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Randomized Algorithms

Exercise Sheet 6

Due: November 30, 2015
at 10:15, in class

Exercise 6.1 (10 points)
Recall from the definition of a randomized circuit family for a function f, the following two properties:

1. If $f_n(x_1, \ldots, x_n) = 0$, then the output of the circuit is 0, regardless of the values of random inputs $r_1, \ldots, r_m$.
2. If $f_n(x_1, \ldots, x_n) = 1$, the output of the circuit is 1 with probability at least $\frac{1}{2}$.

In class, it was shown that if a boolean function has a randomized polynomial-sized circuit family, then it also has a polynomial-sized circuit family. What happens to this proof if in the definition of a randomized circuit family as referred to above, in the second property the fraction $\frac{1}{2}$ would be replaced by $\frac{1}{k}$ for some $k > 2$? (Also comment on the size of the resulting deterministic circuit.)

Exercise 6.2 (10 points)
Consider a round table with n seats. We now randomly assign $m \leq n$ persons to the chairs. What is the expected number of persons without a direct neighbor on the left-hand seat and right-hand seat? (Note that there can be at most one person on each seat.)

Exercise 6.3 (10 points)
Consider the problem where we throw n balls independently and uniformly at random into n bins. Let X be the random variable giving the number of bins containing more than one ball.

Show that

$$\lim_{n\to\infty} E[X]/n \geq 1/c,$$

for some constant $c > 1$.

Exercise 6.4 (10 points)
Consider the following balls-and-bin game. We have a single bin containing one black ball and one white ball. We repeatedly do the following: choose one ball from the bin uniformly at random, check its colour and then put the ball back in the bin together with a new ball of the same colour (you keep a stack of white and black balls somewhere separately). Repeat this until there are n balls in the bin.

Show that the number of white balls in the bin is equally likely to be any number from $1, 2, \ldots, n-1$.