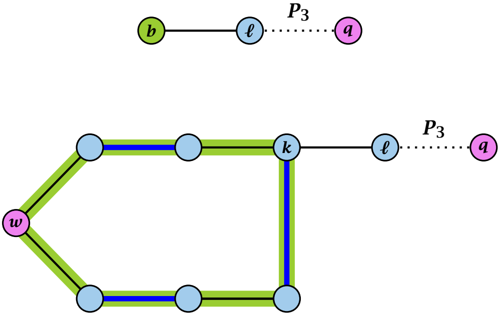

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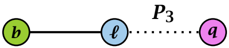

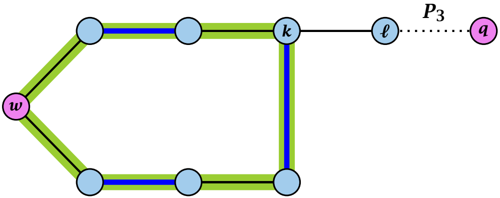

- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 95
If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

- If P does not contain a node from B there is nothing to prove.
- We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

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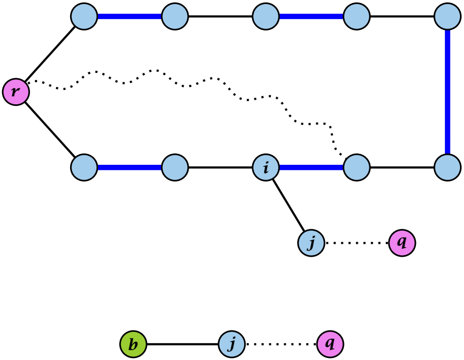
Correctness

Illustration for Case 1:



Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

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Algorithm 25 search(r, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
found $\leftarrow$ false
unlabel all nodes;
give an even label to r and initialize *list* $\leftarrow \{r\}$
while *list* $\neq \emptyset$ **do**
    delete a node i from *list*
    examine(i, *found*)
    if *found* $=$ true **then return**

Search for an augmenting path starting at r.

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$A(i)$ contains neighbours of node i.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

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found is just a Boolean that allows to abort the search process...

Algorithm 25 search(r, *found*)

```
set Ā(i) ← A(i) for all nodes i
found ← false
unlabel all nodes;
give an even label to r and initialize list ← {r}
while list ≠ ∅ do
    delete a node i from list
    examine(i, found)
    if found = true then return
```

In the beginning no node is in the tree.

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Put the root in the tree.

list could also be a set or a stack.

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while $list \neq \emptyset$ **do**
    delete a node i from $list$
    examine(i, $found$)
    if $found = \text{true}$ **then return**

As long as there are nodes with unexamined neighbours...

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...examine the next one

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while *list* $\neq \emptyset$ **do**
    delete a node i from *list*
    examine(i, *found*)
    if *found* $=$ true **then return**

If you found augmenting path
abort and start from next root.

Algorithm 26 examine(i, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

Examine the neighbours of a node i

Algorithm 26 examine(i, *found*)

```
for all j ∈ Ā(i) do
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        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

For all neighbours j do...

Algorithm 26 examine(i, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

You have found a blossom...

Algorithm 26 examine(i, *found*)

for all $j \in \bar{A}(i)$ **do**
    if j is even **then** contract(i, j) and **return**
    if j is unmatched **then**
        $q \leftarrow j$;
        $\text{pred}(q) \leftarrow i$;
        $\mathit{found} \leftarrow \text{true}$;
        return
    if j is matched and unlabeled **then**
        $\text{pred}(j) \leftarrow i$;
        $\text{pred}(\text{mate}(j)) \leftarrow j$;
        add $\text{mate}(j)$ to *list*

You have found a free node which gives you an augmenting path.

Algorithm 26 examine(i, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

If you find a matched node that is not in the tree you grow...

Algorithm 26 examine(i, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

mate(j) is a new node from which you can grow further.

Algorithm 27 contract(i, j)

trace pred-indices of i and j to identify a blossom B
create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
label b even and add to *list*
update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in B
delete nodes in B from the graph

Contract blossom identified by nodes i and j

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

b will be an even node, and it has unexamined neighbours.

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Every node that was adjacent to a node in B is now adjacent to b

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
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6: delete nodes in B from the graph

Only for making a blossom expansion easier.

Algorithm 27 contract(i, j)

