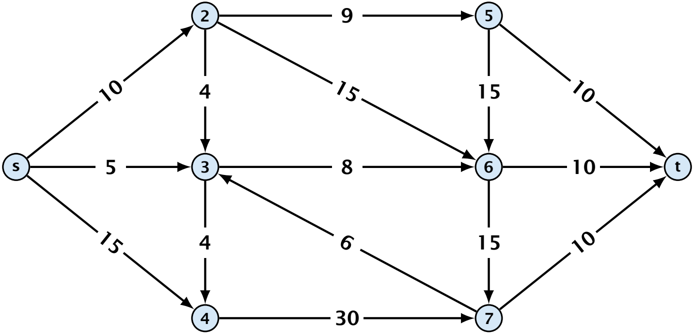
10 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



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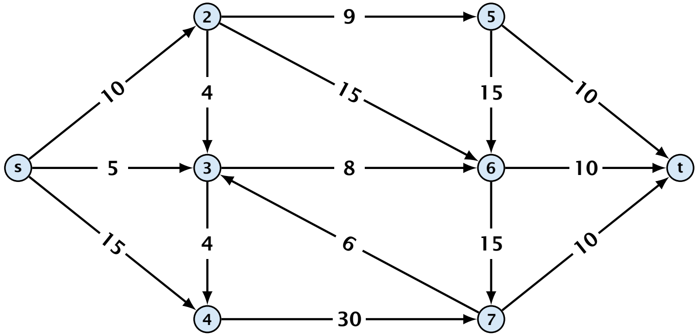

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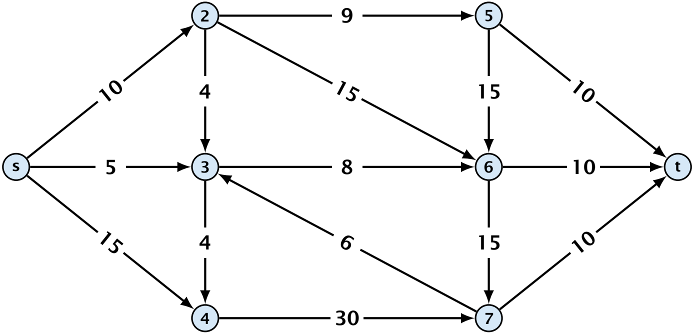

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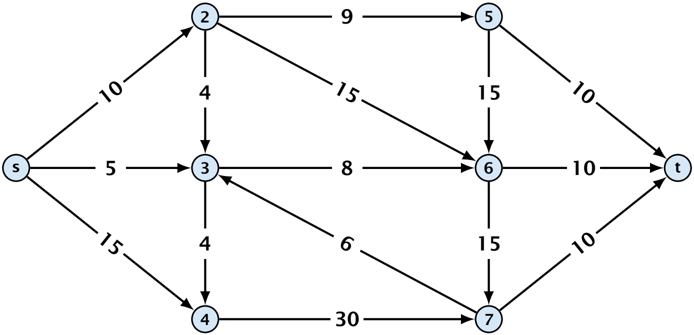

Cuts

Definition 1

An (s,t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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The capacity of a cut A is defined as

$$\mathrm{cap}(A, V \setminus A) := \sum_{e \in \mathrm{out}(A)} c(e) \;,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

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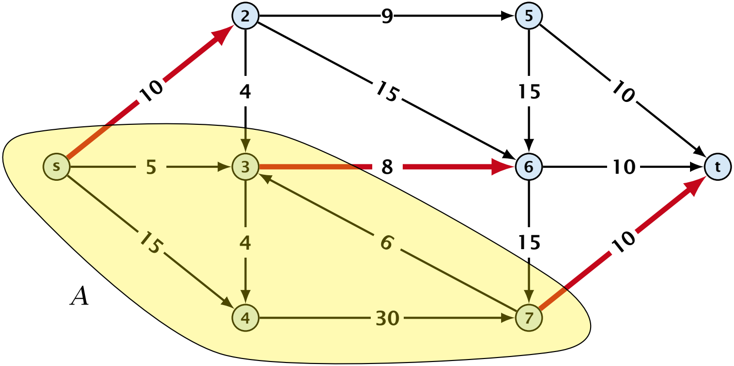
Minimum Cut Problem: Find an (s, t)-cut with minimum capacity.

Cuts

Example 3



The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 4

An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

   $$0 \le f(e) \le c(e) \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

   $$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$

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Flows

Definition 5
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e) \enspace .$$

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Flows

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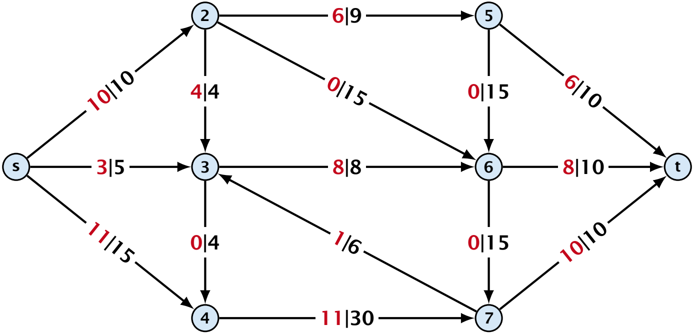

The value of the flow is $\text{val}(f) = 24$.

Flows

Lemma 7 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \enspace .$$

Proof.

$\mathrm{val}(f)$

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$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

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$$\begin{aligned}
\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(s)} f(e) \\
&= \sum_{e\in\mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}} \Bigg(\sum_{e\in\mathrm{out}(v)} f(e) - \sum_{e\in\mathrm{in}(v)} f(e) \Bigg)
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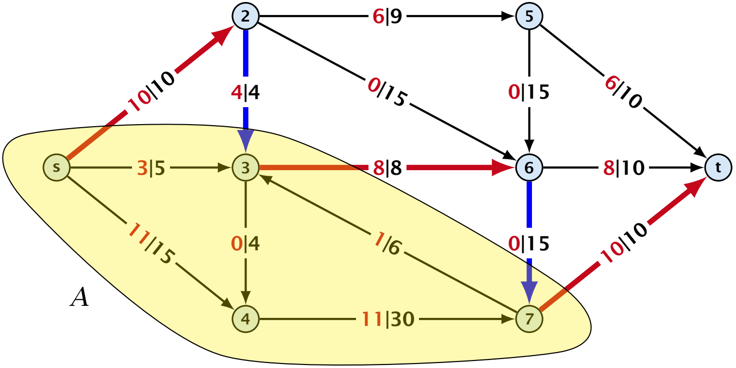
The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 8



2
5
6|9
10|10
4|4
0|15
0|15
6|10
s
3
6
t
3|5
8|8
8|10
11|15
0|4
1/6
0|15
10|10
A
4
7
11|30

Corollary 9

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

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Then f is a maximum flow.

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