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Winter term 2019/20
Assignment 04
November 4, 2019

Efficient Algorithms and Data Structures I

Deadline: November 11, 10:15 am in the ***Efficient Algorithms*** *mailbox.*

Homework 1 (4 Points)

Solve the following recurrence relation using the characteristic polynomial:

$$a_n = -4a_{n-1} - 4a_{n-2} \text{ for } n \geq 2, \quad a_0 = 6, a_1 = -20 \ .$$

Homework 2 (4 Points)

We want to calculate the value of $a_n = \sum_{i=1}^{n} i^2$ by setting up a recurrence relation

(a) State the recurrence relation for a_n w.r.t. a_{n-1} for $n \geq 1$.

(b) Transform this recurrence relation into a linear homogeneous recurrence relation via the method developed in the lecture.

(c) Solve the recurrence relation using the method of the characteristic polynomial.

Homework 3 (6 Points)

This exercise will provide an alternative method for analyzing homogeneous linear recurrences. Consider the following well-known recurrence:

$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n \geq 2 \ .$$

with $a_0 = 0$, $a_1 = 1$.
Let $b_n = a_{n-1}$ for $n \geq 1$ and $\vec{x}_n = (a_{n+1}, b_{n+1})^T$ for $n \geq 0$.

(a) Determine the matrix $M \in \mathbb{R}^{2\times 2}$ so that $\vec{x}_n = M \cdot \vec{x}_{n-1}$ for $n \geq 1$.

(b) Show that $\mathcal{O}(\log n)$ matrix multiplications suffice for computing X^n for a square matrix X.

(c) Determine the eigenvalues of M and, for each eigenvalue, determine a corresponding eigenvector.

(d) Clearly $\vec{x}_0 = (1, 0)^T$. Use the eigenvectors and eigenvalues of M to solve the recurrence relation derived in the first part and give the closed form of a_n.

   Hint: If $\vec{x}_0$ were an eigenvector to eigenvalue λ, what would $\vec{x}_n$ be?

Homework 4 (6 Points)

The honey bee Armin randomly snacks on his two favorite flowers, a red rose and a yellow tulip. Whenever he is on the rose, he stays there with probability 0.9 and moves to the tulip with probability 0.1. If he is on the tulip, he stays there with probability 0.8 and moves to the rose with probability 0.2.

1. Let r_n and t_n denote the probability that Armin is after timestep n on the rose and the tulip, respectively. Express r_n and t_n using a two-dimensional recurrence relation for $n \geq 1$, i.e. find a matrix P such that

$$\begin{pmatrix} r_n \\ t_n \end{pmatrix} = P \cdot \begin{pmatrix} r_{n-1} \\ t_{n-1} \end{pmatrix} \ .$$

   Also draw the corresponding Markov chain.

2. Beatrix, the queen of the hive, asks for a closed form of r_n and t_n, assuming that Armin starts on the rose. Use the eigenvectors of P.

3. Determine the limiting distribution of Armin over the flowers. How is it linked to the eigenvectors of P?

Hint: Use the approach from Homework 3.

Tutorial Exercise 1

Solve the following recurrence relation using generating functions:

$$a_n = a_{n-1} + 2^{n-1} \text{ for } n \geq 1 \text{ with } a_0 = 2 \ .$$

[The PageRank Computation] is essentially the determination of the limiting distribution of a random walk on the web graph.

– Page, Brin, Motwani, Winograd