

Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> w_i	<i>losses</i> ℓ_i	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

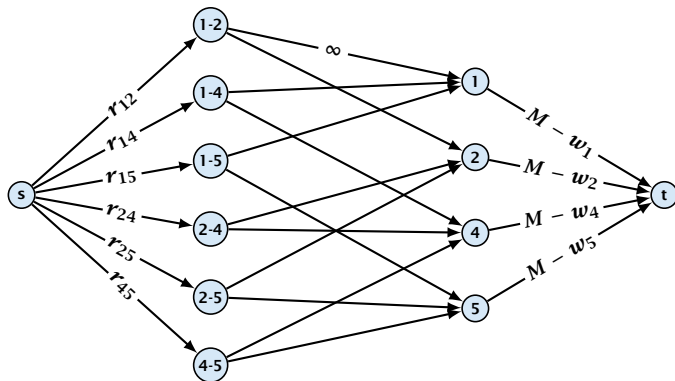
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Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y , r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

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Flow network for $z = 3$. M is number of wins Team 3 can still obtain.

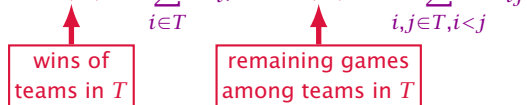


Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

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A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$.

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- ▶ This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

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Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is *integral*.
- ▶ For every pairing x - y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- ▶ This is less than $M - w_x$ because of capacity constraints.
- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- ▶ Hence, team z is not eliminated.

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