6.4 Generating Functions

Definition 4 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is
$$F(z) := \sum_{n\geq 0} a_n z^n;$$
- exponential generating function (exponentielle Erzeugendenfunktion) is
$$F(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$

Example 5

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is
$$F(z) = 1 .$$

2. The generating function of the sequence $(1, 1, 1, \ldots)$ is
$$F(z) = \frac{1}{1 - z} .$$

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

What does $\sum_{n\ge 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\ge 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\ge 0}^{\infty} z^n \right) = 1 \ .$$

This is well-defined.

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n\geq 0} z^n = \frac{1}{1-z} \; .$$

We can compute the derivative:

$$\underbrace{\sum_{n\geq 1} nz^{n-1}}_{\sum_{n\geq 0}(n+1)z^n} = \frac{1}{(1-z)^2}$$

Hence, the generating function of the sequence $a_n = n+1$ is $1/(1-z)^2$.

Formally the derivative of a formal power series $\sum_{n\geq 0} a_n z^n$ is defined as $\sum_{n\geq 0} n a_n z^{n-1}$.

The known rules for differentiation work for this definition. In particular, e.g. the derivative of $\frac{1}{1-z}$ is $\frac{1}{(1-z)^2}$.

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

6.4 Generating Functions

We can repeat this

$$\sum_{n\geq 0}(n+1)z^n = \frac{1}{(1-z)^2} \ .$$

Derivative:

$$\underbrace{\sum_{n\geq 1} n(n+1)z^{n-1}}_{\sum_{n\geq 0}(n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\ge k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} = \sum_{n\ge 0}(n+k)\cdot\ldots\cdot(n+1)z^n$$
$$= \frac{k!}{(1-z)^{k+1}}\ .$$

Hence:

$$\sum_{n\ge 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}}\ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\begin{aligned}\sum_{n\geq 0} nz^n &= \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\end{aligned}$$

The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\
&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\
&= zA(z) + \sum_{n\geq 0} z^n \\
&= zA(z) + \frac{1}{1-z}
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n\geq 0} (n+1) z^n$$

Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
n	$\frac{z}{(1-z)^2}$
a^n	$\frac{1}{1-az}$
n^2	$\frac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	e^z

Some Generating Functions

n-th sequence element	*generating function*
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\frac{F(z)}{1-z}$
nf_n	$z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables
6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n\geq 1} a_n z^n$$

2. Plug in:

$$A(z) = 1 + \sum_{n\geq 1} (2a_{n-1}) z^n$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1} \\ &= 1 + 2z\sum_{n\geq 0}a_n z^n \\ &= 1 + 2z\cdot A(z) \end{aligned}$$

4. Solve for $A(z)$.

$$A(z) = \frac{1}{1-2z}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

Example: $a_n = 3a_{n-1} + n,\ a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$
\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} a_n z^n \\
&= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n \\
&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

   We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

   This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \\ &= (A+3B)z^2 + (-2A-4B-3C)z + (A+B+C) \end{aligned}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$
\begin{aligned}
A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\
&= \frac{7}{4} \cdot \sum_{n\ge 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\ge 0} z^n - \frac{1}{2} \cdot \sum_{n\ge 0} (n+1) z^n \\
&= \sum_{n\ge 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n \\
&= \sum_{n\ge 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4} \Big) z^n
\end{aligned}
$$

6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$.