Technische Universität München
Fakultät für Informatik
Lehrstuhl für Algorithmen und Komplexität
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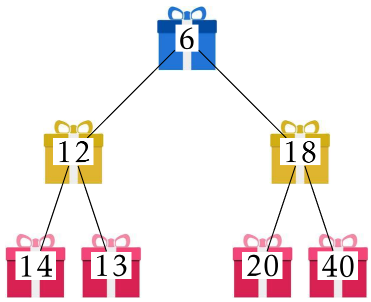
Winter term 2019/20
Assignment 10
December 16, 2019

Efficient Algorithms and Data Structures I

Deadline: January 7, 10:15 am in the ***Efficient Algorithms*** *mailbox.*

Homework 1 (4 Points)

Cueball is preparing christmas by arranging his presents in a binary heap. The presents are labeled with the price he bought them for. Initially, his heap looks like as follows:



(a) Cueball must replace one of the presents in the last minute, so he has to update his heap! Show how the heap looks like after each operation.

   (i) Insert 5

   (ii) Delete 40

(b) Cueball's sister wants to rearrange the heap, yet maintain its heap property. How many valid heaps over the set $\{5, 6, 12, 13, 14, 18, 20\}$ exist?

Homework 2 (5 Points)

Santa's n elves $E_1, \ldots, E_n$ have a reindeer riding contest! At the start elves E_i and E_{i+1} are adjacent to each other. They start riding from a straight line at some angle ϕ_i (determined by their reindeer) and keeps riding in a straight line along this direction at a constant speed $s_i > 0$. Whenever an elf E_j comes across the path traversed by any other elf E_i, we say that E_i defeated E_j and in that case, E_j stops riding.

(a) We call the point where E_i defeats E_j as the point of ambush $A_{i,j} \in \mathbb{R}^2$. Show that if $A_{i',j'}$ is a point of ambush which occurs closest to the start line, then i' and j' are consecutive integers.

   Assume here that all elves start in the same direction (all angles between 0 and 180 degrees), and that no more than 2 elves meet at the same point.

(b) Show how to enumerate in $\mathcal{O}(n \log n)$ time all events where one elf defeats another.

Homework 3 (5 Points)

Santa loves to give gifts that children like! This is why he notes for every child c its happiness score $h_c \in \mathbb{Z}$ after receiving the gift. His data scientist elves are tasked with analyzing these values.
This year, Santa tells the elves that he is interested in the median happiness of the children. He also says that he needs constant updates on the median happiness so far. The elves now want to design a data structure that

- Allows them to insert a new happiness score in $\mathcal{O}(\log n)$ steps,
- allows them to report the current median happiness in $\mathcal{O}(1)$ steps.

Help them design the necessary data structure. Explain your data structure precisely and prove that it reaches the required time bounds.
Note: Let r_i be the element of rank i in a given list. If n is odd, the median is element $r_{(n+1)/2}$. If n is even, the median is $(r_{n/2} + r_{n/2+1})/2$. You may assume that all elements your data structure encounters are distinct.
Hint: One possible solution to this problem uses two binary heaps.

Homework 4 (6 Points)

Santa Claus says that $f(n) \in \overset{\infty}{\Omega}(g(n))$ if there exists a positive constant c such that

$$f(n) \geq c \cdot g(n) \geq 0 \qquad \text{for infinitely many integers } n.$$

(a) Rudolph the red nosed reindeer says that Ω and $\overset{\infty}{\Omega}$ are identical. Prove him wrong by giving two nonnegative functions $f(n)$ and $g(n)$, such that $f(n) \in \overset{\infty}{\Omega}(g(n))$ but $f(n) \notin \Omega(g(n))$.

(b) Find inputs that cause DELETE-MIN, DECREASE-KEY, and DELETE to run in $\Omega(\log n)$ time for a binomial heap.

(c) Santa asks you to explain why running times of INSERT, MINIMUM, and MERGE are $\overset{\infty}{\Omega}(\log n)$ but not $\Omega(\log n)$ for a binomial heap. Can you help him?

Bonus Homework 1 (10 Bonus Points)

Note: Bonus points improve your score for both semester halves!
Answer the following questions. For true/false questions, you must explain your answer, otherwise no points are given.

(a) Suggest a quote for me to put at the end of the final exam. (Great answers get an extra award!)

(b) True/False: $n^{1/2+1/\log n} \in \Theta(\sqrt{n})$.

(c) True/False: A FIND-operation on a splay tree may increase the height of the tree.

(d) True/False: Any 2-independent set of hash functions is a universal set of hash functions.

(e) Which simple modification allows Binomial Heaps to have the MINIMUM() operation run in $\mathcal{O}(1)$?

Bonus Homework 2 (6 Bonus Points)

Last year's exam contained the following statement:

> After 1000 inserts, the expected number of lists in a randomized skiplist is at least 10.

The question was removed from grading, because the proof was deemed too difficult. Give a proof to the claim. Your proof should not rely on the use of a calculator / WolframAlpha…; one should be able to fully understand it by just using pen and paper.

Tutorial Exercise 1

For any positive integer n, show a sequence of Fibonacci heap operations that creates a Fibonacci heap consisting of just one tree that is a linear chain of n nodes.

> On December 25, Isaac Newton's birthday, we celebrate the existence of comprehensible physical laws. [...] One way to celebrate Grav-Mass is to decorate a tree with apples and other fruits. Glue them or attach them, but not too well! The idea is that occasionally a fruit should fall. Put them on the tree no more than 2 feet up, so that they won't get damaged or hurt anybody when they fall. Investigating and perfecting the methods for doing this is a great way expose a child to the process of scientifically studying the behavior of the physical world.
>
> – R. Stallman