# Overview: Shortest Augmenting Paths

**Lemma 1**

*The length of the shortest augmenting path never decreases.*

**Lemma 2**

*After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.*

# Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

**Theorem 3**

*The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.*

**Proof.**

- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

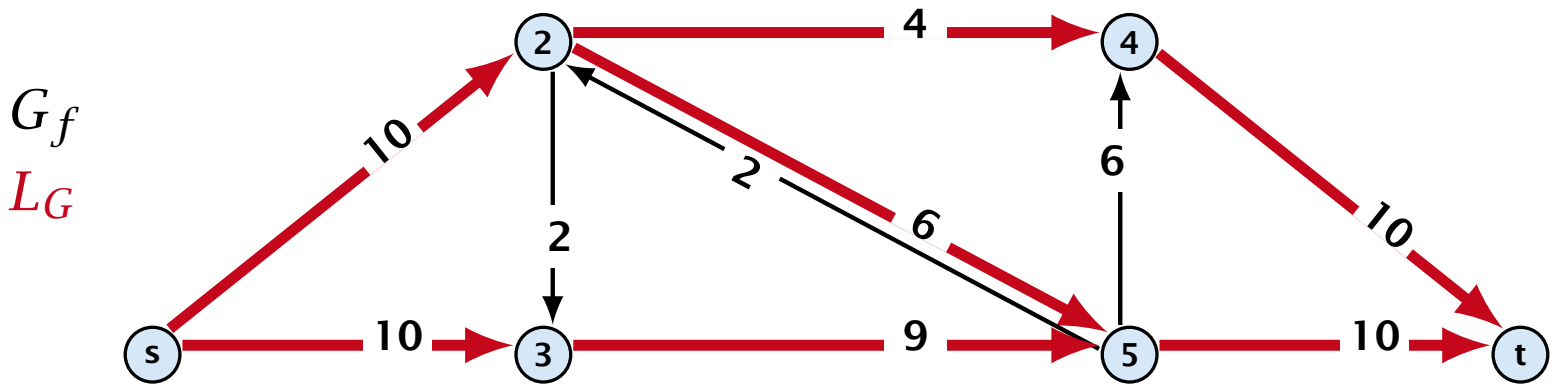
□

# Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest $s$-$v$ path in $G_f$.

Let $L_G$ denote the subgraph of the residual graph $G_f$ that contains only those edges $(u,v)$ with $\ell(v) = \ell(u) + 1$.

A path $P$ is a shortest $s$-$u$ path in $G_f$ if it is a an $s$-$u$ path in $L_G$.



In the following we assume that the residual graph $G_f$ does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

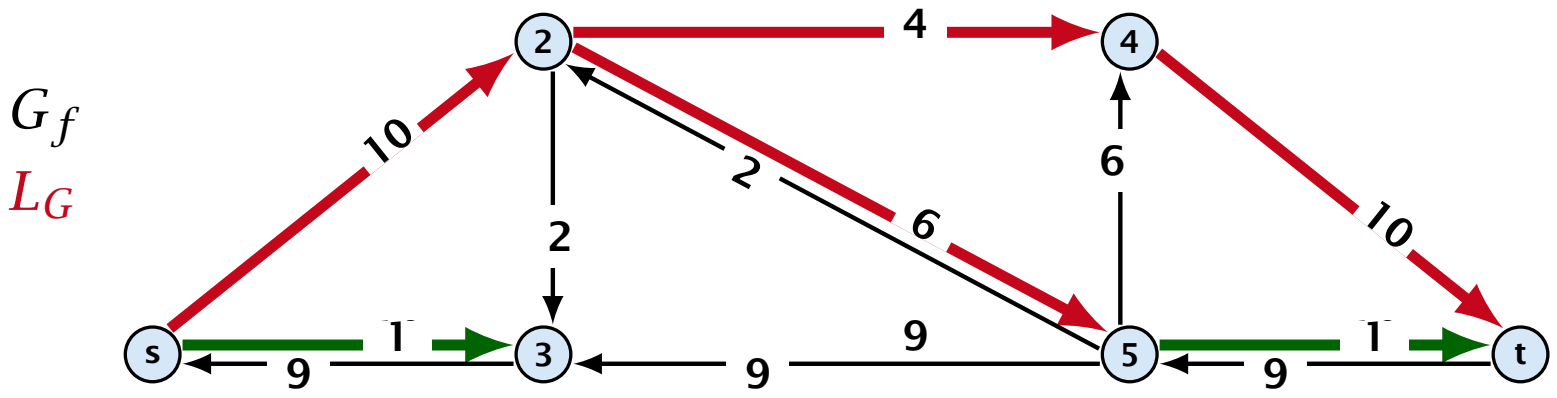
# Shortest Augmenting Path

**First Lemma:**
The length of the shortest augmenting path never decreases.

After an augmentation $G_f$ changes as follows:

- Bottleneck edges on the chosen path are deleted.
- Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between $s$ and $t$.