```
trace pred-indices of i and j to identify a blossom B
create new node b and set Ā(b) ← ∪_{x∈B} Ā(x)
label b even and add to list
update Ā(j) ← Ā(j) ∪ {b} for each j ∈ Ā(b)
form a circular double linked list of nodes in B
delete nodes in B from the graph
```

Only delete links from nodes not in B to B.

When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.
- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.
- There are at most n contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.
- In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

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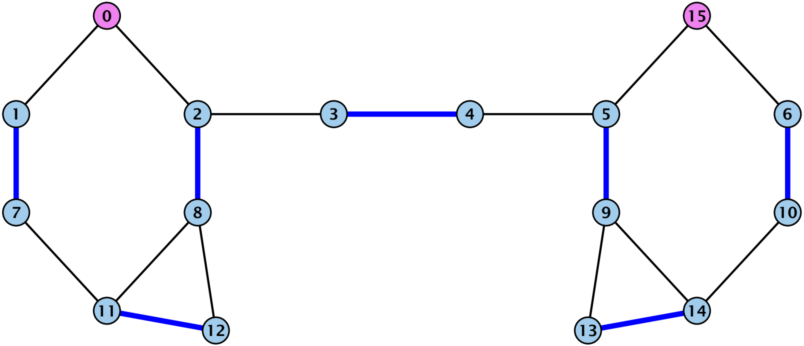
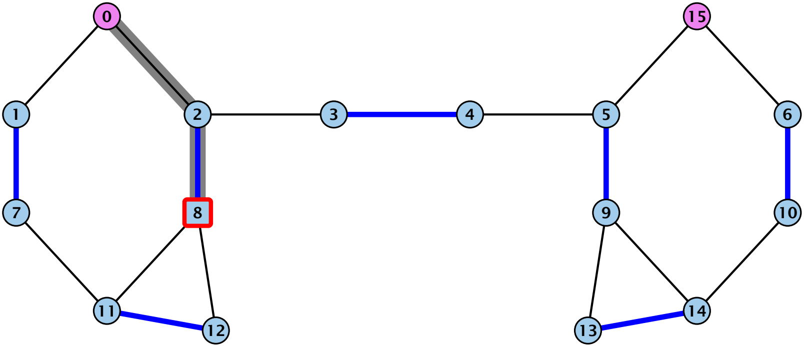
Example: Blossom Algorithm



Example: Blossom Algorithm



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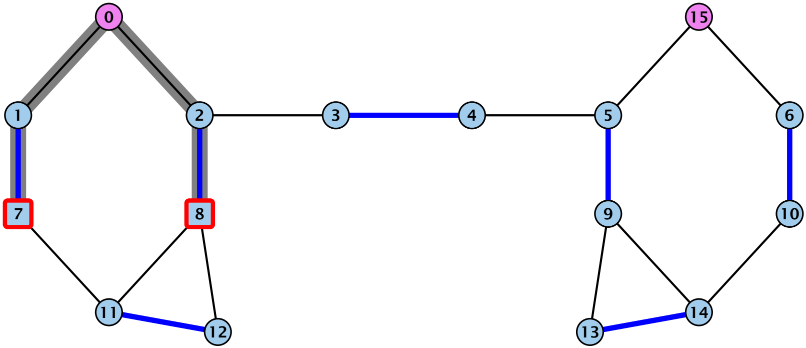

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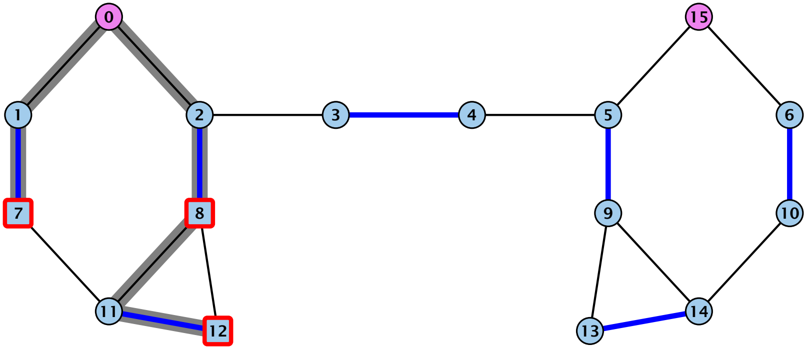

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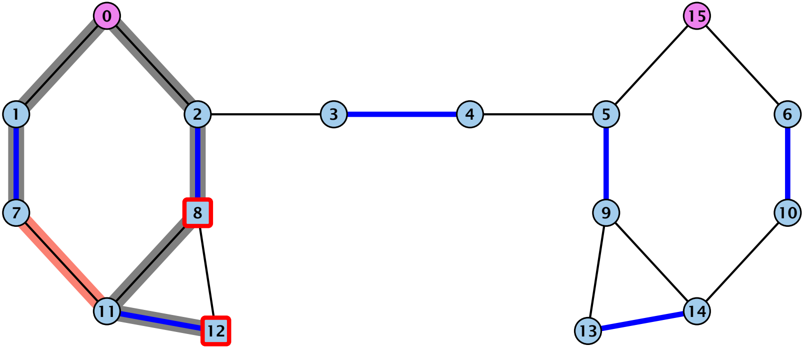

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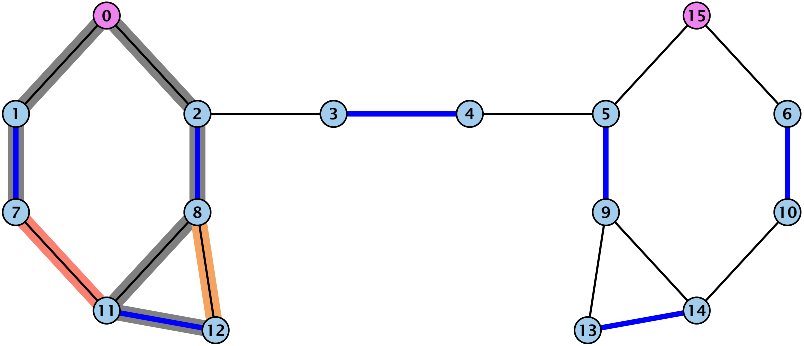

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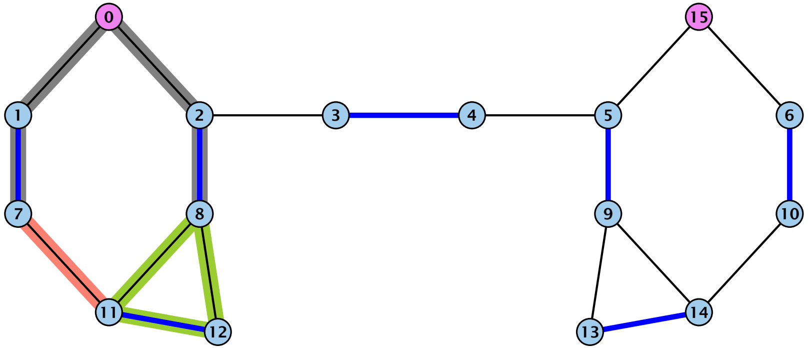

