13.2 Relabel to Front

Algorithm 21 relabel-to-front(G, s, t)

initialize preflow
initialize node list L containing $V \setminus \{s, t\}$ in any order
foreach $u \in V \setminus \{s, t\}$ **do**
    $u.\mathit{current\text{-}neighbour} \leftarrow u.\mathit{neighbour\text{-}list\text{-}head}$
$u \leftarrow L.\mathit{head}$
while $u \neq \mathrm{null}$ **do**
    $\mathit{old\text{-}height} \leftarrow \ell(u)$
    discharge(u)
    if $\ell(u) > \mathit{old\text{-}height}$ **then** // relabel happened
        move u to the front of L
    $u \leftarrow u.\mathit{next}$

Lemma 1 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1. - Pushes do no create any new admissible edges. Therefore, if discharge$()$ does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).

       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

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Proof:

- Maintenance:
  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

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Lemma 2
There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \mathrm{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

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Lemma 3
The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.current\text{-}neighbour$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

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Note that by definition a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can at the same time be a non-saturating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 4

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

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Lemma 5
The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 6
The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.