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Randomized Algorithms

Exercise Sheet 13

Due: February 1, 2016
at 10:15, in class

Exercise 13.1 (10 points)
Prove that, for every integer n, there exists a way to 2-color the edges of K_x so that there is no monochromatic clique of size k when

$$x = n - \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Hint: Start by 2-coloring the edges of K_n, then fix things up.

Exercise 13.2 (10 points)
A hypergraph H is a pair of sets (V, E), where V is the set of vertices and E is the set of hyperedges. Every hyperedge in E is a subset of V. In particular, an r-uniform hypergraph is one where the size of each edge is r. For example, a 2-uniform hypergraph is just a standard graph. A dominating set in a hypergraph H is a set of vertices $S \subset V$ such that $e \cap S \neq \emptyset$ for every edge $e \in E$. That is, S hits every edge of the hypergraph.

Let $H = (V, E)$ be an r-uniform hypergraph with n vertices and m edges. Show that there is a dominating set of size at most $np + (1-p)^r m$ for every real number $0 \leq p \leq 1$.
Hint: Start by sampling each vertex to be in the dominating set with probability p. Then modify this set to make sure that you end up with a dominating set.

Exercise 13.3 (10 points)
Let T be a binary search tree in which all the keys are distinct. Consider a leaf x of T and let y be its parent. Show that $key(y)$ is either the smallest key in T larger than $key(x)$ or the largest key in T smaller than $key(x)$.

Exercise 13.4 (10 points)
We have shown that for any element in a set S of size n, the expected depth of a random treap for S is $O(\ln n)$. Show that the depth is $O(\ln n)$ with high probability.
Hint: Recall that the depth of an element x_m can be written as the sum of indicator variables $x_{i,m}$ for the event "x_i is an ancestor of x_m". Start by showing that

$$P[depth(x_m) \leq 7(H_m + H_{n-m+1} - 1)] \geq 1 - (1/n)^6.$$

Then use this fact to prove that with high probability all nodes x_m have a depth of at most $7(H_m + H_{n-m+1} - 1)$ simultaneously.