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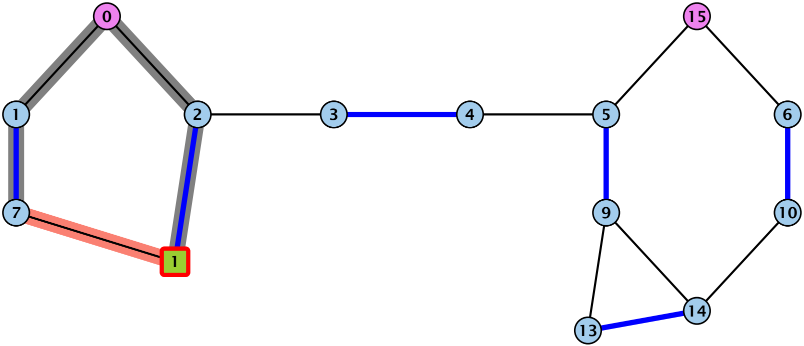

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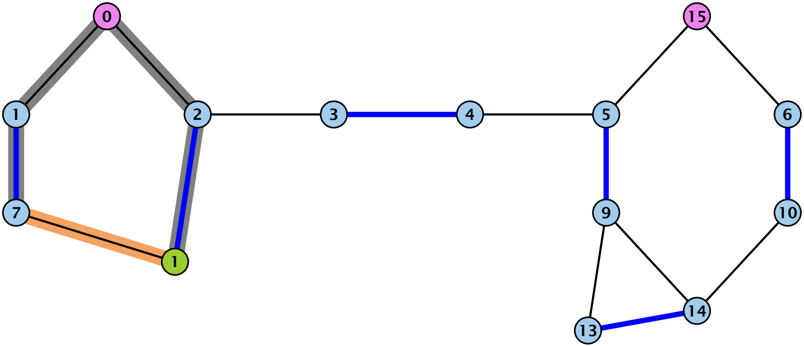

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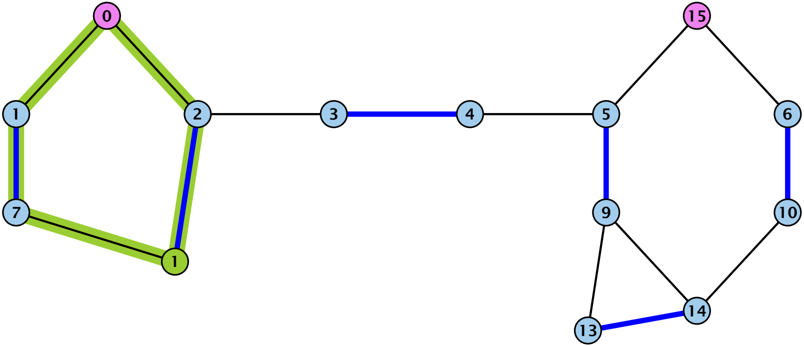

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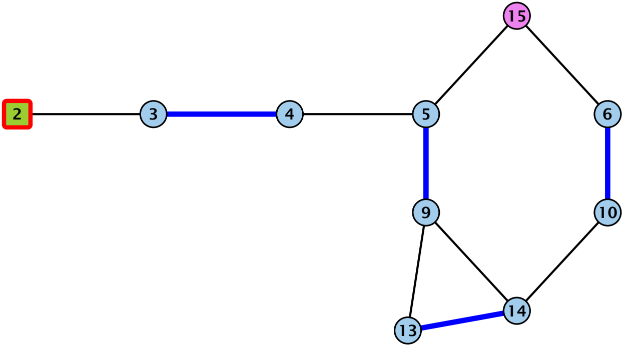

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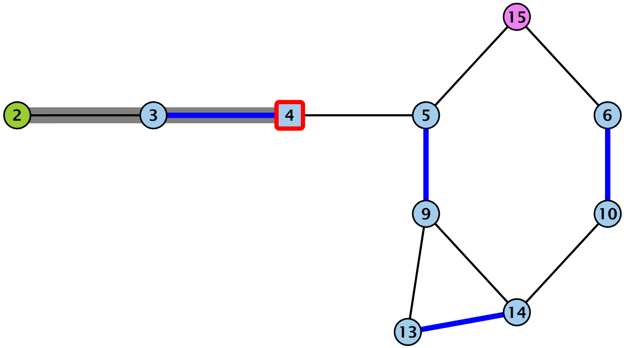

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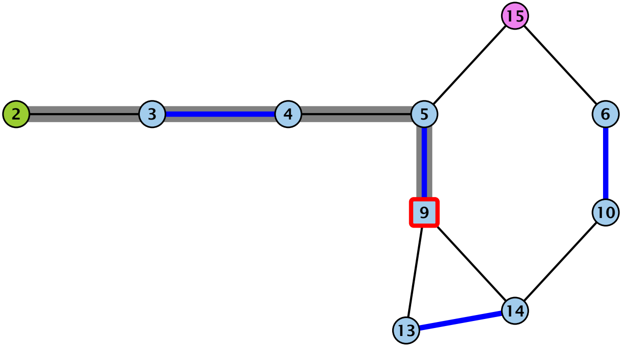

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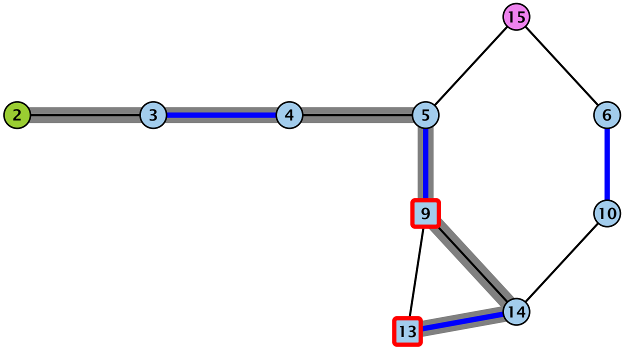

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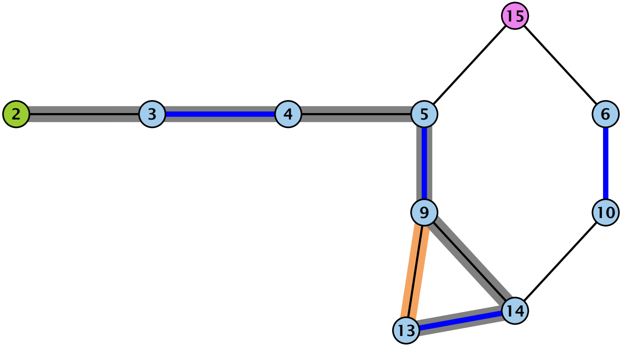

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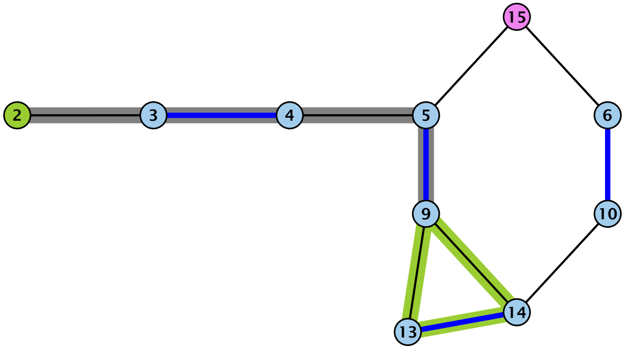

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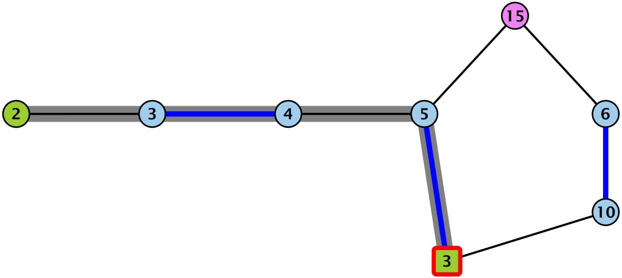

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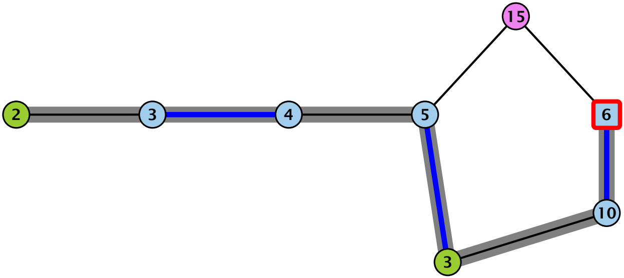

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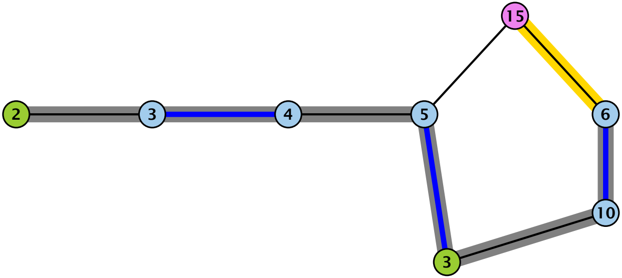

