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Randomized Algorithms

Exercise Sheet 8

Due: December 15, 2014

Exercise 1: (10 points)
Consider a bin which initially contains one black ball and one white ball. We repeatedly do the following: choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same color. We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n-1$. (*Hint: Induction.*)

Exercise 2: (10 points)
Compute the probability that there are three people who share the same birthday in a lecture hall with $n = 100$ people. Assume that no person is born in a leap year, a person is equally likely to be born on any day of the year and the birthdays of different people are independent.

Exercise 3: (10 points)
There are m hospitals each one with a certain number of available intern positions and n graduating medical students who are interested in joining one of the hospitals. Each hospital has a ranking of the students in decreasing order of preference and each student has a ranking of the hospitals in decreasing order of preference. Assume that there are more graduating students than available positions in the m hospitals. The objective is to assign each student to at most one hospital in a way that all the available positions of all hospitals are filled. Obviously, there will be some students who will not get assigned to any hospital. An assignment of students to hospitals is stable if neither of the following situations occurs

- First type of instability: There are two students s, s' and a hospital h such that
  - s is assigned to h,
  - s' is assigned to no hospital, and
  - h prefers s' to s
- Second type of instability: There are two students s, s' and two hospitals h, h', such that
  - s is assigned to h,
  - s' is assigned to h',
  - h prefers s' to s, and
  - s' prefers h to h'.

Propose an algorithm for finding stable assignment of students to hospitals and prove that your algorithm is correct.

Exercise 4: (10 points)
Consider the following process which is related to the coupon collector problem. There are n bins and n players, where each player has an infinite supply of balls. Initially, all bins are empty. The process proceeds in a sequence of rounds. In each round, each player throws a ball into an empty bin chosen uniformly at random among all currently empty bins. Show that the expected number of rounds before every bin is non-empty is $O(\log^* n)$, where $\log^* n$ is the iterated logarithm of n defined as follows:

$$\log^* n = \begin{cases} 1 + \log^*(\log n), & \text{if } n > 1 \\ 0, & \text{if } n \leq 1 \end{cases}$$

Hint: For any integer $i \geq 0$, we define

$$f(i+1) = 2^{f(i)} \textit{ for } i \geq 1 \qquad f(0) = 1$$

During the process, we say that the system is in the state S_i if it holds that

$$\frac{n}{f(i+1)} \leq \textit{number of empty bins} < \frac{n}{f(i)}$$