6.2 Master Theorem

Lemma 1

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.
If $f(n) = \mathcal{O}(n^{\log_b(a) - \epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.
If $f(n) = \Omega(n^{\log_b(a) + \epsilon})$ and for sufficiently large n
$af\left(\frac{n}{b}\right) \leq c f(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:
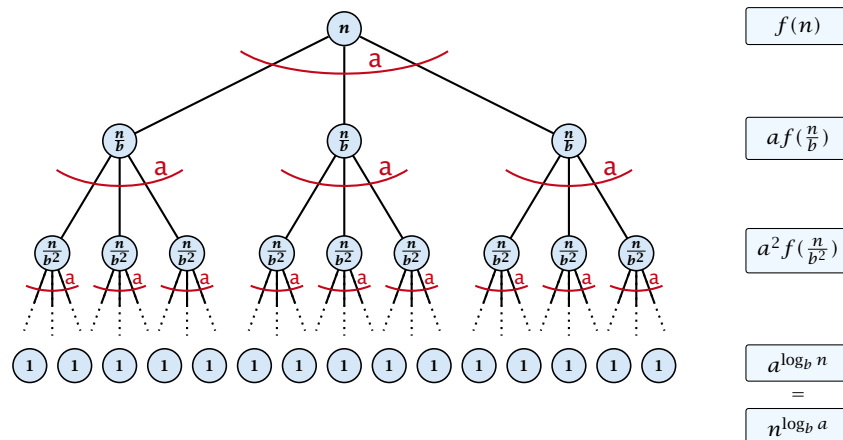
6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \le \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \ge cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\ge c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k}\ell^{k+1}$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$ $\leq \dfrac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$

Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

$\overline{\left\lfloor \text{Where did we use } f(n) \geq \Omega(n^{\log_b a + \epsilon})? \right\rfloor}$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

```
1 1 0 1 1 0 1 0 1     A
1 0 0 0 1 0 0 1 1     B
  1 0 0 1 1 0 1 1 1
1 0 1 1 0 0 1 0 0 0
```

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

```
    1 0 0 0 1 × 1 0 1 1
    ─────────────────
            1 0 0 0 1
          1 0 0 0 1 0
        0 0 0 0 0 0 0
      1 0 0 0 1 0 0 0
    ─────────────────
      1 0 1 1 1 0 1 1
```

- This is also nown as the "school method" for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▸ Computing intermediate results: $\mathcal{O}(nm)$.
- ▸ Adding m numbers of length $\leq 2n$:
  $\mathcal{O}((m + n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:
Suppose that integers A and B are of length $n = 2^k$, for some k.

| B_1 | B_0 | × | A_1 | A_0 |

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

1:	**if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2:	**return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3:	split A into A_0 and A_1	$\mathcal{O}(n)$
4:	split B into B_0 and B_1	$\mathcal{O}(n)$
5:	$Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6:	$Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7:	$Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
8:	**return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▸ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▸ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▸ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

$\Rightarrow$ Not better then the "school method".

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1 \qquad \overbrace{}^{= Z_2} \overbrace{}^{= Z_0}$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1} - \underbrace{A_0 B_0}$$

Hence,

Algorithm 4 mult(A, B)

1:	**if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2:	**return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3:	split A into A_0 and A_1	$\mathcal{O}(n)$
4:	split B into B_0 and B_1	$\mathcal{O}(n)$
5:	$Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6:	$Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7:	$Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
8:	**return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▸ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▸ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▸ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".