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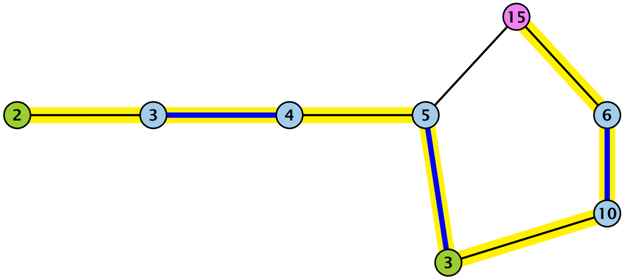

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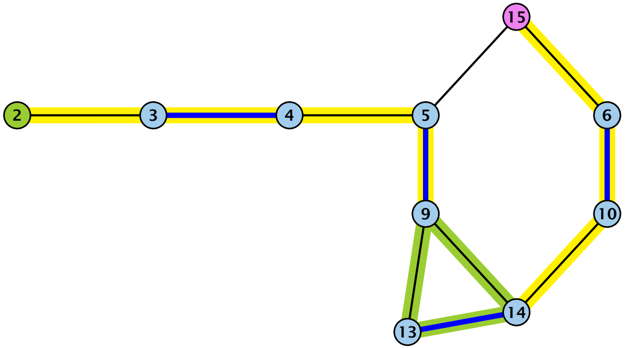

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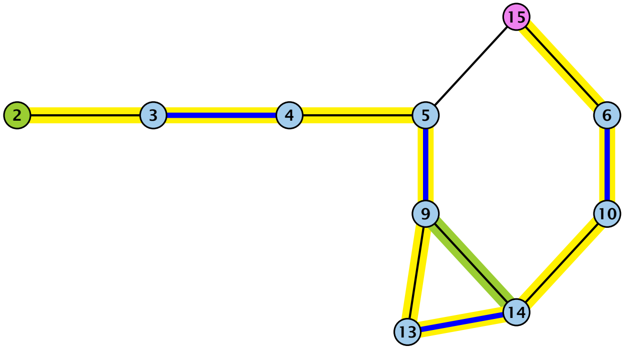

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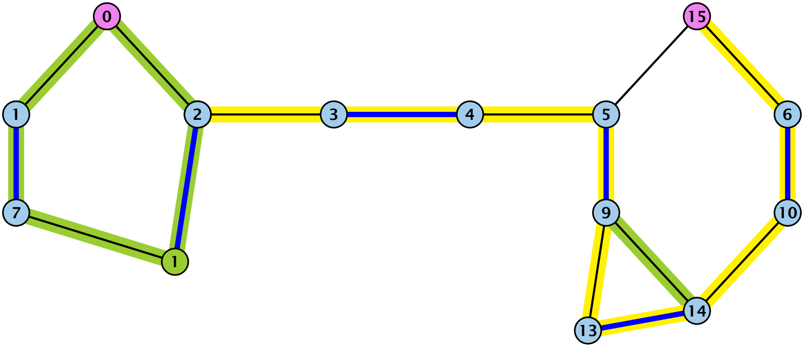

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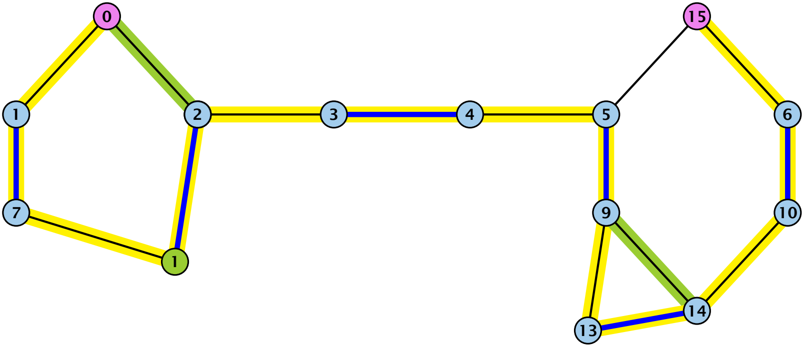

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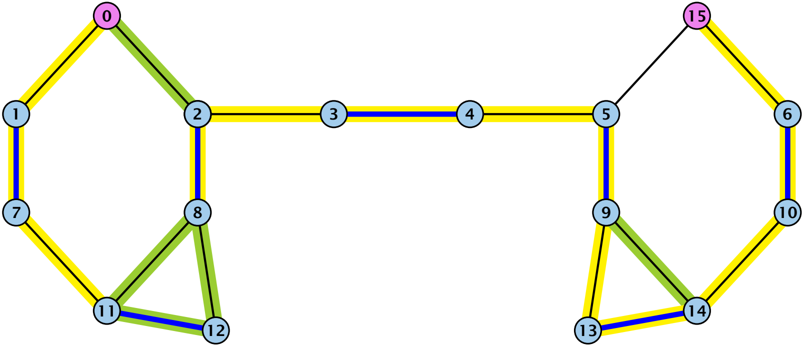

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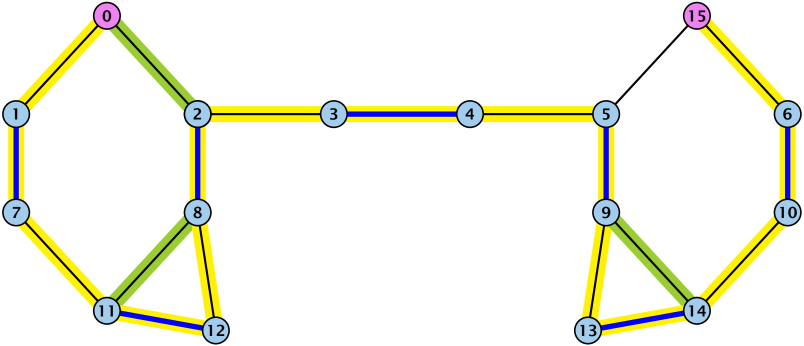

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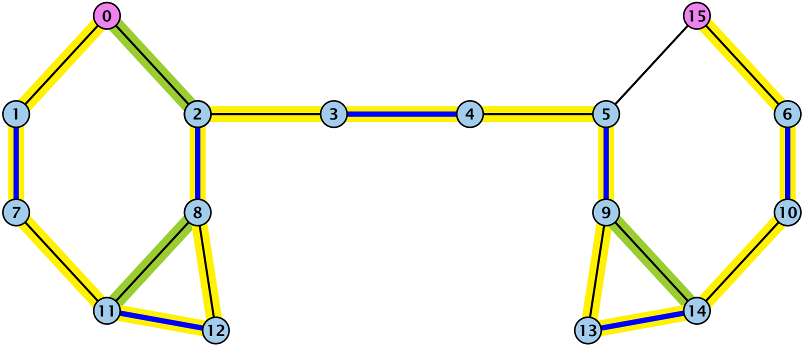

A Fast Matching Algorithm

Algorithm 28 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \emptyset$
2: **repeat**
3: let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be maximal set of
4: vertex-disjoint, shortest augmenting path w.r.t. M.
5: $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
6: **until** $\mathcal{P} = \emptyset$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis Hopcroft-Karp

Lemma 96

Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- The connected components of G are cycles and paths.