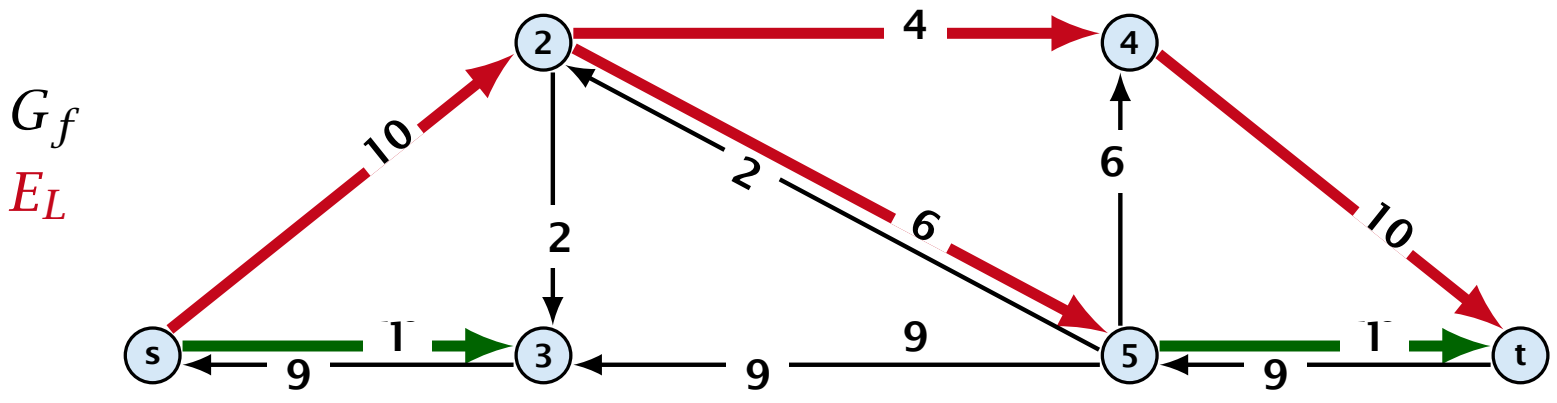
# Shortest Augmenting Path

**Second Lemma:** After at most $m$ augmentations the length of the shortest augmenting path strictly increases.

Let $E_L$ denote the set of edges in graph $L_G$ at the beginning of a round when the distance between $s$ and $t$ is $k$.

An $s$-$t$ path in $G_f$ that uses edges not in $E_L$ has length larger than $k$, even when considering edges added to $G_f$ during the round.

In each augmentation one edge is deleted from $E_L$.



# Shortest Augmenting Paths

**Theorem 4**

*The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.*

**Theorem 5 (without proof)**

*There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.*

**Note:**

There always exists a set of $m$ augmentations that gives a maximum flow (why?).

# Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

# Shortest Augmenting Paths

We maintain a subset $E_L$ of the edges of $G_f$ with the guarantee that a shortest $s$-$t$ path using only edges from $E_L$ is a shortest augmenting path.

With each augmentation some edges are deleted from $E_L$.

When $E_L$ does not contain an $s$-$t$ path anymore the distance between $s$ and $t$ strictly increases.

Note that $E_L$ is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between $s$ and $t$ in $G_f$ is $k$.

$E_L$ is initialized as the level graph $L_G$.

Perform a DFS search to find a path from $s$ to $t$ using edges from $E_L$.

Either you find $t$ after at most $n$ steps, or you end at a node $v$ that does not have any outgoing edges.

You can delete incoming edges of $v$ from $E_L$.

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between $s$ and $t$ strictly increases.

Initializing $E_L$ for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching $t$) or unsuccessful) decreases the number of edges in $E_L$ and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph $G_f$ and has to check whether the edge is still in $E_L$ for the next search.

There are at most $n$ phases. Hence, total cost is $\mathcal{O}(mn^2)$.