- The graph contains $k \triangleq |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. M.

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Analysis Hopcroft-Karp

- Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).
- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \dots \cup P_k) = M \oplus P_1 \oplus \dots \oplus P_k$.
- Let P be an augmenting path in M'.

Lemma 97

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \dots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

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Analysis Hopcroft-Karp

Proof.

- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.
- Hence, the set contains at least $k+1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.
- Each of these paths is of length at least ℓ.

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- If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.
- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \leq k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$.

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Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Proof.

The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

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Analysis Hopcroft-Karp

Lemma 99
The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ *phases.*

Proof.

- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|}+1) \leq \sqrt{|V|}$ additional augmentations.

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Analysis Hopcroft-Karp

Lemma 100

One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

construct a “level graph” G':

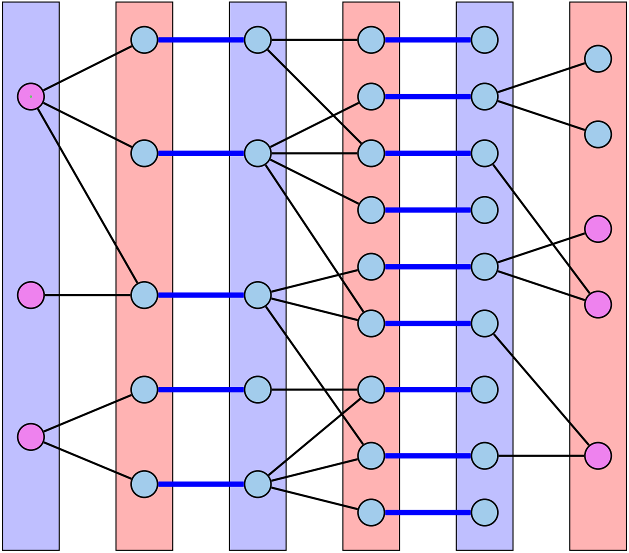
- construct Level 0 that includes all free vertices on left side L
- construct Level 1 containing all neighbors of Level 0
- construct Level 2 containing matching neighbors of Level 1
- construct Level 3 containing all neighbors of Level 2
- ...
- stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

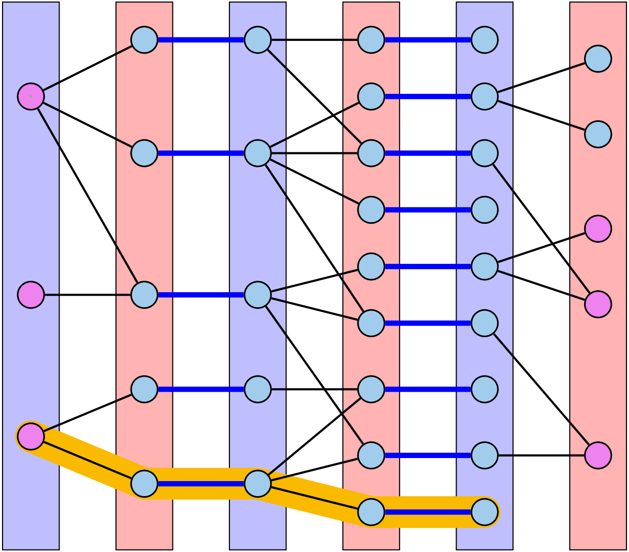
Analysis Hopcroft-Karp

- a shortest augmenting path must go from Level 0 to the last layer constructed
- it can only use edges between layers
- construct a maxim**al** set of vertex disjoint augmenting path connecting the layers
- for this, go forward until you either reach a free vertex or you reach a "dead end" v
- if you reach a free vertex delete the augmenting path and all incident edges from the graph
- if you reach a dead end backtrack and delete v together with its incident edges

Analysis Hopcroft-Karp



Analysis Hopcroft-Karp



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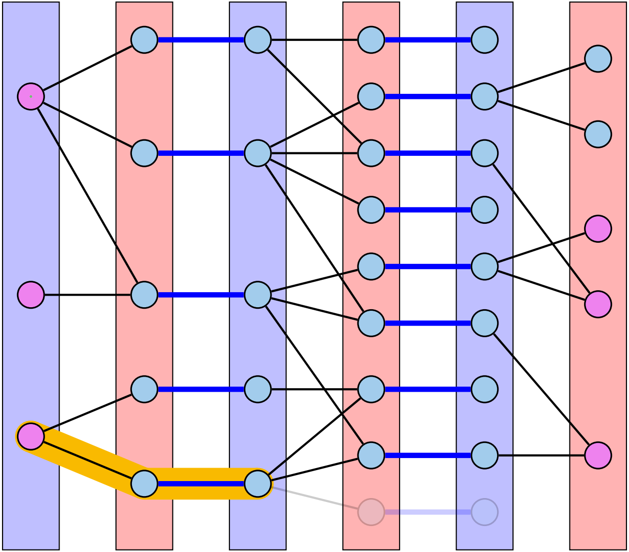

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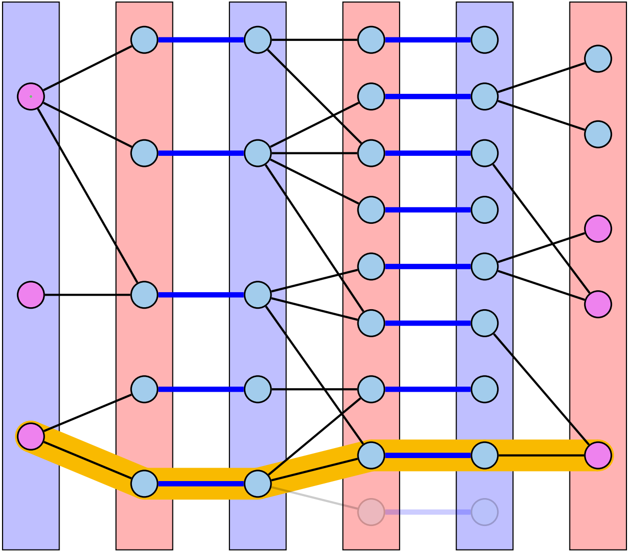

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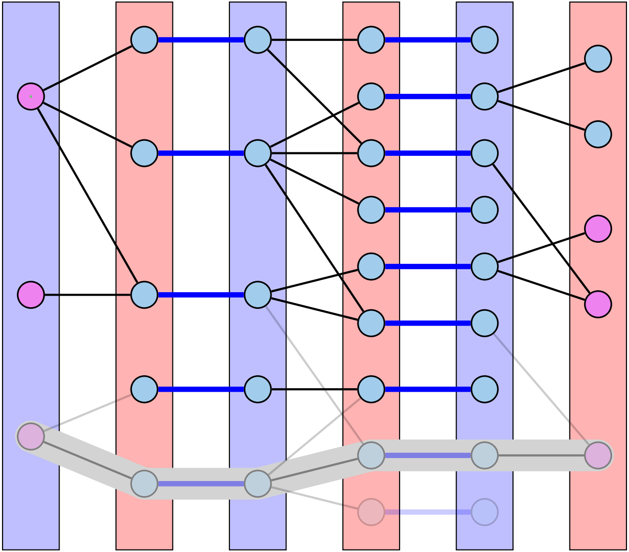

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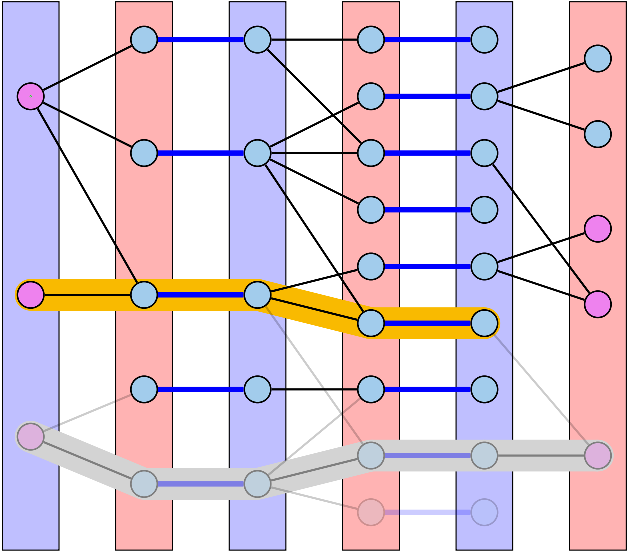

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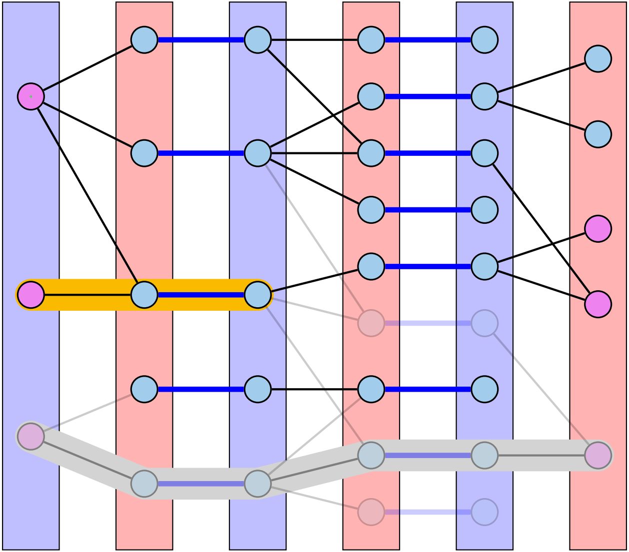

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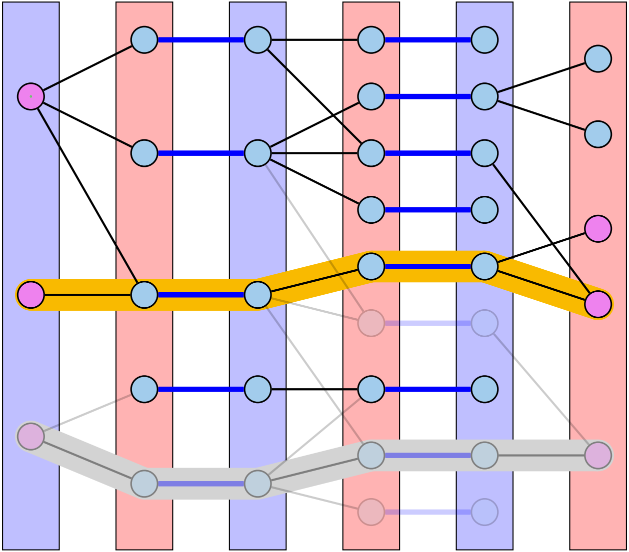

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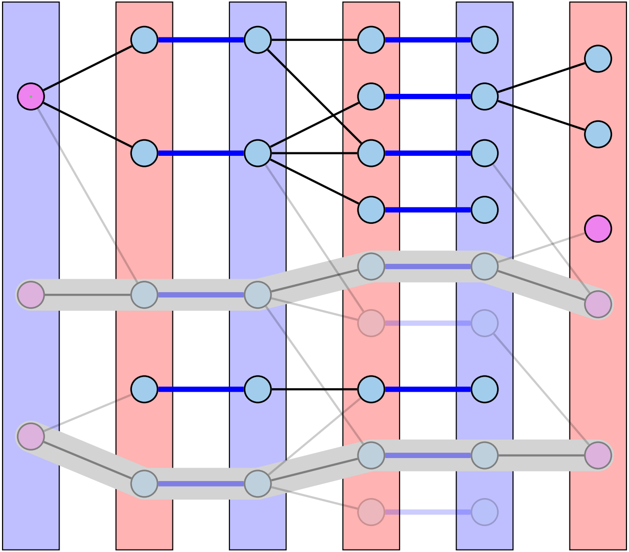

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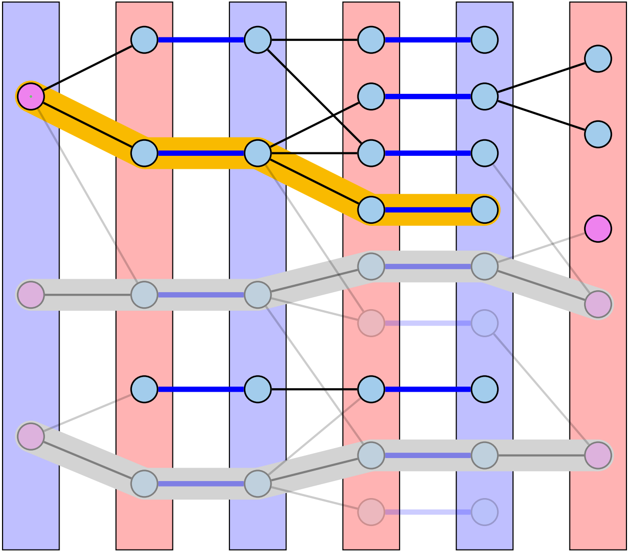

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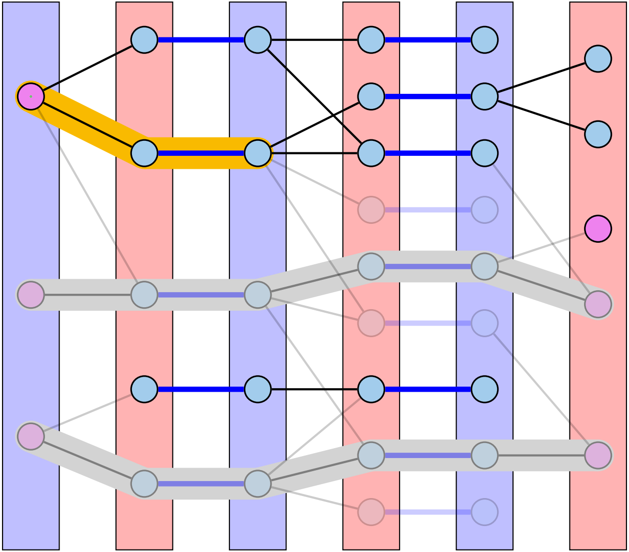

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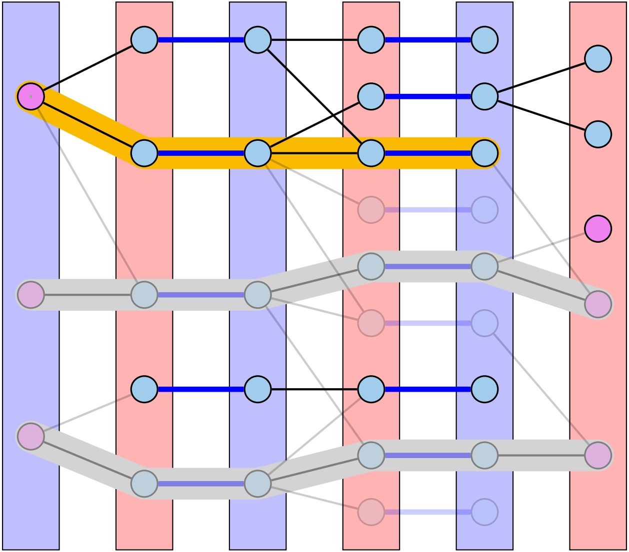

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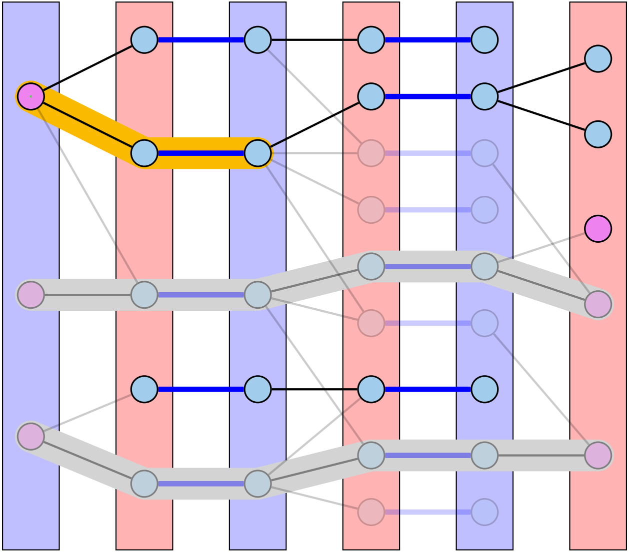

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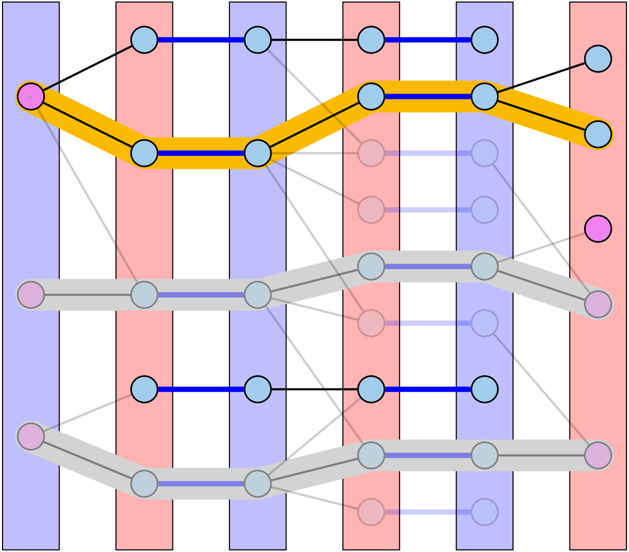

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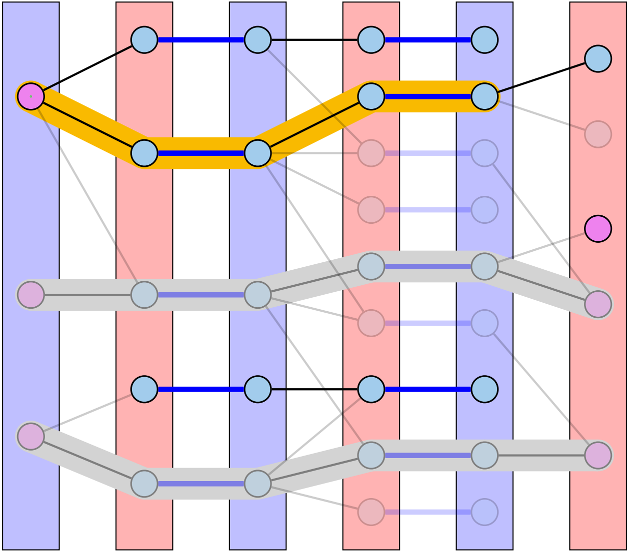

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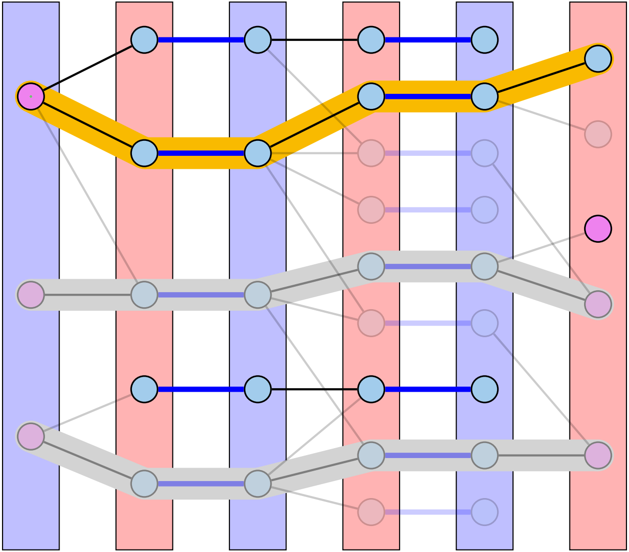

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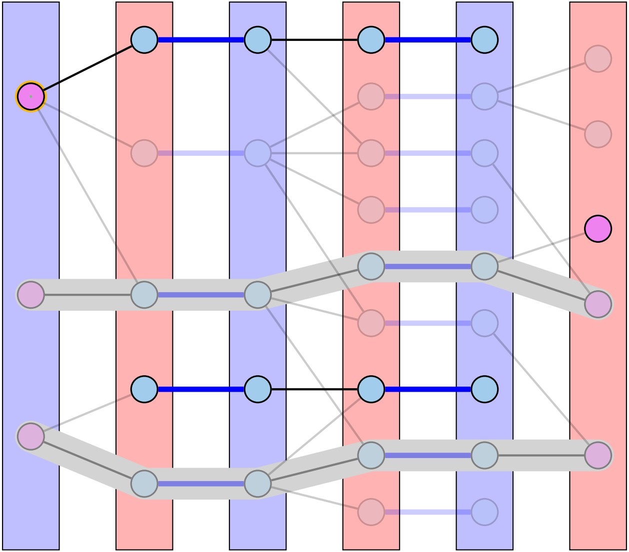

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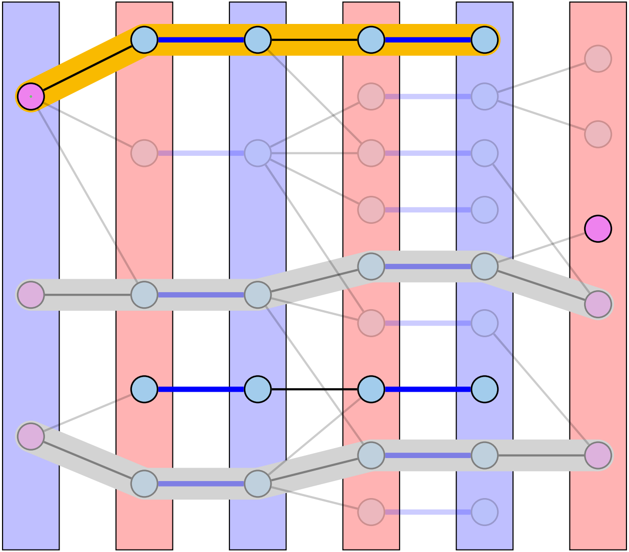

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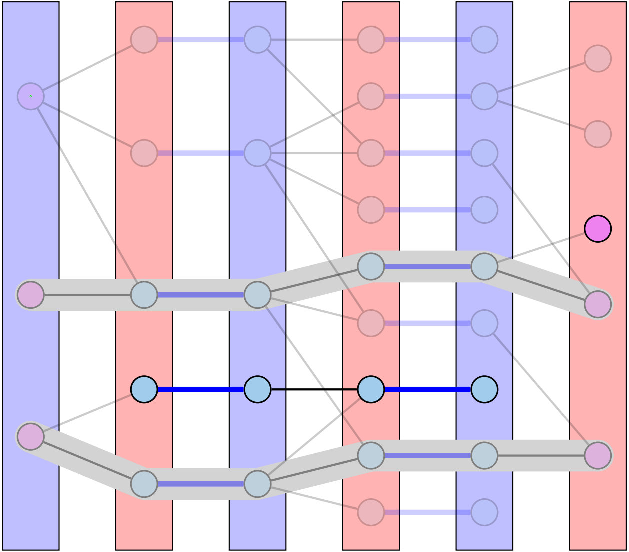

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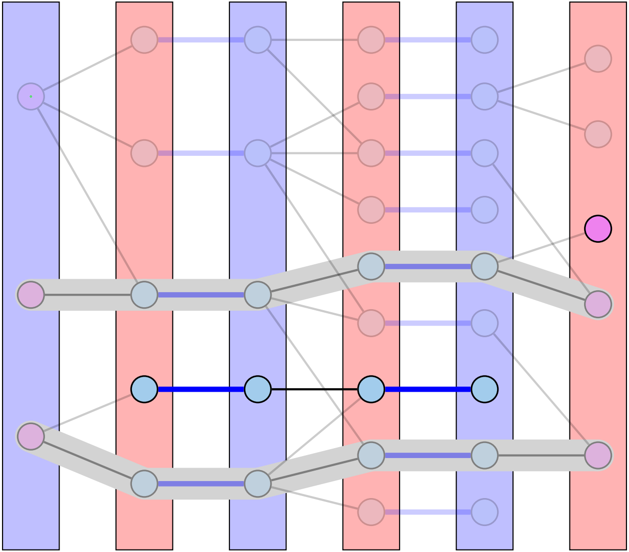

Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $\mathcal{O}(mn)$

- a search (successful or unsuccessful) takes time $\mathcal{O}(n)$
- a search deletes at least one edge from the level graph

there are at most n phases

Time: $\mathcal{O}(mn^2)$.

Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$

- an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases

- after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph
- hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.